A Novel Numerical Method for Exact Model Matching Problem with Stability ⋆

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Abstract

In systems design the exact model matching problem with stability consists of compensating a given system, using a realizable control law of a specified structure, in order to ensure the stability of the compensated system and achieve a target closed-loop transfer function. In this paper we develop a novel numerical method to verify the solvability of the problem for left invertible systems and further construct a desired solution. Our method has a complexity which is cubic in the state dimension of the system and the desired model and can be implemented in a numerically reliable way. *Copyright@2002 IFAC*

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1 Introduction

In this paper we study the solution of the exact model matching problem with stability for a system (A, B, C) characterized by

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times q}$, $C \in \mathbf{R}^{p\times n}$, and where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^q$ is the input and $y \in \mathbf{R}^p$ is the output of the system. We consider a linear state feedback control law of the form

$$u = Fx + Gv, \tag{2}$$

where $F \in \mathbf{R}^{q\times n}$, $G \in \mathbf{R}^{q\times q}$ and $v \in \mathbf{R}^q$ is the new input. Then the closed-loop system of (1)-(2) is

$$\dot{x} = (A + BF)x + BGv, \quad y = Cx. \tag{3}$$

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The transfer function matrix of the closed-loop system (3) from the new input v to the output y is then given by

$$M_{F,G}(s) := C(sI_n - A - BF)^{-1}BG. \tag{4}$$

In this paper we state the exact model matching problem with stability as follows :

Definition 1 *Given system (1) and a strictly proper transfer matrix $M(s)$, the exact model matching problem with stability consists of finding matrices $F \in \mathbf{R}^{q\times n}$ and $G \in \mathbf{R}^{q\times q}$ with G nonsingular such that*

$$M_{F,G}(s) = M(s), \tag{5}$$

and furthermore the matrix $A + BF$ is stable (i.e. has all its eigenvalues in the open left half plane).

The exact model matching problem is of both theoretical and practical importance since a number of control problems can be linked to it [6]. Among those are the decoupling problem, the model tracking problem, and model reference adaptive control. The exact model matching problem was originally presented without stability requirement in [20], where a solution was given for the case of invertible systems, using feedback invariants, a coordinate transformation and a set of polynomial matrix equations. In [19,18] the restriction of invertibility was eliminated and the problem was reduced to that of solving a system of linear algebraic equations. The problem was treated in [14] by means of polynomial matrix equations under the assumption that the open-loop system is invertible. A time-domain solution was developed in [16], which led to a set of nonlinear matrix equations yielding the feedback law. In [13] a solution was given in the frequency domain which reduced the problem to that of solving a large set of linear equations. In [8,12], the problem was studied via a polynomial approximation approach. Further related results were given in [7,9,10,17,21,15].

The exact model matching problem has thus been studied extensively in the last three decades but several issues have not been addressed appropriately :
(i) numerically verifiable necessary and sufficient solvability conditions for the exact model matching problem for a general linear time-invariant system and a given general proper model are still not available in the literature : existing results transform the underlying problem to another kind of problem such as nonlinear equations or rational/polynomial matrix equations;
(ii) solutions based on rational/polynomial matrix equations [8,11,12,14,15,20] or nonlinear matrix equations [16] with constant unknowns, do not lead to numerically reliable methods for solving the exact model matching;
(iii) results in [13,19,18] are delicate from a numerical point of view because the resulting linear systems of equations are much larger than the state dimension of the models $M_0(s) := C(sI_m - A)^{-1}B$ and $M(s)$ and the computations of the involved coefficient matrices are poorly conditioned.
(iv) although the model matching problem in H_∞ control can be solved using algebraic Riccati equations, this does not appear to be the case – as far as we know – for the exact model matching problem.
Therefore, we believe there is still a lack of numerically reliable methods for solving it. In this paper we thus revisit the exact model matching problem with stability from a numerical point of view.

We first point out that if the original system (1) is unstabilizable then the problem has no solution. On the other hand, if its uncontrollable modes are stable, then they play no role in the problem. We can therefore assume *without loss of generality* that the system (1) is *controllable* (but not necessarily observable). We will denote its transfer function by $M_0(s) = C(sI_n - A)^{-1}B$. If the given realization of the desired transfer matrix $M(s)$ is not minimal, we can always obtain one via a numerically stable procedure [4]. We can therefore assume *without loss of generality* that we are given a *minimal* realization of the desired model $M(s)$: $M(s) = \mathcal{C}(sI_m - \mathcal{A})^{-1}\mathcal{B}$, with $\mathcal{A} \in \mathbf{R}^{m\times m}$, $\mathcal{B} \in \mathbf{R}^{m\times q}$, and $\mathcal{C} \in \mathbf{R}^{p\times m}$. Notice that if $M(s)$ is left invertible then $M_0(s)$ must also be left invertible since this property is not affected by the matrices F and G in (5), and that $m \leq n$ is required since $M(s)$ is also given by (4).

The main purpose of this paper is to develop a new method to verify the solvability and compute a desired solution for the exact model matching problem with stability for left invertible systems. In contrast to [15], we do not require the computation of the zero structure of the systems $M_0(s)$, $M(s)$ and $\left[\; M_0(s) \;\; M(s) \;\right]$. Furthermore, our method has a computational complexity which is cubic in the state dimensions of $M_0(s)$ and $M(s)$ and can be implemented using orthogonal transformations only.

We will denote the complex plane by $\mathbf{C}$, the closed right half plane by $\mathbf{C}_{unst}$, the spectrum of a square matrix A by $\sigma(A)$, and the spectrum of a regular pencil $sE - A$ by $\sigma(E, A)$. We will call $\cap_{K\in\mathbf{R}^{n\times p}}\sigma(A + KC)$ the unobservable spectrum of the pair (A, C) and

$$\text{rank}_g(R(s)) := \text{rank}(R(s)) \text{ for almost all } s \in \mathbf{C}$$

the *generic rank* of a rational matrix $R(s)$.

Lemma 2 *Let $\{A, B, C\}$ be a realization of dimension n for which the pair (A, B) is controllable. Let $\{A_{22}, B_2, C_2\}$ be a minimal realization of dimension n_2 of $C(sI_n - A)^{-1}B$. Then*

(i) $\sigma(A)$ is the union of $\sigma(A_{22})$ and the unobservable spectrum of the pair (A, C),
(ii) the invariant zeros of the system $\{A, B, C\}$ are the union of the invariant zeros of the system $\{A_{22}, B_2, C_2\}$ and the unobservable spectrum of the pair (A, C).

Proof. The proof is easy and thus omitted it here. □

2 Reduction to invertible sub-blocks

The exact model matching problem with stability is better understood for square invertible systems than for left invertible ones (see e.g. [15]). This observation motivates us to reduce the underlying problem to one for an invertible system. For this purpose, we show in this section that there always exists an orthogonal transformation $\mathcal{W}$ such that the top $q \times q$ block of $\mathcal{W}M(s)$ is invertible. Rephrasing the original problem in this coordinate system will eventually lead to a closed form solution which will be given in the next section.

Theorem 3 *Given left invertible matrices $M(s) = \mathcal{C}(sI_m - \mathcal{A})^{-1}\mathcal{B}$ and $M_0(s) = C(sI_n - A)^{-1}B$ there always exists an orthogonal transformation $\mathcal{W}$ such that $\mathcal{W}M(s)$ and $\mathcal{W}M_0(s)$ have the following generalized state space realizations, obtained under orthogonal coordinate transformations $\mathcal{U}, \mathcal{V}^\star_{b,g}(\text{Ker}C), U$ and V :*

(i) $\mathcal{W}M(s) = \mathcal{C}_{new}(s\mathcal{E}_{new} - \mathcal{A}_{new})^{-1}\mathcal{B}_{new}$

where

$$\begin{cases} \mathcal{A}_{new} - s\mathcal{E}_{new} = \mathcal{U}(\mathcal{A} - sI_m)\mathcal{V}^\star_{b,g}(\mathrm{Ker}C) \\ \quad = \begin{matrix} m_1 \\ m_2 \\ q \end{matrix} \overset{\begin{matrix} m_1 & m_2 & q \end{matrix}}{\begin{bmatrix} \mathcal{A}_{11} - s\mathcal{E}_{11} & \mathcal{A}_{12} - s\mathcal{E}_{12} & \mathcal{A}_{13} - s\mathcal{E}_{13} \\ 0 & \mathcal{A}_{22} - s\mathcal{E}_{22} & \mathcal{A}_{23} - s\mathcal{E}_{23} \\ \mathcal{A}_{31} - s\mathcal{E}_{31} & \mathcal{A}_{32} - s\mathcal{E}_{32} & \mathcal{A}_{33} - s\mathcal{E}_{33} \end{bmatrix}}, \\ \mathcal{B}_{new} = \mathcal{U}\mathcal{B} = \begin{matrix} m_1 \\ m_2 \\ q \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{B}_3 \end{bmatrix}, \\ \mathcal{C}_{new} = \mathcal{W}\mathcal{C}\mathcal{V}^\star_{b,g}(\mathrm{Ker}C) = \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} m_1 & m_2 & q \end{matrix}}{\begin{bmatrix} 0 & 0 & \mathcal{C}_{13} \\ 0 & \mathcal{C}_{22} & \mathcal{C}_{23} \end{bmatrix}}, \end{cases} \tag{6}$$

$$\mathrm{rank}(\mathcal{B}_3) = \mathrm{rank}(\mathcal{C}_{13}) = q, \ \mathrm{rank}_g(\mathcal{A}_{11} - s\mathcal{E}_{11}) = m_1, \tag{7}$$

$$\mathrm{rank} \begin{bmatrix} \mathcal{A}_{22} - s\mathcal{E}_{22} \\ \mathcal{C}_{22} \end{bmatrix} = \mathrm{rank}(\mathcal{E}_{22}) = m_2, \quad \forall s \in \mathbf{C}. \tag{8}$$

(ii) $\mathcal{W}M_0(s) = C_{new}(sE_{new} - A_{new})^{-1}B_{new}$
where

$$\begin{cases} A_{new} - sE_{new} = U(A - sI_n)V \\ \quad = \begin{matrix} n_1 \\ n_2 \\ q \end{matrix} \overset{\begin{matrix} n_1 & n_2 & q \end{matrix}}{\begin{bmatrix} A_{11} - sE_{11} & A_{12} - sE_{12} & A_{13} - sE_{13} \\ 0 & A_{22} - sE_{22} & A_{23} - sE_{23} \\ A_{31} - sE_{31} & A_{32} - sE_{32} & A_{33} - sE_{33} \end{bmatrix}}, \\ B_{new} = UB = \begin{matrix} n_1 \\ n_2 \\ q \end{matrix} \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix}, \\ C_{new} = \mathcal{W}CV = \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} n_1 & n_2 & q \end{matrix}}{\begin{bmatrix} 0 & 0 & C_{13} \\ 0 & C_{22} & C_{23} \end{bmatrix}} \begin{matrix} \}q \\ \}p-q \end{matrix}, \end{cases} \tag{9}$$

$$\mathrm{rank}(B_3) = q, \ \mathrm{rank}_g(A_{11} - sE_{11}) = n_1, \tag{10}$$

$$\mathrm{rank} \begin{bmatrix} A_{22} - sE_{22} & A_{23} - sE_{23} \\ 0 & C_{13} \\ C_{22} & C_{23} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} E_{22} & E_{23} \\ 0 & C_{13} \\ C_{22} & C_{23} \end{bmatrix} = n_2 + q, \ \forall s \in \mathbf{C}. \tag{11}$$

Proof. The forms (6) and (9) are direct consequences of the well-established generalized upper triangular form of an arbitrary matrix pencil [4]. We omit the derivation of the forms (6-8) and (9-11) since they can be constructed using techniques similar to those of [2]. □

The form (6) provides not only the invariant zero structure of $M(s)$ but also links two sub-blocks of $\mathcal{W}M(s)$, as shown in the following corollary.

Corollary 4 *With the notation of Theorem 3, the following properties hold :*

(a) the invariant zeros of $M(s)$ are given by the finite generalized eigenvalues of $s\mathcal{E}_{11} - \mathcal{A}_{11}$,
(b) $\begin{bmatrix} I_q & 0_{q\times(p-q)} \end{bmatrix} \mathcal{W}M(s)$ is invertible,
(c) $\begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{W}M(s) = \mathcal{Z}(s) \begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M(s)$,
where $\mathcal{Z}(s) = [\mathcal{C}_{23} - \mathcal{C}_{22}(s\mathcal{E}_{22} - \mathcal{A}_{22})^{-1}(s\mathcal{E}_{23} - \mathcal{A}_{23})]\mathcal{C}_{13}^{-1}$.

Proof. Parts (a) and (b) follow from conditions (7) and (8), respectively. For part (c) we have by a simple calculation that

$$\mathcal{Z}(s) \begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M(s)$$

$$= \left[\begin{array}{cccc|c} \mathcal{A}_{22} - s\mathcal{E}_{22} & 0 & 0 & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & \mathcal{A}_{11} - s\mathcal{E}_{11} & \mathcal{A}_{12} - s\mathcal{E}_{12} & \mathcal{A}_{13} - s\mathcal{E}_{13} & 0 \\ 0 & 0 & \mathcal{A}_{22} - s\mathcal{E}_{22} & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & \mathcal{A}_{31} - s\mathcal{E}_{31} & \mathcal{A}_{32} - s\mathcal{E}_{32} & \mathcal{A}_{33} - s\mathcal{E}_{33} & \mathcal{B}_3 \\ \hline \mathcal{C}_{22} & 0 & 0 & \mathcal{C}_{23} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} \mathcal{A}_{22} - s\mathcal{E}_{22} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}_{11} - s\mathcal{E}_{11} & \mathcal{A}_{12} - s\mathcal{E}_{12} & \mathcal{A}_{13} - s\mathcal{E}_{13} & 0 \\ 0 & 0 & \mathcal{A}_{22} - s\mathcal{E}_{22} & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & \mathcal{A}_{31} - s\mathcal{E}_{31} & \mathcal{A}_{32} - s\mathcal{E}_{32} & \mathcal{A}_{33} - s\mathcal{E}_{33} & \mathcal{B}_3 \\ \hline \mathcal{C}_{22} & 0 & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{W}M(s). \quad \square$$

The transfer function
$\mathcal{W}M_{F,G}(s) = C_{new}(sE_{new} - A_{new} - B_{new}FV)^{-1}B_{new}G$
is supposed to be equal to $\mathcal{W}M(s)$, so we are looking for an analogous result to Corollary 4 for this transfer function which depends on F and G.

Corollary 5 *With the notation of Theorem 3 and with G invertible, the following properties hold :*

(a) the invariant zeros of $M_{F,G}(s)$ are given by the finite generalized eigenvalues of $sE_{11} - A_{11}$,
(b) $\begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M_{F,G}(s)$ is invertible for some F, G iff

$$\mathrm{rank}(C_{13}) = q, \quad \mathrm{rank}_g(A_{22} - sE_{22}) = n_2, \tag{12}$$

(c) if rank conditions (12) hold, then
$\begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{W}M_{F,G}(s) = Z(s) \begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M_{F,G}(s)$,
where
$Z(s) = [C_{23} - C_{22}(sE_{22} - A_{22})^{-1}(sE_{23} - A_{23})]C_{13}^{-1}$.

Proof. We first point out that the realization of $\mathcal{W}M_{F,G}$ has the same form as that of $\mathcal{W}M(s)$ and that properties (10) and (11) of Theorem 3 are not affected by the feedback matrices F and G. Part (a) then follows from conditions (10) and (11). We need now in addition that (12) holds in order to prove part (b). Finally, if (12) holds, the proof of part (c) is similar to that of part (c) in Corollary 4. $\square$

In order to have $M_{F,G}(s) = M(s)$ one needs to satisfy simultaneously

$$\begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M_{F,G}(s) = \begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M(s), \tag{13}$$

$$\begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{W}M_{F,G}(s) = \begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{W}M(s). \tag{14}$$

(13) implies that $\begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M_{F,G}(s)$ must be invertible since $\begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M(s)$ is invertible. By Corollary 5, conditions (12) are then satisfied. (14) is then satisfied if and only if $Z(s) = \mathcal{Z}(s)$ since $\begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{W}M(s)$ is invertible. This then finally leads to the following theorem.

Theorem 6 *With the notation of Theorem 3 and with G nonsingular, let the RQ factorization of $\begin{bmatrix} C_{13} & \mathcal{C}_{13} \end{bmatrix}$ be*

$$\begin{bmatrix} C_{13} & \mathcal{C}_{13} \end{bmatrix} Z = \begin{bmatrix} R & 0 \end{bmatrix}, \ Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

where R is nonsingular, $Z_{11} \in \mathbf{R}^{q \times q}$ and Z is orthogonal. Then $M_{F,G}(s) = M(s)$ iff the condition (12) holds,

$$\operatorname{rank}_g \begin{bmatrix} \mathcal{A}_{22} - s\mathcal{E}_{22} & 0 & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & A_{22} - sE_{22} & 0 & A_{23} - sE_{23} \\ 0 & 0 & \mathcal{C}_{13} & C_{13} \\ \mathcal{C}_{22} & C_{22} & \mathcal{C}_{23} & C_{23} \end{bmatrix}$$

$$= m_2 + n_2 + q, \tag{15}$$

and

$$\operatorname{rank}_g (\begin{bmatrix} A_{11} - sE_{11} & A_{12} - sE_{12} & 0 & 0 & (A_{13} - sE_{13})Z_{12} \\ 0 & A_{22} - sE_{22} & 0 & 0 & (A_{23} - sE_{23})Z_{12} \\ 0 & 0 & \mathcal{A}_{11} - s\mathcal{E}_{11} & \mathcal{A}_{12} - s\mathcal{E}_{12} & (\mathcal{A}_{13} - s\mathcal{E}_{13})Z_{22} \\ 0 & 0 & 0 & \mathcal{A}_{22} - s\mathcal{E}_{22} & (\mathcal{A}_{23} - s\mathcal{E}_{23})Z_{22} \\ \hat{F}_1 + A_{31} - sE_{31} & \hat{F}_2 + A_{32} - sE_{32} & \hat{G}(\mathcal{A}_{31} - s\mathcal{E}_{31}) & \hat{G}(\mathcal{A}_{32} - s\mathcal{E}_{32}) & \hat{F}_3 Z_{12} + D(s) \end{bmatrix}$$

$$= (n-q) + (m-q), \tag{16}$$

where

$$\begin{cases} \overset{\;n_1\quad\; n_2\quad\; q}{\begin{bmatrix} \hat{F}_1 & \hat{F}_2 & \hat{F}_3 \end{bmatrix}} := \hat{F} := B_3 F V, \ \hat{G}\mathcal{B}_3 := B_3 G, \\ D(s) := (A_{33} - sE_{33})Z_{12} + \hat{G}(\mathcal{A}_{33} - s\mathcal{E}_{33})Z_{22}. \end{cases} \tag{17}$$

Proof. Obviously, $M_{F,G}(s) = M(s)$ and hence (13) and (14) hold with G invertible iff

$$\begin{bmatrix} I_q & 0 \end{bmatrix} C_{new}(sE_{new} - A_{new} - B_{new}FV)^{-1} B_{new} G$$
$$= \begin{bmatrix} I_q & 0 \end{bmatrix} \mathcal{C}_{new}(s\mathcal{E}_{new} - \mathcal{A}_{new})^{-1} \mathcal{B}_{new}, \tag{18}$$

$$\begin{bmatrix} 0 & I_{p-q} \end{bmatrix} C_{new}(sE_{new} - A_{new} - B_{new}FV)^{-1} B_{new} G$$
$$= \begin{bmatrix} 0 & I_{p-q} \end{bmatrix} \mathcal{C}_{new}(s\mathcal{E}_{new} - \mathcal{A}_{new})^{-1} \mathcal{B}_{new}.$$

As pointed out above, this is equivalent to conditions (12) and condition (13), and $Z(s) = \mathcal{Z}(s)$. Since

$$\operatorname{rank}_g (\begin{bmatrix} I_{n+m} & 0 & 0 \\ 0 & I_q & 0 \end{bmatrix}$$
$$\times \begin{bmatrix} A_{new} - sE_{new} + B_{new}FV & 0 & B_{new}G \\ 0 & \mathcal{A}_{new} - s\mathcal{E}_{new} & -\mathcal{B}_{new} \\ C_{new} & \mathcal{C}_{new} & 0 \end{bmatrix})$$
$$= n + m +$$
$$\operatorname{rank}_g \{ \begin{bmatrix} I_q & 0 \end{bmatrix} [C_{new}(sE_{new} - A_{new} - B_{new}FV)^{-1} B_{new} G$$
$$- \mathcal{C}_{new}(s\mathcal{E}_{new} - \mathcal{A}_{new})^{-1} \mathcal{B}_{new}] \},$$

so, a simple calculation yields that (18), (16) and (13) are equivalent. It thus only remains to show that $Z(s) = \mathcal{Z}(s)$ is equivalent to (15) when (16) holds. It is easy to see from $\operatorname{rank}_g(A_{22} - sE_{22}) = n_2$ and $\operatorname{rank}_g(\mathcal{A}_{22} - s\mathcal{E}_{22}) = m_2$ that

$$\operatorname{rank}_g \begin{bmatrix} \mathcal{A}_{22} - s\mathcal{E}_{22} & 0 & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & A_{22} - sE_{22} & 0 & A_{23} - sE_{23} \\ 0 & 0 & \mathcal{C}_{13} & C_{13} \\ \mathcal{C}_{22} & C_{22} & \mathcal{C}_{23} & C_{23} \end{bmatrix}$$

$$= m_2 + n_2 + q + \operatorname{rank}_g(Z(s) - \mathcal{Z}(s)),$$

which completes the proof. $\square$

We have thus reduced the exact model matching problem (*without* stability) to a set of rank conditions. Conditions (12) and (15) can be verified easily [4]. Finally, condition (16) depends on F and G and we discus in the next section how to check it.

3 Conditions for F and G

In this section we derive solvability conditions for F and G related to rank condition (16), under the assumption that condition (12) already holds.

Lemma 7 *With notation of Theorem 6 and assuming that (12) holds, denote by $\Phi - s\Theta$ the pencil*

$$\begin{bmatrix} A_{11}-sE_{11} & A_{12}-sE_{12} & 0 & 0 & (A_{13}-sE_{13})Z_{12} \\ 0 & A_{22}-sE_{22} & 0 & 0 & (A_{23}-sE_{23})Z_{12} \\ 0 & 0 & \mathcal{A}_{11}-s\mathcal{E}_{11} & \mathcal{A}_{12}-s\mathcal{E}_{12} & (\mathcal{A}_{13}-s\mathcal{E}_{13})Z_{22} \\ 0 & 0 & 0 & \mathcal{A}_{22}-s\mathcal{E}_{22} & (\mathcal{A}_{23}-s\mathcal{E}_{23})Z_{22} \end{bmatrix}.$$

Then there exist orthogonal matrices P and Q such that

$$P(\Phi - s\Theta)Q = \begin{bmatrix} \Phi_{11}-s\Theta_{11} & \Phi_{12} & \Phi_{13}-s\Theta_{13} \\ 0 & 0 & \Phi_{23}-s\Theta_{23} \end{bmatrix}, \quad (19)$$

where $\Phi_{11} \in \mathbf{R}^{\tau_1 \times \tau_1}$, $\Phi_{12} \in \mathbf{R}^{\tau_1 \times q}$,

$$\operatorname{rank} \begin{bmatrix} \Phi_{11}-s\Theta_{11} & \Phi_{12} \end{bmatrix} = \operatorname{rank}(\Theta_{11}) = \tau_1, \ \forall s \in \mathbf{C}, \quad (20)$$

$$\operatorname{rank}_g(\Phi_{23}-s\Theta_{23}) = n+m-2q-\tau_1. \quad (21)$$

Proof. It follows from Theorem 3 that $\mathcal{E}_{22}$, $A_{11}-sE_{11}$ and $\mathcal{A}_{11}-s\mathcal{E}_{11}$ are invertible, and from (12) that $A_{22}-sE_{22}$ is also invertible. This implies that $\Phi - s\Theta$ is of full column normal rank. Therefore, the orthogonal matrices P and Q and the form (19) with properties (20) and (21) can be obtained by computing the generalized upper triangular form of the pencil $\Phi - s\Theta$ [4]. □

Let us now apply the transformation Q also to the following submatrices derived from (16)

$$\begin{bmatrix} A_{31}-sE_{31} & A_{32}-sE_{32} & 0 & 0 & (A_{33}-sE_{33})Z_{12} \\ I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_{12} \\ 0 & 0 & \mathcal{A}_{31}-s\mathcal{E}_{31} & \mathcal{A}_{32}-s\mathcal{E}_{32} & (\mathcal{A}_{33}-s\mathcal{E}_{33})Z_{22} \end{bmatrix} Q = \begin{array}{c} \begin{array}{ccc} \tau_1 & q & n+m-2q-\tau_1 \end{array} \\ \begin{bmatrix} \Phi_{31}-s\Theta_{31} & \Phi_{32}-s\Theta_{32} & \Phi_{33}-s\Theta_{33} \\ J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \\ \Pi_{31}-s\Xi_{31} & \Pi_{32}-s\Xi_{32} & \Pi_{33}-s\Xi_{33} \end{bmatrix} \end{array}. \quad (22)$$

Eliminating the block column corresponding to the invertible pencil $\Phi_{23}-s\Theta_{23}$ we finally obtain that the rank condition (16) is equivalent to:

$$\tau_1 = \operatorname{rank}_g \begin{bmatrix} \Phi_{11}-s\Theta_{11} & \Phi_{12} \\ X - s(\Phi_{31}+\hat{G}\Xi_{31}) & Y - s(\Theta_{32}+\hat{G}\Xi_{32}) \end{bmatrix}$$

$$\text{where } \begin{cases} \tau_1 = \operatorname{rank}(\Theta_{11}), \\ X = \Phi_{31} + \hat{G}\Pi_{31} + \sum_{i=1}^{3} \hat{F}_i J_{i1}, \\ Y = \Phi_{32} + \hat{G}\Pi_{32} + \sum_{i=1}^{3} \hat{F}_i J_{i2}. \end{cases}$$

Using [3], we obtain rank conditions for $\hat{F}$ and $\hat{G}$:

$$\Theta_{32} + \hat{G}\Xi_{32} = 0, \quad \operatorname{rank} \begin{bmatrix} \Theta_{11} & \Phi_{11} & \Phi_{12} \\ \Theta_{31}+\hat{G}\Xi_{31} & X & Y \end{bmatrix} = \tau_1.$$

Consider the RQ factorization

$$\begin{bmatrix} \Theta_{11} & \Phi_{11} & \Phi_{12} \end{bmatrix} W = \begin{bmatrix} \Delta & 0 \end{bmatrix},$$

where $\Delta \in \mathbf{R}^{\tau_1 \times \tau_1}$ and W is orthogonal. Denote

$$\begin{bmatrix} \Xi_{31} & \Pi_{31} & \Pi_{32} \\ \Theta_{31} & \Phi_{31} & \Phi_{32} \\ 0 & J_{11} & J_{12} \\ 0 & J_{21} & J_{22} \\ 0 & J_{31} & J_{32} \end{bmatrix} W = \begin{array}{c} \\ q \\ q \\ n_1+n_2+q \end{array} \begin{array}{c} \begin{array}{cc} \tau_1 & \tau_1+q \end{array} \\ \begin{bmatrix} \times & M \\ \times & L \\ \times & N \end{bmatrix} \end{array}. \quad (23)$$

we obtain the set of linear equations

$$\hat{G}M + \hat{F}N + L = 0, \quad \hat{G}\Xi_{32} + \Theta_{32} = 0. \quad (24)$$

The above analysis leads to the following theorem.

Theorem 8 *With the notation above, condition (16) holds with G nonsingular iff equations (24) are satisfied by matrices $\hat{F}$, $\hat{G}$ with $\hat{G}$ nonsingular. In such a case, a desired feedback matrix pair (F, G) to (16) with G nonsingular, is obtained from (17).*

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4 Stability Condition of $A + BF$

Up to now, we have not included the condition of stability of $A + BF$, which we will discuss in this section. In the following theorem we show that only the anti-stable eigenspaces of the pencils $sE_{11} - A_{11}$ and $s\mathcal{E}_{11} - \mathcal{A}_{11}$ play a role in the stability requirement of the exact model matching problem for the system (1) and the desired model $M(s)$. Our proof is new and different from that in [15]. It clarifies the role played by the anti-stable invariant zeros in the exact model matching with stability.

Theorem 9 *With the notation of Theorem 3, let F and G with G nonsingular satisfy $M_{F,G}(s) = M(s)$. Then $A+BF$ is stable iff $\mathcal{A}$ is stable and $\sigma(E_{11}, A_{11}) \cap \mathbf{C}_{unst} = \sigma(\mathcal{E}_{11}, \mathcal{A}_{11}) \cap \mathbf{C}_{unst}$.*

Proof. The equality $M_{F,G}(s) = M(s)$ implies that $C(sI_n - A - BF)^{-1}BG = \mathcal{C}(sI_m - \mathcal{A})^{-1}\mathcal{B}$, where $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ is a minimal realization and hence $n \geq m$. Let us define $\sigma_{A+BF,C}$ as the unobservable spectrum of the pair $(A + BF, C)$. The controllability of the pair $(A + BF, BG)$ is preserved and Corollaries 4 and 5 imply that the invariant zeros of $\{A + BF, BG, C\}$ and of $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ are given by the finite eigenvalues of $(sE_{11} - A_{11})$ and $(s\mathcal{E}_{11} - \mathcal{A}_{11})$, respectively. According to Lemma 2 we thus have both identities

$$\begin{cases} \sigma(A + BF) = \sigma(\mathcal{A}) \cup \sigma_{A+BF,C}, \\ \sigma(A_{11}, E_{11}) = \sigma(\mathcal{A}_{11}, \mathcal{E}_{11}) \cup \sigma_{A+BF,C}. \end{cases} \tag{25}$$

We now prove necessity and sufficiency of the theorem.
Necessity: Let $A+BF$ be stable then the first identity in (25) implies that both $\mathcal{A}$ and the unobservable eigenvalues of $(A+BF, C)$ are stable. The latter and the second identity in (25) then also imply that $\sigma(A_{11}, E_{11})$ and $\sigma(\mathcal{A}_{11}, \mathcal{E}_{11})$ have the same unstable eigenvalues.
Sufficiency: Let $\mathcal{A}$ be stable and let $\sigma(A_{11}, E_{11})$ and $\sigma(\mathcal{A}_{11}, \mathcal{E}_{11})$ have the same unstable eigenvalues. Then the second identity in (25) implies that $\sigma_{A+BF,C}$ is stable. Using this and the first identity in (25) it then also follows that $A + BF$ is stable. $\square$

The following main theorem summarizes all results in which we used the notation introduced earlier.

Theorem 10 *Assume that system (1) and the desired model $M(s)$ are left invertible. Then there exist matrices $F \in \mathbf{R}^{q\times n}$ and $G \in \mathbf{R}^{q\times q}$ with G nonsingular such that $M_{F,G}(s) = M(s)$ and $A + BF$ is stable iff*

(a) $\mathcal{A}$ is stable, conditions (12) and (15) hold, and $\sigma(E_{11}, A_{11}) \cap \mathbf{C}_{unst} = \sigma(\mathcal{E}_{11}, \mathcal{A}_{11}) \cap \mathbf{C}_{unst}$;
(b) Equations (24) have a solution $\hat{F}$, $\hat{G}$ with $\hat{G}$ nonsingular.

In case (a) and (b) hold, a desired feedback matrix pair (F, G) is obtained from (17).

5 Numerical method and example

As a consequence of Theorem 10 we have the following algorithmic implementation.

Algorithm 5

Input: $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times q}, C \in \mathbf{R}^{p\times n}$, $\mathcal{A} \in \mathbf{R}^{m\times m}, \mathcal{B} \in \mathbf{R}^{m\times q}, \mathcal{C} \in \mathbf{R}^{p\times m}$ such that the system (1) and the model $M(s)$ are left invertible.

Output: $F \in \mathbf{R}^{q\times n}$ and $G \in \mathbf{R}^{q\times q}$ such that (5) holds and $A + BF$ is stable, if such a solution exists.

Step 1: (a) Compute orthogonal matrices $\mathcal{W}, \mathcal{U}, \mathcal{V}^\star_{b,g}(\mathrm{Ker}C)$, U and V such that

$$\mathcal{A}_{new} - s\mathcal{E}_{new} := \mathcal{U}(\mathcal{A} - sI_m)\mathcal{V}^\star_{b,g}(\mathrm{Ker}C)$$

$$= \begin{matrix} m_1 \\ m_2 \\ q \end{matrix} \overset{\begin{matrix} m_1 & \qquad m_2 & \qquad q \end{matrix}}{\begin{bmatrix} \mathcal{A}_{11} - s\mathcal{E}_{11} & \mathcal{A}_{12} - s\mathcal{E}_{12} & \mathcal{A}_{13} - s\mathcal{E}_{13} \\ 0 & \mathcal{A}_{22} - s\mathcal{E}_{22} & \mathcal{A}_{23} - s\mathcal{E}_{23} \\ \mathcal{A}_{31} - s\mathcal{E}_{31} & \mathcal{A}_{32} - s\mathcal{E}_{32} & \mathcal{A}_{33} - s\mathcal{E}_{33} \end{bmatrix}},$$

$$\mathcal{B}_{new} = \mathcal{U}\mathcal{B} = \begin{matrix} m_1 \\ m_2 \\ q \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{B}_3 \end{bmatrix},$$

$$\mathcal{C}_{new} = \mathcal{W}\mathcal{C}\mathcal{V}^\star_{b,g}(\mathrm{Ker}C) = \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} m_1 & m_2 & q \end{matrix}}{\begin{bmatrix} 0 & 0 & \mathcal{C}_{13} \\ 0 & \mathcal{C}_{22} & \mathcal{C}_{23} \end{bmatrix}},$$

$$A_{new} - sE_{new} := U(A - sI_n)V$$

$$= \begin{matrix} n_1 \\ n_2 \\ q \end{matrix} \overset{\begin{matrix} n_1 & \qquad n_2 & \qquad q \end{matrix}}{\begin{bmatrix} A_{11} - sE_{11} & A_{12} - sE_{12} & A_{13} - sE_{13} \\ 0 & A_{22} - sE_{22} & A_{23} - sE_{23} \\ A_{31} - sE_{31} & A_{32} - sE_{32} & A_{33} - sE_{33} \end{bmatrix}},$$

$$B_{new} = UB = \begin{matrix} n_1 \\ n_2 \\ q \end{matrix} \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix},$$

$$C_{new} = \mathcal{W}CV = \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} n_1 & n_2 & q \end{matrix}}{\begin{bmatrix} 0 & 0 & C_{13} \\ 0 & C_{22} & C_{23} \end{bmatrix}},$$

where
$\mathrm{rank}(\mathcal{B}_3) = \mathrm{rank}(\mathcal{C}_{13}) = q,\ \mathrm{rank}_g(\mathcal{A}_{11} - s\mathcal{E}_{11}) = m_1,$
$\mathrm{rank}\begin{bmatrix} \mathcal{A}_{22} - s\mathcal{E}_{22} \\ \mathcal{C}_{22} \end{bmatrix} = \mathrm{rank}(\mathcal{E}_{22}) = m_2, \quad \forall s \in \mathbf{C},$
$\mathrm{rank}(B_3) = q,\ \mathrm{rank}_g(A_{11} - sE_{11}) = n_1,$
$\mathrm{rank}\begin{bmatrix} A_{22} - sE_{22} & A_{23} - sE_{23} \\ 0 & C_{13} \\ C_{22} & C_{23} \end{bmatrix} = \mathrm{rank}\begin{bmatrix} E_{22} & E_{23} \\ 0 & C_{13} \\ C_{22} & C_{23} \end{bmatrix}$
$= n_2 + q, \quad \forall s \in \mathbf{C}.$

(b) Verify the stability of $\mathcal{A}$ and the following conditions:
$\mathrm{rank}(C_{13}) = q,\ \mathrm{rank}_g(A_{22} - sE_{22}) = n_2,$
$\sigma(E_{11}, A_{11}) \cap \mathbf{C}_{unst} = \sigma(\mathcal{E}_{11}, \mathcal{A}_{11}) \cap \mathbf{C}_{unst},$

$$\mathrm{rank}_g \begin{bmatrix} \mathcal{A}_{22} - s\mathcal{E}_{22} & 0 & \mathcal{A}_{23} - s\mathcal{E}_{23} & 0 \\ 0 & A_{22} - sE_{22} & 0 & A_{23} - sE_{23} \\ 0 & 0 & \mathcal{C}_{13} & C_{13} \\ \mathcal{C}_{22} & C_{22} & \mathcal{C}_{23} & C_{23} \end{bmatrix}$$
$= m_2 + n_2 + q$

by using the algorithm in [4] for computing the generalized upper triangular form of an arbitrary pencil. If they hold, continue; else, the model matching problem with stability is not solvable and so stop.

Step 2: (a) Compute and partition an orthogonal matrix Z with $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ and $Z_{11} \in \mathbf{R}^{q\times q}$ such that

$$\begin{bmatrix} C_{13} & \mathcal{C}_{13} \end{bmatrix} Z = \begin{bmatrix} R & 0 \end{bmatrix}, \quad R \in \mathbf{R}^{q\times q}, \quad \mathrm{rank}(R) = q,$$

(b) Compute orthogonal matrices P and Q such that

$$P \begin{bmatrix} A_{11}-sE_{11} & A_{12}-sE_{12} & 0 & 0 & (A_{13}-sE_{13})Z_{12} \\ 0 & A_{22}-sE_{22} & 0 & 0 & (A_{23}-sE_{23})Z_{12} \\ 0 & 0 & \mathcal{A}_{11}-s\mathcal{E}_{11} & \mathcal{A}_{12}-s\mathcal{E}_{12} & (\mathcal{A}_{13}-s\mathcal{E}_{13})Z_{22} \\ 0 & 0 & 0 & \mathcal{A}_{22}-s\mathcal{E}_{22} & (\mathcal{A}_{23}-s\mathcal{E}_{23})Z_{22} \end{bmatrix} Q$$

$$= \begin{bmatrix} \Phi_{11}-s\Theta_{11} & \Phi_{12} & \Phi_{13}-s\Theta_{13} \\ 0 & 0 & \Phi_{23}-s\Theta_{23} \end{bmatrix},$$

where $\Phi_{11} \in \mathbf{R}^{\tau_1\times\tau_1}$, $\Phi_{12} \in \mathbf{R}^{\tau_1\times q}$,
$\mathrm{rank}\begin{bmatrix} \Phi_{11}-s\Theta_{11} & \Phi_{12} \end{bmatrix} = \mathrm{rank}(\Theta_{11}) = \tau_1, \ \forall s \in \mathbf{C}$,
$\mathrm{rank}_g(\Phi_{23}-s\Theta_{23}) = n+m-2q-\tau_1$.

(c) Compute

$$\begin{bmatrix} A_{31}-sE_{31} & A_{32}-sE_{32} & 0 & 0 & (A_{33}-sE_{33})Z_{12} \\ I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_{12} \\ 0 & 0 & \mathcal{A}_{31}-s\mathcal{E}_{31} & \mathcal{A}_{32}-s\mathcal{E}_{32} & (\mathcal{A}_{33}-s\mathcal{E}_{33})Z_{22} \end{bmatrix} Q$$

$$= \begin{array}{c} \\ q \\ n_1 \\ n_2 \\ q \\ q \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & q & n+m-2q-\tau_1 \end{array} \\ \begin{bmatrix} \Phi_{31}-s\Theta_{31} & \Phi_{32}-s\Theta_{32} & \Phi_{33}-s\Theta_{33} \\ J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \\ \Pi_{31}-s\Xi_{31} & \Pi_{32}-s\Xi_{32} & \Pi_{33}-s\Xi_{33} \end{bmatrix} \end{array}.$$

(d) Compute orthogonal matrix W such that
$\begin{bmatrix} \Theta_{11} & \Phi_{11} & \Phi_{12} \end{bmatrix} W = \begin{bmatrix} \Delta & 0 \end{bmatrix}$,
where $\Delta \in \mathbf{R}^{\tau_1\times\tau_1}$ is nonsingular.

(e) Compute

$$\begin{bmatrix} \Xi_{31} & \Pi_{31} & \Pi_{32} \\ \Theta_{31} & \Phi_{31} & \Phi_{32} \\ 0 & J_{11} & J_{12} \\ 0 & J_{21} & J_{22} \\ 0 & J_{31} & J_{32} \end{bmatrix} W = \begin{array}{c} \\ q \\ q \\ n_1+n_2+q \end{array} \begin{array}{c} \begin{array}{cc} \tau_1 & \tau_1+q \end{array} \\ \begin{bmatrix} \times & M \\ \times & L \\ \times & N \end{bmatrix} \end{array}.$$

Step 3: Solve the equations
$\hat{G}M + \hat{F}N + L = 0, \quad \hat{G}\Xi_{32} + \Theta_{32} = 0,$
using QR factorization. If a solution pair $(\hat{F}, \hat{G})$ with $\hat{G}$ nonsingular exists, continue; else, the model matching problem with stability is not solvable and so stop.

Step 4: Compute (F, G) by solving equations
$B_3FV = \hat{F}, \quad B_3G = \hat{G}\mathcal{B}_3$. Output F and G.

The following comments can be given regarding the numerical properties of the above algorithm :

(i) all steps require only orthogonal transformations which can be carried out in a numerically backward stable manner; the overall numerical stability of the algorithm does not automatically follow from this, but it guarantees that the substeps are solved in a numerically stable manner;

(ii) algorithms with cubic computational complexity in the dimensions of the matrices that are involved are given in [1] for the computation of the generalized upper triangular form and the staircase form of a matrix pencil, for the Schur form of square matrix and for the QR/RQ factorization of an arbitrary matrix [5];

(iii) since all steps involve matrices of dimensions of the same order as those of the original systems, the overall complexity of Algorithm 5 is thus at most cubic in the system dimensions of the given systems.

Example 1 *We apply Algorithm 5 to a random example in which system (1) and the desired model $M(s)$ are not square. All computation were done on a computer with relative accuracy 2.2204×10^{-16}. To guarantee that there is a solution to the exact model matching problem with stability, we chose matrices (A, B, C) equal to*

$$\mathcal{J}(\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \mathcal{A} \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \mathcal{B} \end{bmatrix}\mathcal{D})\mathcal{J}^{-1}, \ \mathcal{J}\begin{bmatrix} \tilde{B}_1 \\ \mathcal{B} \end{bmatrix}\mathcal{K}, \ \begin{bmatrix} 0 & \mathcal{C} \end{bmatrix}\mathcal{J}^{-1}.$$

where all matrices in the right hand side are random, $\mathcal{A}$ is stable, $\tilde{A}_{11}$ is unstable, $\mathcal{J}$ and $\mathcal{K}$ are nonsingular, and $\mathrm{cond}(\mathcal{J}, 2) < 10$. The minimality of the triplets $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (A, B, C) follows from the randndomness of their elements. Note that A is unstable because of $\tilde{A}_{11}$. Using Algorithm 5 with a relative rank tolerance of 10^{-12}, we obtained a solution F, G. We verified that $A+BF$ is stable as requested, and furthermore, $C(sI-A-BF)^{-1}BG \approx M(s)$ in the sense that the error $E(s) := \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B} - C(sI-A-BF)^{-1}BG$ is small compared to $M(s)$. Indeed, $E(s) = 0$ provided its first 10 Markov parameters are zero:

$$E_i := \begin{bmatrix} \mathcal{C} & -C \end{bmatrix} \begin{bmatrix} \mathcal{A} & 0 \\ 0 & A+BF \end{bmatrix}^i \begin{bmatrix} \mathcal{B} \\ BG \end{bmatrix} = 0, \ i = 0, \cdots, 9$$

and the computed versions of these expressions satisfied:

$$\|E_i\|_\infty \le 2.3\times 10^{-14}\|\mathcal{C}\mathcal{A}^i\mathcal{B}\|_\infty, \quad i = 0, \cdots, 9.$$

6 Concluding remarks

In this paper we have studied the exact model matching problem with stability for left invertible systems. Based on matrix pencil theory, we have developed a new numerical method to verify the solvability of the underlying problem and further construct a desired solution. Our new method can be implemented via a numerically reliable manner, and its computational complexity is cubic in the sizes of the system (1) and the desired model $M(s)$. The results trivially extend to the discrete-time case as well. The generalization to the case where the systems (1) and $M(s)$ are not left invertible is still a challenging problem which is definitely worthy of further study.

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