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A switched system approach to the decidability of consensus*

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Abstract— **The convergence to consensus of all products of a given set of matrices is known to be algorithmically decidable when all matrices in the set are stochastic. We formulate this question as a stability problem for switched systems, and show that the decidability result remains valid for more general classes than stochastic matrices. Our results make use of a general theorem of Lagarias and Wang on the convergence of switched systems, and allow showing as a byproduct that the bound provided by this theorem is tight.**

INTRODUCTION

A consensus system represents a group of agents trying to agree with each other on some common value. These systems, and in particular linear ones, have attracted an important research attention because they are commonly used in a variety of distributed computation schemes. The possible applications range from coordination of autonomous platoons of vehicles (for example in [1]) to data fusion in systems with distributed measurements [2], distributed optimization [3] or coordination of multiagent systems (see [4] and references therein). See also [5] for a survey.

In many consensus systems, the agents update their value as linear combinations of the values of agents with which they can communicate:

$$x_i(t) = \sum_j a_{ij}(t) x_j(t-1) \tag{1}$$

where x_i is the value of agent i and a_{ij} represents the way agent j influences agent i. This interaction between agents typically depends on their value at each particular time, leading to non-linear dynamical systems [4, 6]. Agents following these dynamics tend to be more and more in agreement, in the sense that their values generally get closer to each other. Deciding whether they will always converge to a state of consensus (i.e. a state in which all agents have the same value) is however a hard problem (see for instance [7] for a quite simple model for which no conditions for convergence to consensus are known).

In some situations, even if it is hard to explicit the complete sequence of matrices $A(t)$ corresponding to System (1), it may be possible to guarantee that these matrices stay in some set S. One of the strongest convergence notions is the convergence of System (1) for any sequence of transition matrices in S. Blondel and Olshevsky have studied [8] the complexity of deciding the convergence in that sense. They showed that this is NP-hard but decidable. The decidability proof, based on earlier work by Paz [9], shows with combinatorial arguments that if there is an initial condition and a sequence of transition matrices in the set S such that the system does not converge, then there is an initial condition and a *periodic* sequence of transition matrices such that the system does not converge. The period of this cyclic sequence is bounded by a number that depends only on the number of agents n.

These linear consensus systems are if fact linear *switched* systems for which there is a rich literature (see for example [10, 11, 12]). However, the results about consensus and those about switched systems are quite different in nature. Results on consensus are often based on topological conditions on the communication graph between the agents (see for example [4, 13, 14] and references therein) while in switched systems theory, many stability conditions have been proposed in terms of Lyapunov theorems, which translate into LMIs, or convex optimization problems [10].

Lagarias and Wang proved a finiteness result for stability analysis [15] that has some similarities to the one of Blondel and Olshevsky for consensus. They showed that for some matrix sets (sets such that there is a induced polytope norm in which $\forall A \in S,\ \|A\| \leq 1$, see Subsection II-B for a precise statement), then either all infinite products converge to zero or there is an infinite *periodic* product that does not converge to zero.

In this article, we study the problem of convergence to a common eigenvector of a set of matrices. We reformulate the problem of convergence to a common eigenvector as a stability problem (a similar idea has been used for consensus by Jadbabaie et al. [4]), to which we can apply the result of [15]. This allows obtaining a general decidability result for the convergence to a common eigenvector which can be particularized to recover the decidability result of [8, 9].

As we have said, we use a theorem of Lagarias and Wang to prove our result. In the case of stochastic matrices, our result provides a tight bound in that for all dimensions n there exists a set of $n \times n$ matrices such that our bound is attained. Using that fact, we can conclude that the bound in the theorem of Lagarias and Wang is tight in the sense that for each n there exists a norm and a matrix set such that the matrix set attains the bound.

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Outline

In Section I, we introduce the problems that we will treat. We prove our general result in Section II and show how it applies to a simple example. In Section III, we show how the decidability result of [8] is a particular case of our result. We also show the tightness of the bound of the theorem of [15].

I. Problem setting

We study switched systems of the form

$$x(t) = A_{\sigma_t} x(t-1), \ \ A_{\sigma_t} \in S \tag{2}$$

where $S = \{A_1, \dots, A_m\}$ is a finite set of $n \times n$ matrices that share a common eigenvector of eigenvalue 1: $\forall A_i \in S, \quad A_i v = v$ and $\sigma : \mathbb{N} \to \{1 \dots m\} : t \to \sigma_t$ is an infinite sequence of indices. We don't suppose that the update matrices A_{σ_t} are known, they can have complicated dynamics (possibly depending on x), but we suppose that we can guarantee that the transition matrix is always in the set S. We want to determine, by analyzing only the set S, if every possible trajectory of the system converges to a vector parallel to v, that is, if for any $x(0)$ and any sequence σ, the sequence $(x(t))_t$ generated by (2) converges to a vector parallel to v.

Therefore we study the following problem.

Problem 1 (Convergence to a common eigenvector): For a given set S of matrices that share a common eigenvalue-eigenvector pair $(1, v)$ does the sequence $(x(t))_t$ converge to a vector parallel to v for any $x(0)$ and any sequence σ?

Consensus systems are a particular case of System (2). In consensus systems, $x_i(t)$ represents the value of agent i. We suppose that each agent computes its new value as a weighted average of other agents values. Therefore, the transition matrices are row-stochastic matrices (matrices with non-negative elements and satisfying $A\mathbf{1} = \mathbf{1}$ with $\mathbf{1} = \begin{pmatrix}1 & \dots & 1\end{pmatrix}^T$).

We obtain the same equation as System (2) but now, the set S is a set of stochastic transitions matrices. The problem of deciding convergence to consensus is the following.

Problem 2 (Convergence of consensus systems): For a given finite set S of stochastic matrices, does the sequence $(x(t))_t$ converge to consensus (a vector parallel to $\mathbf{1}$) for any $x(0)$ and any sequence σ?

II. Convergence of switched systems to a common eigenvector

In this section we study the convergence of System (2) to a common eigenvector v of the matrices in S. More precisely, we give in Corollary 1, Subsection II-D necessary conditions for the decidability of Problem 1.

Our main theorem uses two main notions: polytope norms and faces of a polytope.

Definition 1 (Polytope norms): A norm in $\mathbb{R}^n$ is a polytope norm if its unit ball $\{x | \ \|x\| \leq 1\}$ can be characterized by a finite set of linear inequalities. We also call a polytope norm a matrix norm that is induced by a polytope vector norm.

Definition 2: A subset F of a polytope $\mathcal{P}$ is called a face if it can be represented as

$$F = \mathcal{P} \cap \{x \mid Bx = c\}$$

where $B \in \mathbb{R}^{n-d \times n}$ and $c \in \mathbb{R}^{n-d}$ are such that

$$\forall x \in \mathcal{P}, \ Bx \leq c.$$

We call d the *dimension* of the face.

We can now state our main theorem.

Theorem 1: Let $S = \{A_1, \dots, A_m\}$ be a finite set of matrices with a common eigenvector v of eigenvalue 1. Suppose that there is a polytope norm $\|.\|$ such that

$$\forall A_i \in S, \ \forall x \in v^\perp, \ \|x^T A_i\| \leq \|x\|$$

and define

$$L = \frac{1}{2} \sum_{d=0}^{n-1} F(d, \mathcal{P})$$

where $F(d, \mathcal{P})$ is the number of faces of dimension d of the polytope $\mathcal{P}$ defined as the intersection of the unit ball of $\|.\|$ and the space $v^\perp$.

(i) Either for any sequence σ and any $x(0)$, the sequence $(x(t))_t$ generated by (2) converges to a vector of span$\{v\}$

(ii) or there is a product $\Pi = A_{\xi_1} \dots A_{\xi_l}$ of $l \leq L$ matrices from S such that the system

$$x(s) = \Pi x(s-1) \tag{3}$$

does not converge to v for some initial condition.

We will prove Theorem 1 in Subsection II-C.

A. Reformulation as a stability problem

In this section, we perform an algebraic transformation of System (2) that allows reformulating Problem 1 as one of convergence to zero of all infinite products of an auxiliary set of matrices. This transformation has already be suggested in [4, 13]. We will see that the problem of convergence to zero can be treated with classical tools from switched systems theory.

Lemma 1: Let $S = \{A_1, \dots, A_m\}$ be a finite matrix set such that $\forall A_i \in S, \ \ A_i v = v$. For a given σ, System (2) converges to the common eigenvector for all initial conditions if and only if the left-infinite product $\dots A_{\sigma_1} A_{\sigma_0}$ converges to a rank-one matrix of the form vy^T:

$$\forall x_0, \exists a \in \mathbb{R}, \ \lim_{t \to \infty} x(t) = av$$
$$\Leftrightarrow \exists w \in \mathbb{R}^n, \ \lim_{t \to \infty} A_{\sigma_t} A_{\sigma_{t-1}} \dots A_{\sigma_0} = vw^T.$$

Proof:

- $\Rightarrow$ Suppose that for any initial condition, there is an a such that x converges to av. We can define a function $x_0 \to \lim_{t \to \infty} x(t)$. This function is clearly linear. It can therefore be represented by a matrix and this matrix is the limit $\lim_{t\to\infty} A_{\sigma_t} A_{\sigma_{t-1}} \dots A_{\sigma_0}$. Because the image of the application is always in span(v), the matrix is necessarily equal to vw^T for some w.

- $\Leftarrow$ is immediate.

■

Let P be a given $(n-1)\times n$ full row rank matrix satisfying $Pv = 0$ (any matrix whose rows are a basis of $v^\perp$ satisfies these conditions, thus such a matrix P exists).

Lemma 2: With P defined as above, for any matrix A_i such that $A_i v = v$, there exists a unique $(n-1)\times(n-1)$ matrix A'_i that is solution of $PA_i = A'_i P$.

Proof: Because $Pv = 0$ and v is an eigenvector of A_i, the rows of PA_i are orthogonal to v. On the other hand, the rows of P span the space orthogonal to v by definition. Each row of PA_i can thus be expressed by a unique combination of the rows of P and therefore there exists a unique matrix A'_i such that $PA_i = A'_i P$. ■

Lemma 3: Let $S = \{A_1, \ldots, A_m\}$ be a set of matrices with a common eigenvector v corresponding to an eigenvalue 1 and let $S' = \{A'_1, \ldots, A'_m\}$ be as defined above. For any sequence σ, the following assertions are equivalent.

- The left-infinite product converges to the space of the matrices of the form vw^T, which is equivalent to:
$$\lim_{t\to\infty} PA_{\sigma_t}\ldots A_{\sigma_0} = 0.$$
- The infinite product of matrices from S' converges to zero:
$$\lim_{t\to\infty} A'_{\sigma_t}\ldots A'_{\sigma_0} = 0.$$

Proof:

From the definition of the A'_i's, the second assertion is equivalent to
$$\lim_{t\to\infty} A'_{\sigma_t}\ldots A'_{\sigma_0}P = 0$$
which, due to the rank of P, is equivalent to
$$\lim_{t\to\infty} A'_{\sigma_t}\ldots A'_{\sigma_0} = 0.$$
■

We now prove that if every infinite product of matrices of S becomes arbitrarily close to the space of matrices of the form vw^T, then every infinite product converges to a matrix in that space. This lemma is useful to avoid reaching oscillating or infinite consensus states.

Lemma 4: Let $S = \{A_1, \ldots, A_m\}$ be a set of matrices with a common eigenvector v corresponding to an eigenvalue 1 and let $S' = \{A'_1, \ldots, A'_m\}$ be as defined above. The following assertions are equivalent.

(i) For any sequence σ, the left-infinite product converges to the space of the matrices of the form vw^T, which is equivalent to:
$$\lim_{t\to\infty} PA_{\sigma_t}\ldots A_{\sigma_0} = 0.$$

(ii) For any sequence σ, there exists w such that the left-infinite product converges to the matrix vw^T:
$$\exists w \in \mathbb{R}^n \text{ such that } \lim_{t\to\infty} A_{\sigma_t}\ldots A_{\sigma_0} = vw^T.$$

Proof:

The (i) $\Leftarrow$ (ii) direction is evident, so we only prove the (i) $\Rightarrow$ (ii) direction.

The matrix $A_{\sigma_t}\ldots A_{\sigma_0}$ can be decomposed in $vw^T(t,\sigma) + Z(t,\sigma)$ where $v^T Z = 0$.

Since $A_i v = v \;\; \forall A_i \in S$, we have
$$A_{\sigma_{t+1}}\ldots A_{\sigma_0} = vw^T(t,\sigma) + A_{\sigma_{t+1}}Z(t,\sigma).$$
We can see that
$$Z(t,\sigma) = \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_t}\ldots A_{\sigma_0},$$
and therefore, for $s > t$, we obtain
$$Z(s,\sigma) = \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_s}\ldots A_{\sigma_{t+1}} Z(t,\sigma). \tag{4}$$

We bound the difference between successive iterates:
$$\begin{aligned}
&\|A_{\sigma_{t+1}}\ldots A_{\sigma_0} - A_{\sigma_t}\ldots A_{\sigma_0}\| \\
&= \|vw^T(t,\sigma) + A_{\sigma_{t+1}}Z(t,\sigma) - vw^T(t,\sigma) - Z(t,\sigma)\| \\
&= \|A_{\sigma_{t+1}}Z(t,\sigma) - Z(t,\sigma)\| \\
&\le \|A_{\sigma_{t+1}}Z(t,\sigma)\| + \|Z(t,\sigma)\| \\
&\le (1 + \max_{A\in S}\|A\|)\|Z(t,\sigma)\|.
\end{aligned} \tag{5}$$

We now show that the sequence $(A_{\sigma_t}\ldots A_{\sigma_0})_t$ is a Cauchy sequence. By the triangular inequality and by (4) and (5), we have for any $s > r$,
$$\begin{aligned}
&\|A_{\sigma_s}\ldots A_{\sigma_0} - A_{\sigma_r}\ldots A_{\sigma_0}\| \\
&\le \sum_{t=r}^{s-1} \|A_{\sigma_{t+1}}\ldots A_{\sigma_0} - A_{\sigma_t}\ldots A_{\sigma_0}\| \\
&\le \sum_{t=r}^{s-1} (1 + \max_{A\in S}\|A\|)\|Z(t,\sigma)\| \\
&\le \sum_{t=r}^{s-1} (1 + \max_{A\in S}\|A\|) \left\| \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_t}\ldots A_{\sigma_{r+1}} \right\| \|Z(r,\sigma)\| \\
&\le (1 + \max_{A\in S}\|A\|)\|Z(r,\sigma)\| \sum_{t=r}^{s-1} \left\| \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_t}\ldots A_{\sigma_{r+1}} \right\| \\
&\le (1 + \max_{A\in S}\|A\|)\|Z(r,\sigma)\| \sum_{t=r}^{\infty} \left\| \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_t}\ldots A_{\sigma_{r+1}} \right\|
\end{aligned}$$

We now prove that the series converge. It is well known (Theorem 1 in [10]) that
$$\forall\sigma \lim_{t\to\infty} \|A'_{\sigma_t}\ldots A'_{\sigma_0}\| = 0 \Rightarrow \forall\sigma \|A'_{\sigma_t}\ldots A'_{\sigma_0}\| \le pq^{t+1}$$
for some $p, q \in \mathbb{R}, q < 1$. In turn, this implies that
$$\begin{aligned}
&\left\| \left(I - \frac{vv^T}{v^T v}\right) A_{\sigma_t}\ldots A_{\sigma_{r+1}} \right\| \\
&\le C \left\| A'_{\sigma_t}\ldots A'_{\sigma_{r+1}} \right\| \\
&\le Cpq^{t-r}
\end{aligned}$$
in which the constants depend on S and P (which are fixed in the lemma) and not on t. Because the sequence converges

geometrically, the series converges to something bounded by $Cp\frac{1}{1-q}$.

In turn, for any ε, there exists r such that

$$\|Z(r,\sigma)\| \leq \frac{\varepsilon(1-q)}{Cp(1+\max_{A\in S}\|A\|)},$$

and therefore, we obtain

$$\begin{aligned}&\|A_{\sigma_s}\dots A_{\sigma_0} - A_{\sigma_r}\dots A_{\sigma_0}\|\\ &\leq Cp\frac{1}{1-q}(1+\max_{A\in S}\|A\|)\|Z(r,\sigma)\| \leq \varepsilon.\end{aligned}$$

■

Using Lemma 3 and 4, we see that all trajectories of the original system converge to a vector parallel to v if and only if all products of matrices from a derived matrix set S' converge to zero:

$$\forall\sigma, \lim_{t\to\infty} A'_{\sigma_t}\dots A'_{\sigma_0} = 0.$$

B. Bound on the length of non-converging trajectories

The next theorem, due to Lagarias and Wang [15], provides an interesting finiteness result for the stability of matrix sets.

Theorem 2 ([15]): Let us consider a finite set of matrices $S = \{A_1, \dots A_m\}$. Suppose that there exists a polytope norm such that $\|A_i\| \leq 1$ and let $\mathcal{Q} = \{x \mid \|x\| \leq 1\}$ and $F(d,\mathcal{Q})$ be the number of faces of dimension d of the polytope $\mathcal{Q}$.

(i) Either all infinite products of matrices from S converge to zero

(ii) or there exists a product Π of length l less than

$$M = \frac{1}{2}\sum_{d=0}^{n-1} F(d,\mathcal{Q})$$

that has spectral radius[1] equal to 1.

In Theorem 2, (ii) implies the the existence of an initial condition $x(0)$ such that the system

$$x(s) = \Pi x(s-1)$$

does not converge (because Π has a spectral radius equal to 1). Therefore, Theorem 2 implies the equivalence between

- The existence of a non converging trajectory.
- The existence of a non converging trajectory that is cyclic in the matrices.

We now show that if the set S and the norm $\|.\|$ satisfy conditions of Theorem 1, then there is a norm $\|.\|_P$ such that the set S' (defined as in Subsection II-A) and $\|.\|_P$ satisfy the conditions of Theorem 2.

It is clear that if $\|.\|$ is a (polytope) norm in $\mathbb{R}^n$ then the application

$$\|x\|_P : \mathbb{R}^n \to \mathbb{R} : x \to \|x^T P\|$$

is a (polytope) norm in $\mathbb{R}^{n-1}$. Also, if the norm $\|.\|$ satisfies

$$\forall A_i \in S,\ y \in v^\perp,\ \|y^T A_i\| \leq \|y\|,$$

[1]The spectral radius being the maximum of the absolute values of the eigenvalues.

then $\|.\|_P$ satisfies

$$\forall A'_i \in S',\ x \in \mathbb{R}^{n-1},\ \|x^T A'_i\|_P \leq \|x^T\|_P.$$

Indeed,

$$\|x^T A'_i\|_P = \|x^T A'_i P\| = \|x^T P A_i\| \leq \|x^T P\|.$$

There exists a unique matrix Q satisfying

$$QP = I - \frac{vv^T}{v^T v}.$$

Indeed the rows of P span $v^\perp$, thus each row of $I - \frac{vv^T}{v^Tv}$ can be uniquely expressed as a linear combination of the rows of P.

Lemma 5: The number of faces of the unit ball of $\|.\|_P$ is equal to the number of faces of the intersection of the unit ball of $\|.\|$ with the hyperplane $v^T x = 0$.

Proof: For $x \in v^\perp$, we have

$$\|x\| = \left\| x^T \left(I - \frac{vv^T}{v^T v}\right)\right\| = \|x^T Q P\| = \|x^T Q\|_P$$

which implies

$$\|x\| = 1 \Leftrightarrow \|x^T Q\|_P = 1.$$

Because Q is full (column) rank, this means that the unit ball of $\|.\|_P$ is the image by Q^T of the intersection of the unit ball of $\|.\|$ with the plane $v^T x = 0$. ■

C. Proof of Theorem 1

We are now able to prove Theorem 1.

Proof: We prove that if (ii) is satisfied, then (i) is not. Suppose that there is a product $\Pi = A_{\xi_1}\dots A_{\xi_l}$ of matrices from S such that there is an initial condition x_0 for which the system $x(s) = \Pi x(s-1)$ does not converge. Then System (2) does not converge for initial condition x_0 and sequence of transition matrices $\sigma = \xi_1, \dots, \xi_l, \xi_1, \dots, \xi_l \dots$.

We prove that if (i) is not satisfied, then (ii) is true. Let us suppose that there is a sequence σ and an initial condition x_0 such that System (2) does not converge. Thus, by Lemmas 1 and 3,

$$\lim_{t\to\infty} A'_{\sigma_t}\dots A'_{\sigma_0} \neq 0$$

It is well known [3] that

$$\begin{aligned}&\exists\sigma \text{ such that } \lim_{t\to\infty} A'_{\sigma_t}\dots A'_{\sigma_0} \neq 0\\ \Leftrightarrow &\exists\sigma \text{ such that } \lim_{t\to\infty} A'^T_{\sigma_t}\dots A'^T_{\sigma_0} \neq 0.\end{aligned}$$

By hypothesis of the theorem, there is a norm $\|.\|$ satisfying

$$\forall A_i \in S,\ \forall x \in v^\perp,\ \|x^T A_i\| \leq \|x\|.$$

Therefore $\|.\|_P$ defined as in Subsection II-A satisfies the conditions of Theorem 2. We can therefore apply Theorem 2 on the set $S'^T = \{A_1'^T, \dots A_m'^T\}$ that provides the desired $\Pi'^T = A'^T_{\xi_1}\dots A'^T_{\xi_p}$ with spectral radius equal to 1. Its transpose $A'_{\xi_p}\dots A'_{\xi_1}$ has the same spectral radius 1. Because it has a spectral radius equal to one, there is an initial condition x_0 such that

$$\lim_{t\to\infty} \Pi'^t x_0 \neq 0$$

does not converge to zero. Because P is full rank, there exists y_0 such that

$$x_0 = Py_0.$$

This yields

$$\lim_{t\to\infty} \Pi'^t P y_0 \neq 0.$$

By definition of P and A'_i, the product defined as $\Pi = A_{\xi_p} \dots A_{\xi_1}$ satisfies

$$P\Pi = PA_{\xi_p} \dots A_{\xi_1} = A'_{\xi_p} \dots A'_{\xi_1} P = \Pi' P.$$

Therefore

$$\lim_{t\to\infty} P\Pi^t y_0 \neq 0.$$

By Lemma 5, the length of Π is smaller or equal to the bound of Theorem 1. ∎

Example 1: Let us take the set

$$S = \left\{ A_1 = \begin{pmatrix} 1.6 & -0.3 & 0.7 \\ 1.7 & 0.15 & 0.15 \\ -0.2333\dots & 0.11666\dots & 0.58333\dots \end{pmatrix}, \right.$$
$$\left. A_2 = \begin{pmatrix} 1.7666\dots & -0.38333\dots & 0.0333\dots \\ 1.61666\dots & 0.191666\dots & 0.48333\dots \\ 0.45 & -0.225 & 0.25 \end{pmatrix} \right\}.$$

The two matrices in the set satisfy

$$A_i v = v \text{ for } i \in \{1, 2\}$$

with $v = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$. In the plane $v^T x = x_1 + 2x_2 = 0$, there is a polytope $\mathcal{P}$ (black of Figure 1) such that

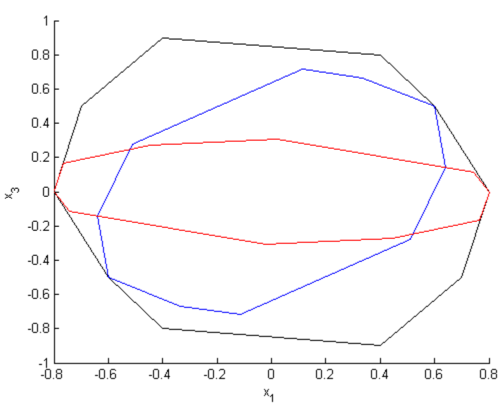
$$\{y \mid y^T = x^T A_1,\ x \in \mathcal{P}\} \in \mathcal{P}$$

(blue) and

$$\{y \mid y^T = x^T A_2,\ x \in \mathcal{P}\} \in \mathcal{P}$$

(red). Because this polytope is symmetric around the origin, there exists a norm such that $\mathcal{P}$ is the intersection of the unit ball of the norm with the plane $v^T x = 0$. Therefore, Theorem 1 applies to this matrix set: either all trajectories of System (2) converge to v or there is a products of matrices A_1 and A_2 that as a second eigenvalue of modulus one.

Fig. 1. Polytope P and its image by A_1^T and A_2^T on the plane $x_1 + 2x_2 = 0$



D. Decidability of the convergence

The next corollary shows sufficient conditions for decidability of Problem 1 by providing an algorithm. Since this algorithm requires computing products of matrices and eigenvalues, we must ensure that this can be done in finite time. One way of guaranteeing this is to assume that the entries are rational or algebraic.

Corollary 1 (Conditions for decidability of Problem 1): For any polytope norm, there exists an algorithm that decides the convergence of System (2) for any finite set S of matrices with algebraic entries satisfying

$$\forall A_i \in S,\ A_i v = v$$

and

$$\forall A_i \in S,\ \forall x \in v^\perp,\ \|x^T A_i\| \leq \|x\|.$$

Proof: By Theorem 1, deciding if all the trajectories of System (2) converge can be achieved by checking that the second largest eigenvalue of every product of length L is smaller than one, which can be done in finite time. ∎

III. Consensus systems

One of the strongest convergence notions that we can consider for System (2) is the convergence for any sequence of transition matrices. In the context of consensus it is often hard to determine how the transition matrix will evolve in S because the transition matrices can depend on the state $x(t)$ [4, 6]. For this reason, Blondel and Olshevsky have proved the decidability of Problem 2. We show here that this result is a corollary of our Theorem 1.

Theorem 3 (Decidability of consensus): Problem 2 is decidable.

Proof: It suffices to prove that the hypotheses of Corollary 1 are satisfied. Indeed stochastic matrices satisfy

$$\forall A_i \in S,\ A_i \mathbf{1} = \mathbf{1} \tag{6}$$

and

$$\forall A \in S,\ \forall x \in \mathbf{1}^\perp,\ \|x^T A\|_1 \leq \|x\|_1. \tag{7}$$

∎

Remark 1: We can see that Conditions (7) and (6) guarantee decidability. There are matrices that are not stochastic that satisfy Conditions (7) and (6), for example,

$$A = \begin{pmatrix} 2/3 & 1/2 & -1/6 \\ 0 & 5/6 & 1/6 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

It is immediate that $A\mathbf{1} = \mathbf{1}$ is satisfied. Moreover, for $x \in \mathbf{1}^\perp$,

$$x^T = x^T \left(I - \frac{\mathbf{1}\mathbf{1}^T}{n} \right),$$

$$\begin{aligned}\|x^T A\|_1 &= \left\| x^T \left(I - \frac{\mathbf{1}\mathbf{1}^T}{n} \right) A \right\|_1 \\ &= \left\| x^T \begin{pmatrix} 1/3 & -1/6 & -1/6 \\ -1/3 & 1/6 & 1/6 \\ 0 & 0 & 0 \end{pmatrix} \right\|_1 \\ &\le \|x\|_1.\end{aligned}$$

We can therefore conclude that stochasticity is not needed for decidability and that Theorem 1 and Corollary 1 are stronger than previously known decidability Theorem 3, even for the norm $\|.\|_1$.

We now derive the exact bound L on the length of the shortest non converging product. For that, we need to count the number of faces of all dimensions of the polytope defined by the intersection of the unit ball of the 1-norm and the space $\mathbf{1}^\perp$.

A. *Counting the number of faces*

Lemma 6: The number of faces of all dimensions of the 1-norm is $3^n - 1$.

Proof: The vertices (faces of dimension 0) of the unit ball of the 1-norm are the basis vectors and their opposite: by choosing $B = e_i$ and $c = 1$ in Definition 2, we have

$$e_i = \mathcal{P} \cap \{x \mid e_i^T x = 1\}$$
$$\forall x \in \mathcal{P},\ e_i^T x \le 1$$

so that each e_i is a face and similarly for $-e_i$. On the other hand, the convex hull of the set $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ is clearly P so that there are no other vertices. Thus, there are $2n$ vertices.

We claim that the faces of dimension $d-1$ are the convex hulls of any set of d vertices that do not contain opposite pairs. Indeed, for any set of d pairwise different indices $\{i_1, \dots, i_d\}$ and any set of signs $\{s_1, \dots, s_d\} \in \{-1, 1\}^d$,

$$\text{conv}(\{s_1 e_{i_1}, \dots, s_d e_{i_d}\}) = \mathcal{P} \cap \{x \mid \sum_{k=1}^{d} s_k e_{i_k}^T x = 1\}$$
$$\forall x \in \mathcal{P},\ \sum_{k=1}^{d} s_k e_{i_k}^T x \le 1.$$

There is no other face of dimension $d-1$ because the faces are convex hulls of the vertices and the inequality

$$\forall x \in \mathcal{P},\ \sum_{k=1}^{d} s_k e_{i_k}^T x \le 1$$

does not hold if for $k_1 \ne k_2$, $i_{k_1} = i_{k_2}$ and $s_{k_1} = -s_{k_2}$. Their number is the number of selections of d basis vectors multiplied by 2^d which is $\binom{n}{d} 2^d$. The total number of faces is therefore

$$\sum_{d=1}^{n} \binom{n}{d} 2^d = 3^n - 1.$$

■

Since we are interested only in the number of faces that have an intersection with the hyperplane $\mathbf{1}^T x = 0$, we have to subtract the number of faces that have no intersection with that hyperplane.

Lemma 7: The number of faces of the intersection of the unit ball of the one norm with the hyperplane $\mathbf{1}^T x = 0$ is $3^n - 2^{n+1} + 1$.

Proof: We claim that a face has no intersection with $\mathbf{1}^T x = 0$ if and only if it has all its vertices in the non-negative orthant or all its vertices in the non-positive orthant. As we have observed in the proof of the previous lemma, the vertices are the basis vectors e_i and their opposite $-e_i$. Clearly $\mathbf{1}^T e_i = 1 > 0$ and $\mathbf{1}^T(-e_i) = -1 < 0$. Therefore, if a face F is defined as convex hull of only the basis vectors, we have $\forall x \in F,\ \mathbf{1}^T x = 1 > 0$, and if it is defined as convex hull of opposite of basis vectors, we have $\forall x \in F,\ \mathbf{1}^T x = -1 < 0$, none of those have thus an intersection with the hyperplane $\mathbf{1}^T x = 0$. On the other hand, a face F defined as the convex hull of both basis vectors and opposite of basis vectors clearly contain x such that $\mathbf{1}^T x = 0$, and intersect thus with the hyperplane $\mathbf{1}^T x = 0$.

In the non-negative orthant, each convex hull of $d+1$ vertices is a face of dimension d. Therefore, there are $\binom{n}{d+1}$ faces of dimension d, so there are $2^n - 1$ faces in total in that orthant. The number of faces in the non-positive orthant is the same. The total number of faces of the unit ball of the 1-norm on the hyperplane $\mathbf{1}^T x = 0$ is therefore

$$3^n - 1 - 2(2^n - 1) = 3^n - 2^{n+1} + 1.$$

■

As a corollary of our main result, we recover the following result from [8, 9].

Theorem 4: Let S be a set of matrices satisfying Conditions (7) and (6). Then

- either for any initial condition $x(0)$ and for any sequence σ the sequence generated by (2) converges to consensus,
- or there is a product Π of length less or equal to $\frac{1}{2}(3^n - 2^{n+1} + 1)$ such that the system

$$x(s) = \Pi x(s-1)$$

does not converge for some initial conditions.

Proof: This is the consequence of Theorem 1 and the fact that the bound is $\frac{1}{2}(3^n - 2^{n+1} + 1)$ by Lemma 7. ■

B. *Tightness of the bound of the theorem of Lagarias and Wang*

It is shown in [4] that for all n, there exists a set of stochastic matrices S such that all infinite products converge to consensus and that the shortest product that has a second eigenvalue of modulus 1 has a length equal to $\frac{1}{2}(3^n - 2^{n+1} + 1)$, the bound of Theorem 4. From that set we can construct the corresponding set S'.

We have thus the following theorem:

Theorem 5 (Tightness of the bound Theorem 2): For any $n \in \mathbb{N}$, there exists a set S of matrices in $\mathbb{R}^{n \times n}$ and a norm $\|.\|$ such that there is an infinite product that does not converge to zero, and the shortest product with spectral radius one has a length of exactly $M(\|.\|)$, the bound defined in Theorem 2.

This proves that for all n there exists a norm and a matrix set such that the bound of Theorem 2 is attained (in the sense that the shortest product Π with $\rho(\Pi) = 1$ has a length equal to the bound).

IV. CONCLUDING REMARKS

In this paper we showed how a convergence result on consensus systems is the consequence of a general theorem in switched systems theory.

We believe that this approach is promising and could be extended to a more quantitative analysis of consensus systems. Indeed, there are many quantitative tools in the theory of switched systems, such as the joint spectral characteristics, that allow a quantitative convergence analysis while analysis of consensus systems is often qualitative (graph-theoretic conditions for convergence, see for example [13, 4, 14] and references therein).

Another interesting feature of our results is that they do not use positivity of the matrices. Indeed, a few positive results (decidability results, finiteness results) on switched systems are available in the literature, but they often rely on the nonnegativity of the matrices [17, 18, 19]. By contrast, here the existence of a finite upper bound on the length of a product which does not lead to consensus relies on an algebraic property of the matrices, and allows taking into account matrices with negative entries.

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