# Global Analysis of Firing Maps

A. Mauroy, J.M. Hendrickx, A. Megretski, and R. Sepulchre

***Abstract*— In this paper, we study the behavior of pulse-coupled integrate-and-fire oscillators. Each oscillator is characterized by a state evolving between two threshold values. As the state reaches the upper threshold, it is reset to the lower threshold and emits a pulse which increments by a constant value the state of every other oscillator.**

**The behavior of the system is described by the so-called firing map: depending on the stability of the firing map, an important dichotomy characterizes the behavior of the oscillators (synchronization or clustering). The firing map is the composition of a linear map with a scalar nonlinearity.**

**After briefly discussing the case of the scalar firing map (corresponding to two oscillators), the stability analysis is extended to the general *n*-dimensional firing map (for $n+1$ oscillators). Different models are considered (leaky oscillators, quadratic oscillators,...), with a particular emphasis on the persistence of the dichotomy in higher dimensions.**

## I. INTRODUCTION

Populations of interacting oscillators have attracted continuing interest among the scientific community for the last decades. While the behavior of a single oscillator is simple, the nonlinear nature of the interactions may trigger complex behaviors of the whole system. Beyond their practical interest, such systems have led to numerous exciting open problems [1], [2], [3].

In the present paper, we focus on systems of integrate-and-fire oscillators [4] characterized by an impulsive coupling. Although the model was first described to study pacemaker cells of the heart [5], it applies to other contexts. In particular, pulse-coupled oscillators are used to simulate neurons [6].

The seminal work of Mirollo and Strogatz [7] reduces the analysis of $n+1$ coupled integrate-and-fire oscillators to the analysis of the firing map, a map describing the $n$ phase differences at successive firing times. The $n+1$-dimensional firing map has a very special structure: it is the composition of a linear (isometric) map and a nonlinear map deriving from the corresponding scalar firing map defined over the phase interval $[0,1]$. The latter can be analytically computed from the scalar differential equation modeling the (integrating) behavior of a single phase oscillator.

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Popular oscillator models include the L(eaky)IF model and the Q(uadratic)IF model. Our previous study [8] has revealed an interesting dichotomy of the LIF model since only two asymptotic behaviors are possible. Either the scalar firing map is contracting and all oscillators asymptotically converge to a phase-locked clustering behavior, or the scalar firing map is expanding and all oscillators eventually synchronize.

In contrast to the LIF model, more general models (including the QIF model) have a scalar firing map that is contracting over some phase interval $[0,\phi]$ and expanding over the complementary interval $[\phi,1]$. Interestingly, numerical simulations suggest that the behavior is still dichotomic for most of the models: phase-locked clustering when the firing map is contracting "on average" and synchrony when the firing map is expanding "on average".

The present paper reports on current progresses to characterize the dichotomic behavior from the global analysis of the firing map. We establish the dichotomy for the scalar firing map ($n=1$) but we show that its extension to higher dimensions depends on finer properties of the scalar firing map.

The paper is organized as follows. In Section II, the pulse-coupled model is described and the firing map is introduced. In particular, the existence and uniqueness of the fixed point is established in full generality. Section III is devoted to the global analysis of the scalar firing map. In Section IV, the general *n*-dimensional firing map is considered. Whereas the dichotomy is shown for some particular firing maps in IV-A, a counterexample is presented in IV-A. In Section V, a local stability result complements the analysis in the *n*-dimensional case. It is established with the help of a technical result, whose proof is performed in Section VI. We conclude with Section VII.

## II. A FIRING MAP TO STUDY PULSE-COUPLED OSCILLATORS

An integrate-and-fire oscillator is described by a scalar state variable $x_i$, which monotonically increases between two thresholds $\underline{x}$ and $\overline{x}$ according to the dynamics $\dot{x}_i = F(x_i)$, $F>0$. Upon reaching the upper threshold $\overline{x}$, the state is reset to the lower threshold $\underline{x}$ and the oscillator is said to "fire". The coupling between the oscillators is impulsive: whenever an oscillator fires, it causes an instantaneous increment $\varepsilon > 0$ to the state of every other oscillator. Any oscillator which exceeds the upper threshold in this process is "absorbed" without firing by the firing oscillator. The oscillator that has just fired and all those that it has absorbed behave then exactly as a single oscillator — that is, they can be replaced by a single oscillator.

For the sake of simplicity, a phase $\phi_i \in [0,1]$ is introduced, which is determined from the state of oscillator $i$ by rescaling in such a way that $\phi_i = 1$ corresponds to the high-threshold and $\dot{\phi}_i = 1/T$, where $T$ is the natural period of the oscillator. The evolution of a single — uncoupled — oscillator is then described by the function

$$x_i = f(\phi_i) \triangleq P^{-1}[T\phi_i], \tag{1}$$

with $P(x)$ defined as the time required by the state $x_i(t)$, solution of $\dot{x}_i = F(x_i)$, $x_i(0) = \underline{x}$, to reach the value $x$ :

$$P(x) = \int_{\underline{x}}^{x} \frac{1}{F(x')} dx', \quad P(\overline{x}) = T. \tag{2}$$

The impulsive coupling only occurs at the precise firing times, where the oscillators receive a phase advance and jump from phase $\phi_i$ to phase $\min(f^{-1}[f(\phi_i)+\varepsilon], \overline{x})$. Between two firings, every phase evolves with a constant velocity $\dot{\phi}_i = 1/T$, without any coupling. The firing map introduced in [7] is the application that maps the (sorted) oscillator phases from one firing instant to their (sorted) phases at the next firing instant. Even though it "omits" the continuous time evolution of the oscillators between the firings, the firing map provides a full characterization of the system evolution.

For two oscillators, the firing map is a one-dimensional map which expresses as

$$\phi^+ = h(\phi) \triangleq f^{-1}[f(1-\phi)+\varepsilon]. \tag{3}$$

Given the phase $\phi$ of oscillator 1 when oscillator 2 fires, the firing map expresses the phase $\phi^+$ of oscillator 2 at the next firing (of oscillator 1), and vice versa. At each iteration, the phase of the firing oscillator ($\phi = 0$) is omitted: the map is scalar. In addition, $h(\cdot)$ is strictly decreasing since the evolution $f$ is strictly increasing.

For $n+1$ distinct oscillators ($n > 1$), the firing map is the $n$-dimensional generalization of the scalar firing map $h(\cdot)$. It is given by

$$\mathbf{H}(\Phi) = \left\{ \begin{array}{rcl} \phi_1^+ & = & h(\phi_n) \\ \phi_2^+ & = & h(\phi_n - \phi_1) \\ & \vdots & \\ \phi_n^+ & = & h(\phi_n - \phi_{n-1}) \end{array} \right. . \tag{4}$$

Whenever an oscillator fires, it receives the index $n+1$ — its phase $\phi_{n+1} = 0$ is omitted in (4) — and the $n$ remaining oscillators receive the index $i$ according to the phase ordering

$$\phi_{n+1} = 0 < \phi_1 < \phi_2 < \cdots < \phi_n < 1. \tag{5}$$

The successive iterations of the firing map $\mathbf{H}(\Phi)$ describe the evolution of the phases at each firing instant. When an absorption occurs, it can be noted that the $n$-dimensional vector is sent onto a $n'$-dimensional space, with $n' < n$. The evolution of the $n'+1$ remaining distinct oscillators is subsequently described by the $n'$-dimensional firing map.

The firing map (4) has remarkable properties. It is the composition $\mathbf{H} = \Delta \circ \mathbf{L}$ of a linear map, characterized by the $n \times n$ matrix

$$\mathbf{L} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{6}$$

with a repeated static nonlinearity $\Delta(\zeta) = [h(\zeta_1) \cdots h(\zeta_n)]$. The linear map $\mathbf{L}$ provides the firing map with a particular chain structure, which implies that the firing map has a unique fixed point $\Phi^*$, as stated in the following result.

*Proposition 1:* Provided that $n < (\overline{x} - \underline{x})/\varepsilon$, the $n$-dimensional firing map (4), which satisfies the phase ordering (5), has a unique fixed point.

*Proof:* See Appendix. ∎

The upper-bound on $n$ ensures that $n+1$ distinct oscillators can coexist over the range $\phi \in [0,1]$. If $n$ was larger than the bound given, no fixed point for the $n$-dimensional firing map could exist since there would always be an absorption within $n$ steps. The unique fixed point of the firing map represents the unique phase-locked configuration of the $n+1$ distinct oscillators spread over the range $[0,1]$. This situation, called clustering, is studied in [8].

## III. GLOBAL ANALYSIS OF THE SCALAR FIRING MAP

We make the following assumption on the integrate-and-fire model $\dot{x}_i = F(x_i)$.

*Assumption 1:* $F(\cdot) : [-1,1] \mapsto \mathbb{R}$ is continuous positive, even, smooth, and strictly monotone on $(0,1]$.

In the sequel, we assume without loss of generality that $[\underline{x}, \overline{x}] \subseteq [-1,1]$, so that $F(\cdot)$ is always defined on $[\underline{x}, \overline{x}]$. Most of the usual integrate-and-fire models satisfy Assumption 1. A representative model of this class is the quadratic integrate-and-fire (QIF) model, which corresponds to $F(x) = S + x^2$, $S > 0$. A so-called *piecewise linear* model is characterized by the dynamics $F(x) = S + \gamma|x|$, $S, \gamma > 0$. The well-known leaky integrate-and-fire (LIF) model, with $F(x) = S + \gamma x$ and $[\underline{x}, \overline{x}] = [0,1]$, appears to be equivalent to the piecewise linear model. (The case $\gamma < 0$ is actually equivalent to the piecewise linear model with thresholds $[\underline{x}, \overline{x}] = [-1,0]$.) In addition, the *exponential model* $F(x) = S\exp(x^2)$, $S > 0$, is also in the class of considered models. (See Fig. 1).

With the dynamics defined above, the scalar firing map has an important property. A well-chosen translation of the scalar firing map, i.e. $h(\phi + \delta)$, has a reflection symmetry with respect to the bisectrix (Fig. 2). This property of the scalar firing map is summarized in the following proposition.

*Proposition 2:* Under Assumption 1, the corresponding scalar firing map (3) satisfies the property

$$h(\cdot + \delta) = h^{-1}(\cdot) - \delta, \tag{7}$$

where $(\delta, \phi_\delta)$ is the unique solution of

$$\left\{ \begin{array}{l} \phi_\delta - h(\phi_\delta) = \delta \\ |h'(\phi_\delta)| = 1 \end{array} \right. . \tag{8}$$

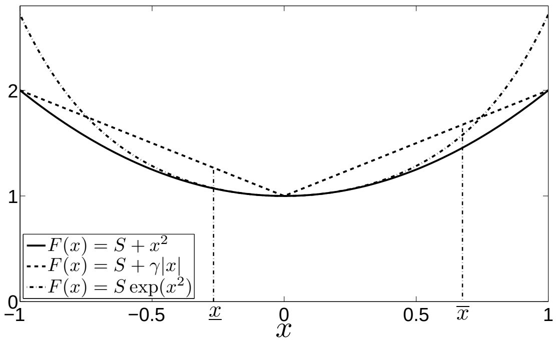


Fig. 1. Three models (quadratic, piecewise linear, and exponential) satisfying Assumption 1. The behavior of the corresponding firing map differs when $\underline{x}+\overline{x} > 0$ or $\underline{x}+\overline{x} < 0$.

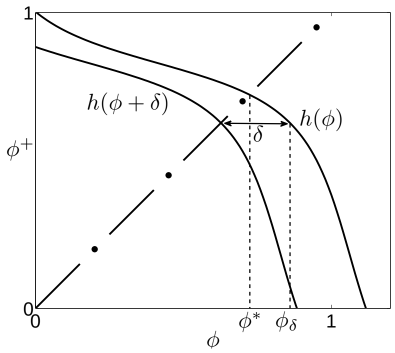


Fig. 2. The scalar firing map has an important property: the map $h(\phi+\delta)$ has a reflection symmetry with respect to the bisectrix.

The value $\delta$ has the same sign as $\underline{x}+\overline{x}$ and is a continuous function of $\underline{x}+\overline{x}$, for a given $F(\cdot)$. Moreover,

$$|h'(\phi)| < 1 \quad \forall \phi < \phi_\delta. \tag{9}$$

*Proof:* Since $F$ is even, it follows from (2) that $P(-x) = -P(x) + P(-\underline{x})$. Consequently, one has

$$f^{-1}(-x) = -f^{-1}(x) + f^{-1}(-\underline{x}) \tag{10}$$

and

$$f(\phi) = -f[-\phi + f^{-1}(-\underline{x})]. \tag{11}$$

Properties (10) and (11) lead to

$$\begin{aligned} h^{-1}(\phi) &= 1 - f^{-1}[f(\phi) - \varepsilon], \\ &= 1 - f^{-1}(-\underline{x}) + f^{-1}[-f(\phi) + \varepsilon], \\ &= 1 - f^{-1}(-\underline{x}) + f^{-1}[f(-\phi + f^{-1}(-\underline{x})) + \varepsilon], \\ &= 1 - f^{-1}(-\underline{x}) + h[\phi + 1 - f^{-1}(-\underline{x})]. \end{aligned}$$

Setting

$$\delta = 1 - f^{-1}(-\underline{x}) = 1 - P(-\underline{x})/T, \tag{12}$$

one obtains (7) and the continuity of $\delta$ with respect to $\underline{x}$ and $\overline{x}$ follows from the continuity of P. In addition, the value $\delta$ is positive if $P(-\underline{x}) < T = P(\overline{x})$, or equivalently if $\underline{x}+\overline{x} > 0$.

It follows from (1) and (2) that

$$f' = T\left(\frac{dP}{dx}\right)^{-1} = TF.$$

From the expression of the scalar firing map (3), one obtains the derivative

$$h'(\phi) = -\frac{f'(1-\phi)}{f'(h(\phi))} = -\frac{F[f(1-\phi)]}{F[f(1-\phi)+\varepsilon]}. \tag{13}$$

Equations (8) and (12) lead to the equality $f[\phi_\delta - 1 + f^{-1}(-\underline{x})] = f(1-\phi_\delta) + \varepsilon$. According to the property (11), one obtains $-f(1-\phi_\delta) = f(1-\phi_\delta) + \varepsilon$ and it follows that $|h'(\phi_\delta)| = 1$. Since $dF/dx > 0$ for $x > 0$, (13) implies that $|h'(\phi)| < 1$ for $\phi < \phi_\delta$. ■

The property (7) is of paramount importance since the condition $\delta > 0$ determines the global stability of the scalar firing map. The following proposition summarizes the result.

*Proposition 3:* If the scalar firing map (3) satisfies (7) and (9) with $\delta > 0$ (resp. $\delta < 0$), then its fixed point is globally attracting (resp. repelling).

*Proof:* We proceed in two steps.

*Step 1:* We show that the firing map $h(\cdot)$ has no nontrivial 2-periodic orbits, that is

$$h(\phi) = h^{-1}(\phi) \tag{14}$$

admits no other solution but the fixed point $\phi^* = h(\phi^*)$. In the plane $(\phi, \phi^+)$, a rotation of $-\pi/4$ of the axes results in a change of variables

$$(\phi, \phi^+) \mapsto (\tilde{\phi}, \tilde{\phi}^+) = \frac{\sqrt{2}}{2}(\phi - \phi^+, \phi + \phi^+).$$

It turns the bisectrix into the $\tilde{\phi}^+$-axis — that is, $\tilde{\phi} = 0$ — and the scalar firing map $h(\phi)$ becomes the map $\tilde{h}(\tilde{\phi})$. The assumption (7) expresses as

$$\tilde{h}\left(\frac{\delta}{\sqrt{2}} + \tilde{\phi}\right) = \tilde{h}\left(\frac{\delta}{\sqrt{2}} - \tilde{\phi}\right) \quad \forall \tilde{\phi}$$

and (9) is rewritten as

$$\tilde{h}'(\tilde{\phi}) > 0 \quad \forall \tilde{\phi} < \frac{\delta}{\sqrt{2}}.$$

These two properties imply that the equation $\tilde{h}(\tilde{\phi}) = \tilde{h}(-\tilde{\phi})$, which is equivalent to (14), has no other solution but $\tilde{\phi} = 0$ (which corresponds to the fixed point $\phi^*$). Hence, the return map $R(\cdot) \triangleq h[h(\cdot)]$ has a unique fixed point.

*Step 2:* We consider the case $\delta > 0$ without loss of generality, since the proof for $\delta < 0$ follows on similar lines. The fixed point $\phi^* = h(\phi^*)$ cannot be larger than $\phi_\delta = h(\phi_\delta) + \delta$ because $\delta > 0$ and $h' < 0$. Then, $\phi^* < \phi_\delta$ implies that $|h'(\phi^*)| < 1$, according to (9). It follows that the derivative of the return map satisfies $R'(\phi^*) = [h'(\phi^*)]^2 < 1$. Moreover, it holds that $R'(\phi) = h'[h(\phi)]h'(\phi) > 0$. Since, in addition, the return map has only one fixed point $R(\phi^*) = \phi^*$, it can be written that

$$\begin{cases} \phi^* < R(\phi) < \phi & \text{if } \phi > \phi^*, \\ \phi < R(\phi) < \phi^* & \text{if } \phi < \phi^*. \end{cases}$$

This leads to

$$|R(\phi)-R(\phi^*)| = |R(\phi)-\phi^*| < |\phi-\phi^*| \quad \forall \phi \neq \phi^*$$

and the fixed point of $R$ is globally attracting. For every $e>0$, there exist integers $N_1$ and $N_2$ such that, for all $\phi$,

$$\begin{cases} |R^N(\phi)-\phi^*| = |h^{2N}(\phi)-\phi^*| < e & \text{for } N \geq N_1, \\ |R^N(h(\phi))-\phi^*| = |h^{2N+1}(\phi)-\phi^*| < e & \text{for } N \geq N_2. \end{cases}$$

It follows that, for every $e>0$, there exists an integer $N_3 = \max(2N_1, 2N_2+1)$ such that, for all $\phi$,

$$|h^N(\phi)-\phi^*| < e \quad \text{for } N \geq N_3.$$

Then, one has $\lim_{N\to\infty} |h^N(\phi)-\phi^*| = 0$, which concludes the proof. ■

The stability of the fixed point leads to an interesting dichotomy. The scalar firing map dictates the evolution of $n+1=2$ pulse-coupled oscillators. Stability of the fixed point thereby means that the two oscillators are asymptotically phase-locked: they have a constant phase difference $\phi^*-0=\phi^*$ at each firing. Anti-stability of the fixed point corresponds to asymptotic synchronization: the phase difference grows at each firing, leading to an eventual absorption, after which the two oscillators remain synchronized.

## IV. GLOBAL ANALYSIS OF $N$-DIMENSIONAL FIRING MAPS

The scalar firing map is characterized by an important dichotomy and it results that the oscillators, after a transient period, are either phased-locked or synchronized. In this section, we discuss the generalization of this dichotomy to arbitrary dimensions. If the dichotomy persists with the $n$-dimensional firing map, the behavior of $n+1$ oscillators is simply characterized. Global stability of the fixed point corresponds to the asymptotic phase-locked clustering behavior of the oscillators, which fire periodically, while global anti-stability of the fixed point corresponds to asymptotic synchronization of all oscillators.

When considering the decomposition $\mathbf{H} = \Delta \circ \mathbf{L}$ of the $n$-dimensional firing map, the linear map $\mathbf{L}$ (6) acts as a mixing map while the repeated static nonlinearity $\Delta(\zeta) = [h(\zeta_1)\cdots h(\zeta_n)]$ provides the firing map with stability properties. The properties of the scalar firing map are therefore intimately linked to the stability of the $n$-dimensional firing map. In the sequel, we show that conditions (7) and (9) are in general not sufficient to ensure the stability of the $n$-dimensional firing map, although they still imply stability under some additional simple assumptions.

### A. Global analysis of firing maps with a contraction property

For dynamics verifying Assumption 1 and defined on $[\underline{x}, \overline{x}] \subseteq [0,1]$ (resp. $\subseteq [-1,0]$), it follows from (13) that the scalar firing map satisfies the property $|h'(\phi)| < 1$ (resp. $|h'(\phi)| > 1$) $\forall \phi \in [0,1]$. The scalar firing map is therefore contracting (resp. expanding) on $[0,1]$, which implies that the fixed point (of the scalar firing map) is globally stable (resp. anti-stable).

The contraction property and the dichotomy shown above for $n=1$ persist in higher dimensions, under an additional assumption on $h''$. This result is summarized in the following theorem [8].

*Theorem 1:* Provided that $h''(\phi) > 0$ for all $\phi$ or $h''(\phi) < 0$ for all $\phi$, the $n$-dimensional firing map (4) is contracting (resp. expanding) with respect to the 1-norm

$$\|\Phi\| = |\phi_1| + |\phi_1-\phi_2| + \cdots + |\phi_{n-1}-\phi_n| + |\phi_n| \tag{15}$$

if $|h'(\phi)| < 1$ (resp. $|h'(\phi)| > 1$) $\forall \phi \in [0,1]$.

One notes that the LIF model $F(x) = S + \gamma x$ satisfies the assumptions of Theorem 1. Indeed, it is equivalent to the piecewise linear model, with $[\underline{x}, \overline{x}] = [0,1]$ (or $[\underline{x}, \overline{x}] = [-1,0]$), and the scalar firing map has the property

$$h''(\phi) = T\gamma^2 \varepsilon \frac{F[f(1-\phi)]}{F[f(1-\phi)+\varepsilon]^2} > 0.$$

The proof of Theorem 1 makes a parallel with the contraction property of the scalar firing map. More precisely, it shows that the linear map $\mathbf{L}$ is an isometry for the distance induced by the above norm while the contraction property of the firing map is determined by the nonlinearity $\Delta(\cdot)$. It can be noted that the assumption on $h''$ is only sufficient (and not necessary) to prove the contraction property with respect to the norm (15). In addition, we suspect that every firing map characterized by $|h'| < 1$ (or $|h'| > 1$) but without verifying the assumptions on $h''$ has a contraction property with respect to a well-chosen norm.

The fact that the global contraction property of the firing map is established by means of a 1-norm rather than a 2-norm points to a potential limitation in studying (4) as an absolute stability problem [9], [10]. For instance, the circle criterion imposes additional slopes restrictions on $h(\cdot)$.

### B. A counterexample for $n=2$

The value $\phi_\delta$ in (9) separates the interval $[0, \phi_\delta]$ over which the scalar firing map is contractive from the interval $[\phi_\delta, 1]$ over which it is expanding. Numerical simulations with many models satisfying Assumption 1 suggest that the dichotomy established for $n=1$ does persist in higher dimensions: the oscillators asymptotically converge to a phase-locked configuration for $(\phi_\delta > \phi^*, \delta > 0)$ (contraction "on average" on $[0, h^{-1}(0)]$) and asymptotically synchronize for $(\phi_\delta < \phi^*, \delta < 0)$ (expansion "on average" on $[0, h^{-1}(0)]$). This is not true, however, without extra assumption on the firing map. This will be illustrated by an explicit construction in the case $n=2$. We will construct an example where $(\phi_\delta \gtrsim \phi^*, \delta \gtrsim 0)$ and where the fixed point is unstable.

For $n=2$, the $2\times 2$ Jacobian matrix of (4) evaluated at the fixed point is given by $\mathbf{J}(\Phi^*) = -\mathbf{D}\mathbf{L}$, where

$$\mathbf{D} = \mathrm{diag}\{|h'(\phi_2^*)| \; |h'(\phi_2^*-\phi_1^*)|\}$$

and with the isometric matrix $\mathbf{L}$ corresponding to (6). One first considers the conservative situation $\hat{\underline{x}} + \hat{\overline{x}} = 0$ characterized by a scalar firing map $\hat{h}$ satisfying (7) with $\delta = 0$, i.e.

$\hat{h}(\phi) = \hat{h}^{-1}(\phi)$. The fixed point verifies $\hat{\phi}_2^* - \hat{\phi}_1^* = \hat{h}(\hat{\phi}_2^*)$ and the product of the entries of $\mathbf{D}$ is

$$|\hat{h}'(\hat{\phi}_2^*)|.|\hat{h}'(\hat{\phi}_2^* - \hat{\phi}_1^*)| = |\hat{h}'(\hat{\phi}_2^*)|.|\hat{h}'[\hat{h}(\hat{\phi}_2^*)]| = 1, \tag{16}$$

since $\hat{h}'(\phi) = [\hat{h}^{-1}(\phi)]' = [\hat{h}'(\hat{h}^{-1}(\phi))]^{-1} = [\hat{h}'(\hat{h}(\phi))]^{-1}$. Consider now a system with the same $F$ and $\underline{x}$ but with a slightly larger $\overline{x}$, so that $\underline{x} + \overline{x} \gtrsim 0$ and $\delta \gtrsim 0$. One focuses on the effect on the product, which writes

$$|h'(\phi_2^*)|.|h'(\phi_2^* - \phi_1^*)| = \frac{F[f(\phi_1^*) - \varepsilon]}{F[f(\phi_1^*)]} \frac{F[f(\phi_2^*) - \varepsilon]}{F[f(\phi_2^*)]}, \tag{17}$$

since the fixed point satisfies

$$\begin{aligned} f(\phi_1^*) - \varepsilon &= f(1 - \phi_2^*), \\ f(\phi_2^*) - \varepsilon &= f(1 + \phi_1^* - \phi_2^*). \end{aligned} \tag{18}$$

Given (1) and (2), the equalities (18) are equivalent to

$$\begin{aligned} P[f(\phi_1^*) - \varepsilon] &= T - P[f(\phi_2^*)], \\ P[f(\phi_2^*) - \varepsilon] - P[f(\phi_1^*)] &= T - P[f(\phi_2^*)]. \end{aligned} \tag{19}$$

If $\overline{x}$ increases, then $T$ increases. Using the fact that $P(x)$ is increasing with respect to $x$, one can deduce from (19) that both values $f(\phi_1^*)$ and $f(\phi_2^*)$ increase. Depending on the derivative

$$\frac{d}{dx}\left[\frac{F(x-\varepsilon)}{F(x)}\right] \tag{20}$$

and on the values $\hat{f}(\hat{\phi}_1^*)$ and $\hat{f}(\hat{\phi}_2^*)$ of the case ($\hat{\underline{x}} + \hat{\overline{x}} = 0$, $\delta = 0$), it is possible that a slight increase of $\overline{x}$ increases the product (17) such that

$$|h'(\phi_2^*)|.|h'(\phi_2^* - \phi_1^*)| > |\hat{h}'(\hat{\phi}_2^*)|.|\hat{h}'(\hat{\phi}_2^* - \hat{\phi}_1^*)| = 1$$

when $\delta \gtrsim 0$. In this case, at least one eigenvalue verifies $|\lambda_i| > 1$ and the fixed point is locally unstable, in spite of the positive value $\delta$ characterizing the scalar firing map. As an example of this situation, we propose the following model.

*Example:* Let the oscillators be characterized by the piecewise linear dynamics $\dot{x} = F(x) = S + \gamma|x|$, $S, \gamma > 0$. Assumption 1 is satisfied and Proposition 2 implies that the scalar firing map has the properties (7) and (9). The derivative (20) satisfies the inequality

$$\frac{d}{dx}\left[\frac{F(x-\varepsilon)}{F(x)}\right] \begin{cases} \leq 0 & \text{if } x \in [0, \varepsilon], \\ > 0 & \text{elsewhere.} \end{cases}$$

We then show that, in the situation $\delta = 0$, neither $\hat{f}(\hat{\phi}_1^*)$ nor $\hat{f}(\hat{\phi}_2^*)$ can lie in $[0, \varepsilon]$. It follows from (12) that $\delta = 0$ implies $\hat{f}^{-1}(-\underline{x}) = 1$. The property (11) thereby becomes $f(\phi) = -f(1-\phi)$ and equation (18) leads to

$$\hat{f}(\hat{\phi}_1^*) - \varepsilon = \hat{f}(1 - \hat{\phi}_2^*) = -\hat{f}(\hat{\phi}_2^*). \tag{21}$$

On the other hand, the second equation in (19) implies that $\hat{f}(\hat{\phi}_2^*) - \varepsilon > \hat{f}(\hat{\phi}_1^*)$. Therefore, if $\hat{f}(\hat{\phi}_1^*) \in [0, \varepsilon]$, then $\hat{f}(\hat{\phi}_2^*) > \hat{f}(\hat{\phi}_1^*) + \varepsilon \geq \varepsilon$, which contradicts (21). And if $\hat{f}(\hat{\phi}_2^*) \in [0, \varepsilon]$, then $\hat{f}(\hat{\phi}_1^*) < \hat{f}(\hat{\phi}_2^*) - \varepsilon \leq 0$, which again contradicts (21). Hence, $|h'(\phi_2^*)|.|h'(\phi_2^* - \phi_1^*)| > 1$ when $\delta \gtrsim 0$ and the fixed point is unstable. Numerical simulations reinforce these observations about the absence of dichotomy. For $\delta \gtrsim 0$, populations of oscillators characterized by the piecewise linear model $F(x) = S + \gamma|x|$ do not exhibit phase-locked clustering configurations, but more complex (even aperiodic) behaviors. □

The above counterexample clearly shows that additional conditions on $h(\cdot)$ must be added to ensure that the dichotomy persists with the $n$-dimensional firing maps.

## V. A LOCAL STABILITY RESULT

A local analysis has highlighted that the dichotomy does not hold for some models satisfying Assumption 1. In this section, we consider the local stability of the fixed point of the $n$-dimensional firing maps in order to determine the conditions for the dichotomy to persist. It has been seen that the fixed point stability, for values $\delta$ close to 0, depends on the derivative (20). In particular, the fixed point (of the 2-dimensional firing map) is stable for all $\delta > 0$ (resp. unstable for all $\delta < 0$) if (20) is strictly negative for all $x \in [-1, 1]$. In this case, it follows that the second derivative $h''(\phi)$ is in turn negative, according to (13). The additional condition

$$h''(\phi) < 0 \quad \forall \phi \tag{22}$$

appears to be a sufficient condition to ensure the dichotomy of the behaviors. When (22) is satisfied, local stability of the $n$-dimensional firing map can actually be established, which is the statement of Theorem 2. As a preliminary to this important result, we need the following proposition, whose proof is performed in Section VI.

*Proposition 4:* Let

$$P(z) = a_n b_n z^n + a_{n-1} b_{n-1} z^{n-1} + \cdots + a_0 b_0.$$

If the following conditions hold

- $a_n > a_{n-1} > \cdots > a_0 > 0$,
- $b_{n-k} = b_k$, $\forall k$,
- The sequence $(b_0, b_1, \ldots, b_n)$ is positive and convex, i.e. $b_k > 0$ for all $k$ and $b_k - b_{k-1} \leq b_{k+1} - b_k$ for $k = 1, \ldots, n-1$,

then all roots of $P$ are strictly in the unit-disk: $P(z) = 0 \Rightarrow |z| < 1$.

The local stability of $n$-dimensional firing maps satisfying (22) is established with the help of Proposition 4.

*Theorem 2:* If the scalar firing map (3) satisfies (7) and (9) with $\delta > 0$ (resp. $\delta < 0$) and if $h''(\phi) < 0$ $\forall \phi$, then the fixed point of the $n$-dimensional firing map (4) is locally stable (resp. unstable).

*Proof:* We consider the case $\delta > 0$ without loss of generality, since the proof for $\delta < 0$ follows on similar lines. The $n \times n$ Jacobian matrix of (4) evaluated at the fixed point is given by $\mathbf{J}(\Phi^*) = -\mathbf{D}\mathbf{L}$, where

$$\mathbf{D} = \text{diag}\{|h'(\phi_n^*)||h'(\phi_n^* - \phi_1^*)| \cdots |h'(\phi_n^* - \phi_{n-1}^*)|\}.$$

The corresponding characteristic polynomial is

$$P(z) = \sum_{k=0}^{n} z^k \prod_{j=k}^{n} p_j, \tag{23}$$

with $p_0 = |h'(\phi_n^*)|$, $p_n = 1$, and $p_j = |h'(\phi_n^* - \phi_j^*)|$ for $j = 1, \ldots, n-1$.

First, one compares the fixed point $\Phi^*$ of the firing map with the fixed point $\hat{\Phi}_j^*$ of a conservative firing map constructed with the scalar map $h(\phi+\delta)$. The coordinates $\phi_j^*$ and $\hat{\phi}_j^*$ respectively solve the sets of equations

$$\left\{ \begin{array}{rcl} \phi_1^* & = & h(\phi_n^*) \\ \phi_2^* & = & h(\phi_n^* - \phi_1^*) \\ & \vdots & \\ \phi_n^* & = & h(\phi_n^* - \phi_{n-1}^*) \end{array} \right. \tag{24}$$

and

$$\left\{ \begin{array}{rcl} \hat{\phi}_1^* & = & h(\hat{\phi}_n^* + \delta) \\ \hat{\phi}_2^* & = & h(\hat{\phi}_n^* - \hat{\phi}_1^* + \delta) \\ & \vdots & \\ \hat{\phi}_n^* & = & h(\hat{\phi}_n^* - \hat{\phi}_{n-1}^* + \delta) \end{array} \right. . \tag{25}$$

Let $B_1(\phi) = \phi$ and $B_i(\phi) = \phi - h[B_{i-1}(\phi)]$ for $i = 2, \ldots, n$. The special structure of (24) and (25) implies that $\phi_n^* = h[B_n(\phi_n^*)]$ and $\hat{\phi}_n^* = h[B_n(\hat{\phi}_n^* + \delta)]$. Next, suppose that $\phi_n^* > \hat{\phi}_n^* + \delta$. Since $h' < 0$ and $B_i' > 0$ (see Appendix), it follows that

$$\phi_n^* = h[B_n(\phi_n^*)] < h[B_n(\hat{\phi}_n^* + \delta)] = \hat{\phi}_n^*,$$

which contradicts the previous assumption. Then, one has $\phi_n^* < \hat{\phi}_n^* + \delta$ and the recursive comparison of (24) and (25), starting from the first line, leads to $\phi_j^* > \hat{\phi}_j^*$ and $\phi_n^* - \phi_j^* < \hat{\phi}_n^* - \hat{\phi}_j^* + \delta$ for $j = 1, \ldots, n-1$. Given the assumption $h'' < 0$, one has $h'(\phi_n^*) > h'(\hat{\phi}_n^* + \delta)$ and $h'(\phi_n^* - \phi_j^*) > h'(\hat{\phi}_n^* - \hat{\phi}_j^* + \delta)$. This can be rewritten as $p_j < \hat{p}_j$ for $j = 0, \cdots, n-1$, where we denote $\hat{p}_0 = |h'(\hat{\phi}_n^* + \delta)|$ and $\hat{p}_j = |h'(\hat{\phi}_n^* - \hat{\phi}_j^* + \delta)|$. Each value $p_j$ can therefore be expressed as the decomposition

$$p_j = \alpha_j \hat{p}_j, \quad j = 0, \cdots, n-1, \tag{26}$$

with $0 < \alpha_j < 1$.

According to (7), one has $h'(\phi + \delta) = 1/h'[h^{-1}(\phi)]$. Similarly to (16), some elementary computations in (25) then lead to the equalities

$$\hat{p}_j \hat{p}_{n-1-j} = 1, \quad j = 0, \cdots, n-1. \tag{27}$$

In addition, it follows from the assumption $h'' < 0$ that

$$0 < \hat{p}_j < \hat{p}_{j-1}, \quad j = 1, \cdots, n-1. \tag{28}$$

Setting $a_n = b_n = 1$ and

$$a_k = \prod_{j=k}^{n-1} \alpha_j, \quad b_k = \prod_{j=k}^{n-1} \hat{p}_j, \quad k = 0, \cdots, n-1,$$

the inequality $0 < \alpha_j < 1$ leads to $a_n > a_{n-1} > \cdots > a_0 > 0$ and (27) implies $b_{n-k} = b_k$. Finally, it follows from $(\hat{p}_k - 1)^2 \geq 0$ and $p_{k-1} > p_k$ (28) that $\hat{p}_k \hat{p}_{k-1} + 1 - 2\hat{p}_k \geq 0$. Since $\hat{p}_k = b_k / b_{k+1}$, one has the convexity condition $b_k - b_{k-1} \leq b_{k+1} - b_k$ for $k = 1, \cdots, n-1$. The polynomial (23) satisfies the assumptions of Proposition 4. This implies that the fixed point is locally stable, which concludes the proof. ■

One easily verifies that the exponential model $\dot{x} = F(x) = S\exp(x^2)$ has a scalar firing map which satisfies the assumptions of Theorem 2. The dichotomy is thus proved (locally) for this model, that is, the behavior of the oscillators is dichotomic, at least for configurations close to the fixed point.

For a certain range of parameters, Theorem 2 also applies to the QIF model $F(x) = S + x^2$, which is similar to the exponential model (see Fig. 1). However, the scalar firing map does not satisfy the property $h''(\phi) > 0$ $\forall \phi$ in full generality, so that the coefficients in the characteristic polynomial (23) cannot be decomposed as (26), with $0 < \alpha_j < 1$. It follows that Theorem 2 does not characterize the behavior of the QIF model in full generality. However, numerical simulations suggest great evidence of the dichotomy in QIF model. It leads to the following conjecture.

*Conjecture 1:* For the QIF model $F(x) = S + x^2$, the fixed point is locally stable (resp. unstable) if $\delta > 0$ (resp. $\delta < 0$) in (7). In other words, if $\underline{x} + \overline{x} > 0$ (resp. $\underline{x} + \overline{x} < 0$), then all the roots of the polynomial (23), with

$$p_j = \frac{F[f(\phi_{j+1}^*) - \varepsilon]}{F[f(\phi_{j+1}^*)]}, \quad j = 0, \ldots, n-1$$

and $p_n = 1$, are inside (resp. outside) the unit-disk: $P(z) = 0 \Rightarrow |z| < 1$ (resp. $|z| > 1$).

The length of interval $I = \{\phi \in [0,1] | h''(\phi) > 0\}$ seems to be an appropriate criterion to determine whether the dichotomy holds. An interval $I = \emptyset$ corresponds to the exponential model and Theorem 2 implies the dichotomy. Conversely, for the dynamics $F(x) = S + \gamma|x|$, the interval $I$ is quite large and the dichotomy does not hold. The QIF model, characterized by a short interval $I$, corresponds to an intermediate situation between the two previous models.

## VI. PROOF OF PROPOSITION 4

The proof of Proposition 4 relies on a geometric property. Suppose that starting from an initial point $s_0$ in the plane, one moves by a distance $c$, calls the arrival point $s_1$, then rotates by an angle $\theta$, moves by a distance $c$ in the new direction, calls $s_2$ the arrival point, and keeps repeating these operations. It is well known that all points $s_k$ lie in that case on a same circle and are thus all at equal distance from the center $g$ of that circle. The next Lemma states that if the distance traveled at each iteration varies, then provided that the sequence of distances is convex and increasing, the sequence of distances between $s_k$ and $g$ is nondecreasing. Its proof, omitted for space reasons, is available in [11].

*Lemma 1:* Let $(c_0, c_1, \ldots, c_m)$ be a nonnegative non-decreasing convex sequence, and fix $\theta \in (0, 2\pi)$. Let $s_{-1} = -c_0/2$, and $s_k = s_{k-1} + c_k e^{ik\theta}$ for all other $k$. Let then $g = i\frac{c_0}{2\tan(\theta/2)}$. There holds

$$|s_{-1} - g| = |s_0 - g| \leq |s_1 - g| \leq \cdots \leq |s_m - g|.$$

To prove Proposition 4, we also need the notion of strict convex hull. The *strict convex hull* of a set of points $s_1,\ldots,s_n \in \mathbb{R}^d$ is the set $\{\sum_i \lambda_i s_i : \lambda_i > 0, \sum_i \lambda_i = 1\}$. The following Lemma, proved in [11], is a consequence of the convexity of the distance.

*Lemma 2:* Let $s_1,\ldots,s_n \in \mathbb{R}^d$ be a set of points that are not all equal and $z$ a point in the same space. If there exists $g \in \mathbb{R}^d$ such that $||z-g||_2 \geq ||s_k - g||_2$ for every $k = 1,\ldots,n$, then $z$ does not belong to the strict convex hull of $s_1,\ldots,s_n$.

We can now prove Proposition 4.

*Proof:* Fix a polynomial $P$. Clearly, $P(re^{i\theta}) \neq 0$ if $\theta = 0$. Fix then a $\theta \in (0,2\pi)$, and let

$$\begin{array}{lll} s_n &=& b_n e^{ni\theta} \\ s_{n-1} &=& b_n e^{ni\theta} + b_{n-1}e^{(n-1)i\theta} \\ s_{n-2} &=& b_n e^{ni\theta} + b_{n-1}e^{(n-1)i\theta} + b_{n-2}e^{(n-2)i\theta} \\ \vdots && \\ s_0 &=& \ldots \end{array} \quad (29)$$

We first prove that 0 does not belong to the strict convex hull of these points. Suppose that $n$ is even, and thus that

$$(b_n, b_{n-1}, \ldots b_0) = (c_m, c_{m-1}, \ldots, c_1, c_0, c_1, \ldots c_m),$$

with $m = n/2$, and where the sequence $c_0, c_1, \ldots, c_m$ is nonnegative, nondecreasing, and convex. Let

$$g' = c_m e^{2mi\theta} + c_{m-1}e^{(2m-1)i\theta} + \cdots + c_1 e^{(m+1)i\theta} + \frac{c_0}{2}e^{mi\theta}.$$

For every $k$, let then $q_k = e^{-mi\theta}(s_k - g')$, with the convention that $s_{n+1} = 0$. The inclusion relations are invariant under rotations and translations, so 0 is in the strict convex hull of $s_0, s_1, \ldots, s_n$ only if $q_{n+1}$ is in the strict convex hull of $q_0, q_1, \ldots, q_n$. Observe that $q_{m+1} = -c_0/2$, and that $q_{m-k} = q_{m+1-k} + c_k e^{-ki\theta}$. Lemma 1 implies then the existence of a $g$ on the imaginary axis such that

$$|q_{m+1} - g| \leq |q_m - g| \leq |q_{m-1} - g| \leq \cdots \leq |q_0 - g|. \quad (30)$$

Moreover, observe that $q_{m+k} = -\overline{q}_{m+1-k}$, i.e., they have the same imaginary part and opposite real parts. There holds therefore $|q_{m+k} - g| = |q_{m+1-k} - g|$ since $g$ has no real part. Together with the inequality (30), this implies that $|q_{n+1} - g| \geq |q_k - g|$ for every $k$. It follows then from Lemma 2 that $q_{n+1}$ is not in the strict convex hull of the $q_k$, and thus that 0 is not in the strict convex hull of the $s_k$ since the inclusion relations are invariant under rotations and translations. If $n$ is odd, a similar argument based on a variation of Lemma 1 can be made, as described in [11].

Suppose now, to obtain a contradiction, that $P(re^{i\theta}) = 0$ for some $r \geq 1$. Dividing $P(re^{i\theta}) = 0$ by $r^n$, we obtain:

$$a_n b_n e^{ni\theta} + a_{n-1}r^{-1}b_{n-1}e^{(n-1)i\theta} + \cdots + a_0 r^{-n} b_0 = 0. \quad (31)$$

Without loss of generality, assume that $a_n = 1$. Observe that since $r \geq 1$ and $1 = a_n > a_{n-1} > \ldots a_0 > 0$, there holds $1 = a_n > a_{n-1}r^{-1} > \cdots > a_0 r^{-n}$. Let $\lambda_0 = a_0 r^{-n}$, and for $k = 1,\ldots,n$, $\lambda_k = a_k r^{k-n} - a_{k-1}r^{k-1-n}$. Clearly, $\lambda_k \in (0,1)$ holds for every $k$, and $\sum_k \lambda_k = a_n = 1$. We can then rewrite equation (31) as

$$\begin{array}{rll} 0 &=& \lambda_n \left(b_n e^{ni\theta}\right) \\ &+& \lambda_{n-1} \left(b_n e^{ni\theta} + b_{n-1}e^{(n-1)i\theta}\right) \\ &\vdots& \\ &+& \lambda_0 \left(b_n e^{ni\theta} + b_{n-1}e^{(n-1)i\theta} + \cdots + b_0\right), \end{array}$$

or equivalently as $0 = \sum_{k=0}^n \lambda_k s_k$, for the $s_k$ defined in (29). This contradicts however the fact that 0 does not belong to the strict convex hull of the $s_k$. Therefore, $P(re^{i\theta}) \neq 0$ if $r \geq 1$ and $\theta \in (0,2\pi)$, which achieves our proof. ■

## VII. CONCLUSIONS

In the present paper, we studied the behavior of pulse-coupled integrate-and-fire oscillators. The evolution of the oscillators is described by the so-called firing map, which has a very special structure. In particular, the stability of the $n$-dimensional firing map is determined by the properties of the corresponding scalar firing map.

For two oscillators, the unique fixed point of the scalar firing map is either globally stable or anti-stable. It results in a dichotomic behavior of the oscillators, which are either asymptotically phase-locked or perfectly synchronized.

The study is extended to the $n$-dimensional firing map. In some particular cases — such as the leaky integrate-and-fire (LIF) oscillators — the firing map has a contraction property and the dichotomic behavior shown for $n = 1$ still persists in higher dimensions. On the other hand, a counterexample is considered, which shows that the dichotomy is not a general property when $n > 1$. In spite of strong numerical evidence, establishing the dichotomy of the quadratic integrate-and-fire (QIF) model remains an open question.

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## APPENDIX

*Proof of Proposition 1*

We proceed in three steps.

*Step 1:* Let $B_1(\phi) = \phi$ and $B_i(\phi) = \phi - h[B_{i-1}(\phi)]$ for $i = 2,\ldots,n+1$. We proof the following property. For $i = 1\ldots,n+1$, there exists a value $0 \leq \phi_i^c$ such that $B_i(\phi_i^c) = 0$ and such that $B_i'(\phi) > 0$ and $0 < B_i(\phi) < 1$ $\forall \phi \in [\phi_i^c, 1)$. This is trivial for $B_1(\phi) = \phi$, with $\phi_1^c = 0$. Considering the property to be true for $B_{i-1}$ with $i \in \{2\ldots,n+1\}$, we proceed by induction. One first obtains

$$B_i'(\phi) = 1 - h'[B_{i-1}(\phi)]B_{i-1}'(\phi) > 0 \quad \forall \phi \in [\phi_{i-1}^c, 1] \quad (32)$$

since $h(\cdot)$, evaluated on $B_{i-1}(\phi) \in [0,1]$, is strictly decreasing. Then, noting that $\phi_{i-1}^c < 1$ and that $h(0) > 1$ by construction, we have

$$B_i(\phi_{i-1}^c) = \phi_{i-1}^c - h[B_{i-1}(\phi_{i-1}^c)] = \phi_{i-1}^c - h(0) < 0. \quad (33)$$

Moreover, one easily computes that

$$B_i(1) = 1 - f^{-1}[\underline{x} + (i-1)\varepsilon] > 0 \tag{34}$$

since $i \leq n+1 < (\overline{x} - \underline{x})/\varepsilon + 1$. As a consequence of (32), (33), (34), and the continuity of $B_i$, there exists a unique $\phi_i^c$, with $\phi_{i-1}^c < \phi_i^c < 1$, such that $B(\phi_i^c) = 0$. Noting that $B_i(1) < 1$, it follows that $0 \leq B_i(\phi) < 1 \ \forall \phi \in [\phi_i^c, 1)$.

*Step 2:* A fixed point of the firing map (4) verifies

$$\phi_i^* = h[B_i(\phi_n^*)], \quad i = 1, \ldots, n. \tag{35}$$

One deduces that the condition (5) implies $0 < \phi_n^* - \phi_i^* < 1$, or equivalently

$$0 < B_i(\phi_n^*) < 1, \quad i = 1, \ldots, n. \tag{36}$$

Moreover, it also holds

$$B_{n+1}(\phi_n^*) = 0. \tag{37}$$

According to the above properties of $B_i$, the value $\phi_n^* = \phi_{n+1}^c$ exists and is the unique solution of (37) which verifies all the conditions (36). The values $\phi_i^*$ are then explicitly determined by (35).

*Step 3:* One verifies that the fixed point does not violate the ordering conditions (5). First, we have

$$\phi_1^* = h[B_1(\phi_n^*)] = h(\phi_n^*) > 0.$$

Furthermore, knowing that

$$h[B_2^*(\phi_n^*)] = h\{\phi_n^* - h[B_1(\phi_n^*)]\} > h(\phi_n^*) = h[B_1^*(\phi_n^*)],$$

the other relations follow by induction. If $h[B_i(\phi_n^*)] < h[B_{i+1}(\phi_n^*)]$ for $i = 1, \ldots, n$, then

$$\begin{aligned} h[B_{i+2}(\phi_n^*)] &= h\{\phi_n^* - h[B_{i+1}(\phi_n^*)]\} \\ &> h\{\phi_n^* - h[B_i(\phi_n^*)]\} = h[B_{i+1}(\phi_n^*)]. \end{aligned}$$

At last, one has $\phi_n^* < 1$. This concludes the proof.

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