Identifiability of dynamical networks with singular noise spectra

Michel Gevers, *Life Fellow, IEEE*
Alexandre Sanfelice Bazanella, *Senior Member, IEEE*
Guilherme Araujo Pimentel

***Abstract*—This paper addresses the problem of identifiability of dynamical networks in the case where the vector of noises on the nodes does not have full rank. In the full rank noise case, network identifiability is defined as the capability of uniquely identifying all three transfer function matrices composing the network from informative data. This includes the noise model, which can be uniquely defined when the noise vector has full rank. When the noise vector has a singular spectrum, it admits an infinite number of different noise models and the definition of network identifiability must be adapted to demand that the correct noise spectrum be identified from informative data rather than a specific noise model. With this new definition, we show that a network with rank reduced noise is identifiable under the same conditions that apply to a network with full rank noise.**

I. Introduction

This paper deals with the identifiability of dynamical networks in which the vector of noise signals acting on the nodes may have a singular spectral density matrix, hereafter called spectrum. An important particular case of this situation, which has received attention in the recent literature, occurs when some nodes of the network are noise-free. In order to understand the problem, and the contribution of this paper, we first briefly recall the progress made in the last five years on the question of identifiability of dynamical networks.

Let us start by defining the class of networks considered in this paper. We consider networks whose nodes are connected by directed edges made up of causal linear time-invariant systems described by their transfer functions. The node signals $w_i(t)$ may, in addition, be excited by known external signals $r_i(t)$, by noise signals $v_i(t)$, or by a combination of both. Thus, the vector $w(t)$ of node signals obeys the following network model $w(t) = G(q)w(t) + K(q)r(t) + v(t)$, where q^{-1} is the backward shift operator, $G(q)$, called the network matrix, describes the internal dynamics of the network, $K(q)$ describes the way in which the external signals act on the nodes, and $v(t)$ is assumed to be a stationary noise vector, of which some elements may be zero, with spectrum $\Phi_v(z)$. The matrices $G(q)$ and $K(q)$, whose elements are causal real rational transfer functions, will be described in more detail in Section II.

M. Gevers is with ICTEAM, Louvain University, B1348 Louvain la Neuve, Belgium e-mail: Michel.Gevers@uclouvain.be.

A. S. Bazanella is with DELAE/UFRGS - Department of Automation and Energy, Universidade Federal do Rio Grande do Sul, Porto Alegre-RS, Brazil e-mail: bazanella@ufrgs.br

G.A. Pimentel is with GACS/PUCRS - Automation and System Control Group, Pontifícia Universidade Católica do Rio Grande do Sul, Porto Alegre-RS, Brazil e-mail: guilherme.pimentel@pucrs.br

A central problem in the identification of such networks from data is the question of network identifiability. It can be simply stated as follows. Given an actual "true" network, and assuming that I am allowed to apply sufficiently rich excitation signals $r(t)$ to it, can I then exactly recover the matrices $G(q)$ and $K(q)$, and the spectrum $\Phi_v(z)$ from the signals $\{w, r\}$ measured on this true network?

The study of this problem has proceeded stepwise, as always. It has first been observed that, generically, this task is impossible unless some prior information is available about some structural properties of the network, such as a priori known elements, diagonal structure, etc [1]. A first series of results on network identifiability have then been obtained under the assumption that the noise spectrum $\Phi_v(\omega)$ is positive definite, i.e. the noise vector $v(t)$ has full rank.

Under the latter assumption, it is well known that the noise vector $v(t)$ admits a uniquely defined model as the output of a white noise driven filter, i.e. $v(t) = H(q)e(t)$, where $H(q)$ is a square, stable and stably invertible real rational matrix with the property that $H(\infty) = I$, and where $e(t)$ is a white noise vector with positive definite covariance matrix Σ. As a result, the network model can then be replaced by $w(t) = G(q)w(t) + K(q)r(t) + H(q)e(t)$, where $H(q)$ is uniquely defined from $\Phi_v(z)$. For networks with full rank noise, the network identifiability question stated above was thus reformulated as: under what conditions (in the form of prior knowledge on the structure of the network) can I uniquely recover the triple $G(q), K(q), H(q)$ from the measured signals $\{w, r\}$, assuming again that I have applied sufficiently rich excitation signals $r(t)$ to the network?

The next observation was to remark that the network model can be rewritten as an Input-Output (I/O) model $w(t) = T(q)r(t) + N(q)e(t)$, where $T(q) = (I - G(q))^{-1}K(q)$ and $N(q) = (I - G(q))^{-1}H(q)$ are transfer function matrices that are uniquely defined by the network matrices $G(q), K(q), H(q)$; in addition, due to a property of the network matrix $G(q)$ to be described later and to $H(\infty) = I$, this I/O model also has the property that $N(\infty) = I$. Now it is well known from system identification theory that, under those conditons, the I/O model $T(q), N(q)$ can be uniquely identified from $\{w, r\}$, assuming again that $r(t)$ is sufficiently rich.

A result of this observation is that the network identifiabiliy property, defined above as the possibility to identify the triplet $[G(q), K(q), H(q)]$ from measured data $\{w, r\}$, could now conveniently be replaced by an easier definition, in terms of mapping between transfer function matrices. More precisely, an alternative and convenient definition of network identifiability is to state that the true network model $[G(q), K(q), H(q)]$ is identifiable if the mapping from $[G(q), K(q), H(q)]$ to $[T(q), N(q)]$ described above is invertible, i.e. if we can recover $[G(q), K(q), H(q)]$ from the identified $[T(q), N(q)]$. It is easy to understand that this is generically impossible, and hence the task is to find conditions on the structure of the true $G(q), K(q)$ and $H(q)$ (in the form of prior knowledge on that structure) such that the mapping becomes invertible. A number of conditions for network identifiability have been obtained for the situation of full rank noise in [1], [2], [3].

Recently, attention has turned to the situation where some nodes are noise-free [4] and to the more general situation, which encompasses the previous one, where the noise vector $v(t)$ does not have full rank [5], [6]. A first solution to this new situation was proposed in [4] by adopting a network identifiability definition that depends on the identification criterion. Another solution was presented in [5], [6] by assuming that a subvector of the noise vector is known to have full rank. The main contribution of the present paper is to show that such restrictions are not necessary. Indeed we shall show that the definitions of network identifiability adopted for the full rank case in [2] and [3] can be naturally extended to the situation of rank reduced noise by exploiting properties of the spectral factorization of singular spectra.

It is well known from spectral factorization theory that, when a stationary noise vector $v(t)$ is not full rank, there is no uniquely defined realization $v(t) = H(q)e(t)$ with $e(t)$ white noise. Thus, it makes no sense to demand that $G(q), K(q), H(q)$ be uniquely recovered from an identified $T(q), N(q)$ since such unique $H(q)$ is not defined. However, what does uniquely characterize the true network is $G(q), K(q)$ and $\Phi_v(z)$. Therefore, in the case where $v(t)$ does not have full rank, the definition of network identifiability has to be adapted. It now refers to whether one can uniquely recover $G(q), K(q)$ and $\Phi_v(z)$ from data $\{w, r\}$. If some of the components of $v(t)$ are zero (i.e. in the case where some nodes are noise-free), then the network will only be called identifiable if the data allow one to discover that the corresponding rows and columns of the identified $\Phi_v(z)$ are zero.

Our first contribution will be to show that the network identifiability problem can again be recast as a two-step problem: first the identification of an I/O model $[T(q), \Phi_{\bar{v}}(z)]$, or $[T(q), N(q), Q]$, where $\Phi_{\bar{v}}(z)$ is the spectrum of the noise $\bar{v}(t)$ of the I/O model, and where $[N(q), Q]$ is a factorisation of $\Phi_{\bar{v}}(z)$ as $\Phi_{\bar{v}}(z) = N(z)QN^T(z^{-1})$; next the recovery of the true $G(q), K(q)$ and $\Phi_v(z)$ from this identified I/O model. An important observation here is that the noise model $[N(q), Q]$ of the I/O model is not unique, because the noise vector of the I/O model is not full rank whenever the noise vector of the network is not full rank. This problem will be addressed using properties of spectral factorization for singular spectra.

Our second contribution will be to show that sets of sufficient conditions that provide network identifiability in the full rank noise case also provide network identifiability in the case where the noise on the network has a singular spectrum. The main message therefore is that, when the noise on the network nodes has a singular spectrum, the tools used for the full rank case have to be slightly adapted but network identifiability conditions can be obtained that are identical to those for the full rank case.

The paper is organized as follows. In Section II we define precisely the class of networks under study, including their rank-defficient noise structure, and define the notation. Next we discuss, in Section III, the factorizations of rational spectra, showing the two families of such factorisations that exist, the relationships between them, and the impossibility of defining uniquely a factorisation for a singular spectrum. Motivated by this impossibility, we adapt the definition of identifiability of networks, and this new definition is presented in Section IV along with our main results. These results show that network identifiability conditions presented previously remain valid under this adapted definition. In order to clarify our theoretical findings, in Section V we present in detail a case study of a network with three nodes, one of which is noise-free, where we perform the identification of the I/O model and then recover the network model and the noise spectrum. Finally, Section VI summarizes our conclusions.

II. Problem statement

We consider a "true" dynamical network

$$w(t) = G^0(q)w(t) + K^0(q)r(t) + v(t) \tag{1}$$

with the following properties:

- $w(t), r(t), v(t)$ are vectors of dimension L
- $G^0(q)$ and $K^0(q)$ are matrices of rational functions
- $G^0(q)$ is the **network matrix**, whose elements $G^0_{ij}(q)$ are proper transfer functions and with $G^0_{ii}(q) = 0$
- there is a delay in every loop from one $w_j(t)$ to itself.
- the network is well-posed so that $(I - G^0(q))^{-1}$ is proper and stable.
- all node signals $w_j(t), j = 1, \ldots, L$ are measurable.
- $r_i(t)$ are known external excitation signals that are available to the user in order to produce informative experiments for identification.
- $v(t) \in \Re^L$ is a wide-sense stationary stochastic process with power spectral density (spectrum) $\Phi_v(z)$.
- the external excitation signals $r_i(t)$ are assumed to be uncorrelated with all noise signals $v_j(t), j = 1, \ldots, L$.
- q^{-1} is the delay operator.

Before we proceed, we recall the notion of an **admissible network matrix**, defined in [3].

Definition 2.1: : A network matrix $G(q)$ is called *admissible* if its diagonal elements are zero, all the $G_{ij}(q)$ are proper, $(I - G(q))^{-1}$ is stable, and there is a delay in every loop going from one $w_j(t)$ to itself.

From now on, we will drop the dependence on t, z and q whenever it creates no ambiguity. We shall consider the general case where the spectrum Φ_v may be singular, and we shall simultaneously treat the case where some of the components of v are zero and where the vector made up of its non-zero components may not be of full rank. We thus split up the noise vector into two subvectors, one for the noisy nodes, one for the noise-free nodes. Without loss of generality, we reorder the nodes such that the last $L-p$ are noise-free. The true network is then defined as follows:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} G^0_{11} & G^0_{12} \\ G^0_{21} & G^0_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} K^0_{11} & K^0_{12} \\ K^0_{21} & K^0_{22} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \quad (2)$$

where $dim(w_1) = dim(v_1) = dim(r_1) = p$, $\dim(w_2) = L-p$, and where the noise vector $v \triangleq \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ is defined by its spectrum

$$\Phi_v(z) = \begin{pmatrix} \Phi_{v_1}(z) & 0 \\ 0 & 0 \end{pmatrix} \quad (3)$$

where v_1 has rank $q \leq p$. The rank of v_1 is defined as the normal rank of its spectrum $\Phi_{v_1}(z)$, i.e. the rank of $\Phi_{v_1}(z)$ at almost all z [7]. The corresponding true I/O model is now:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} T^0_{11} & T^0_{12} \\ T^0_{21} & T^0_{22} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} \quad (4)$$

where $T^0 = (I - G^0)^{-1} K^0$ and the I/O noise $\bar{v} \triangleq \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$ has a spectrum

$$\Phi_{\bar{v}}(z) = (I - G^0(z))^{-1} \Phi_v(z) [(I - G^0(z^{-1}))^{-1}]^T \quad (5)$$

The network identifiability problem with rank-reduced noise is then the following. From (w, r) data, and with r sufficiently rich, we can identify the matrix T^0 and a noise model for the noise $\bar{v}$ of (4) having spectrum $\Phi_{\bar{v}}$. The question is then: under what network identifiability conditions (in the form of prior knowledge on the true network) can we recover the network matrices (G^0, K^0) and a network noise model $v = H(z)e$ with $\text{cov}(e) = \Sigma$ such that

$$H(z)\Sigma H^T(z^{-1}) = \Phi_v(z) = \begin{pmatrix} \Phi_{v_1}(z) & 0 \\ 0 & 0 \end{pmatrix} \quad (6)$$

It is important to note that, in the case of a singular noise spectrum Φ_v there is no true noise model $v = H^0(z)e$ for the true network (2), but only a true noise spectrum defined by (3) with the property that the blocks $(1,2), (2,1)$ and $(2,2)$ are zero because there is no noise on the node signals w_2. As a result, in terms of network identifiability, what matters are conditions that guarantee that the spectrum Φ_v of (3) is recovered exactly despite the fact that there are an infinity of possible noise models for the I/O model (4). This is where the problem of network identifiability with noise-free and/or rank-reduced noise models differs from the full rank noise case, where a unique noise model H can always be used, with the property that H is stable, inversely stable and has $H(\infty) = I_L$.

In order to address this problem, we recall some standard properties of the factorization of singular spectra.

III. Factorization of singular rational spectra

The major early results on factorization of rational spectra can be found in [8], where the continous time case was treated. Extensions to the discrete-time case can be found in [9], [10]. In [11] the set of all equivalent spectral factors was described, based on a discrete-time extension of [8], but only for the case of full rank vector noise processes. Few results on the characterization of all equivalent spectral factors can be found for the case of singular spectra, the reason being that in this case one can define two families of spectral factors, one using square factors, the other one using non-square factors. We now describe these two sets of factorizations.

A. The two families of spectral factors for singular spectra

Let $\Phi_v(z)$ be a real rational spectral density matrix with dimension $L \times L$, and with normal rank $q \leq L$, assuming that the rank is identical almost everywhere. Then essentially two different sets of factorizations can be defined, one with $H(z)$ of dimension $L \times q$ and one with $H(z)$ of dimension $L \times L$.

Factorization 1: [8], [9], [12] $\Phi(z) = H(z)H^T(z^{-1})$ where $H(z)$ is causal, real rational and stable, of dimension $L \times q$, and of normal rank q. This factorization is unique up to right multiplication by a real orthogonal matrix P. Thus, if $H_0(z)$ is one factor, then any other factor is of the form $H(z) = H_0(z)P$ where $PP^T = I_q$.

If the spectrum $\Phi(z)$ has rank q, it can be decomposed as a sum of rank 1 terms: $\Phi(z) = \sum_{l=1}^{q} h_l(z) h_l^T(z^{-1})$. This decomposition is not unique. The $h_l(z)$ in this decomposition correspond to the columns of $H(z)$ in a factorization of type 1 of $\Phi(z) = H(z)H^T(z^{-1})$.

Factorization 2: [10], [13] $\Phi(z) = H(z)\Sigma H^T(z^{-1})$ where $H(z)$ is causal, real rational, stable and inversely stable, of dimension $L \times L$, with $H(\infty) = I_L$. With this factorization, Σ is a real symmetric matrix with dimension $L \times L$, rank equal to q, and is unique.

The use of Factorization 2 lends itself to the use of Prediction Error (PE) identification methods. Indeed, since $H(z)$ is invertible and $H(\infty) = I_L$, predictions can be formed by $\hat{y}(t|t-1) = H^{-1}(q)G(q)u(t) + (1 - H^{-1}(q))y(t)$ and the prediction errors can easily be expressed as $\epsilon(t) = y(t) - \hat{y}(t|t-1)$, where this error is simply zero at the noise-free outputs.

We observe that, unlike the case of a full rank noise process where a uniquely defined factorization exists [11], when the noise has a reduced rank, such unique factorization cannot be defined, whether factorization 1 or factorization 2 is used. Clearly there exists a relationship between these two factorizations, which we establish in the next subsection.

Example 3.1: Consider a noise vector with $L = 3$ such that $v_2(t) = 2v_1(t)$, $v_3(t) \equiv 0$ and let $\phi_1(z)$ be the spectrum of $v_1(t)$. Then $p = 2$, $q = 1$ and the spectrum of this vector is given by

$$\Phi_v(z) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \phi_1(z).$$

Let $g(z)$ be the unique stable and minimum phase spectral factor of $\phi_1(z)$, i.e. $\phi_1(z) = g(z)g(z^{-1})$. Then a Factorisation

2 can be obtained for this spectrum for any $H(z)$ of the form[1]

$$H(z) = \frac{1}{g(\infty)} \begin{bmatrix} g(z) & 0 & a(z) \\ 0 & g(z) & b(z) \\ a(z) & b(z) & c(z) \end{bmatrix}$$

where $b(z)$ is any strictly causal transfer function, $a(z) = -2b(z)$, $a(\infty) = b(\infty) = 0$, $c(z)$ is any causal transfer function such that $c(\infty) = g(\infty)$, while Σ is uniquely defined as

$$\Sigma = \phi_1(\infty) \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As for Factorisation 1, we have

$$H(z) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} g(z)$$

as the unique factorisation since $q = 1$. ■

B. Relationship between the two families of spectral factors

Consider first factorization 1; thus let $\Phi(z) = H(z)H^T(z^{-1})$ where $H(z)$ has dimension $L \times q$ with normal rank q. Let $H_q(z)$ be any $q \times q$ submatrix of $H(z)$ that is nonsingular at ∞, and let S denote the set of corresponding rows. Now define the $L \times L$ matrix $H_1(z)$ as follows

$$H_1(z) \triangleq \begin{bmatrix} H(z) & \bar{H}(z) \end{bmatrix} \tag{7}$$

where the q rows of $\bar{H}(z)$ that belong to the set S are zero, and its $L-q$ remaining rows form any paraunitary matrix $V(z)$, of dimension $(L-q) \times (L-q)$, i.e. $V(z)V^T(z^{-1}) = I_{L-q}$. Since $V(\infty) = I_{L-q}$, it follows from the above construction that $H_1(\infty)$ is nonsingular.

Now define $\tilde{H}(z) \triangleq H_1(z)(H_1(\infty))^{-1}$ and $\Sigma \triangleq H(\infty)H^T(\infty)$. It then follows that

$$\begin{aligned} &\tilde{H}(z)\Sigma\tilde{H}^T(z^{-1}) \\ &\quad = H_1(z)(H_1(\infty))^{-1}H(\infty)H^T(\infty)(H_1(\infty))^{-T}H_1^T(z^{-1}) \\ &\quad = H_1(z)\begin{pmatrix} I_q \\ 0 \end{pmatrix}\begin{pmatrix} I_q & 0 \end{pmatrix}H_1^T(z^{-1}) \\ &\quad = H(z)H^T(z^{-1}) \end{aligned}$$

with the property that $\tilde{H}(\infty) = I_L$. This shows how to construct a factorization of the form 2 from a factorization of the form 1. It also shows that Σ is uniquely defined in factorization 2, while $\tilde{H}(z)$ is not unique by virtue of the freedom in the choice of the paraunitary matrix $V(z)$.

We now start from a $L \times L$ factorization 2 of $\Phi(z)$ and we show how to construct a $L \times q$ factorization of type 1 for $\Phi(z)$. Thus, let $\Phi(z) = H(z)\Sigma H^T(z^{-1})$ with $H(\infty) = I_L$ and $\text{rank}(\Sigma) = q$. We can then factor Σ as $\Sigma = MM^T$ where M is lower triangular with nonnegative diagonal elements:

$$M = \begin{pmatrix} x & 0 & 0 & 0 & 0 & \dots & 0 \\ x & x & 0 & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & x & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & x & \dots & x & 0 & \dots & 0 \end{pmatrix} \triangleq \begin{pmatrix} M_1 & 0_{L\times(L-q)} \end{pmatrix} \tag{8}$$

where $\dim(M_1) = L \times q$ and M_1 has full column rank. Now define $\bar{H}(z) \triangleq H(z)M_1$ and note that $\dim(\bar{H}(z)) = L \times q$ with

$$\begin{aligned} \bar{H}(z)\bar{H}^T(z^{-1}) &= H(z)M_1M_1^TH^T(z^{-1}) \\ &= H(z)MM^TH^T(z^{-1}) \\ &= \Phi(z). \end{aligned}$$

Thus, the factorization $\Phi(z) = \bar{H}(z)\bar{H}^T(z^{-1})$ is of type 1.

IV. Network identifiability with reduced rank noise

As stated in the introduction, in the case of a singular noise spectrum we have to adapt the definition of network identifiability to account for the fact that the noise model is non-unique. On the other hand, it is important to require that a network is called identifiable if the data let us identify the correct noise spectrum, and in particular the fact that some nodes are noise-free, which represents structural information.

Consider the "true" network (1) defined by the triple $\mathcal{S} = [G^0, K^0, \Phi_v]$ and its corresponding true I/O description $[T^0, \Phi_{\bar{v}}]$ defined by

$$\begin{aligned} T^0(z) &\triangleq (I - G^0(z))^{-1}K^0(z) \quad \text{and} \\ \Phi_{\bar{v}}(z) &\triangleq (I - G^0(z))^{-1}\Phi_v(z)(I - G^0(z^{-1}))^{-T} \end{aligned} \tag{9}$$

Just as in previous work on network identifiability, the definition of network identifiability relies on the following observation. Applying a sufficiently rich excitation signal on the true network, and collecting the $\{w, r\}$ data, one can identify an I/O model $[\tilde{T}(z), \tilde{\Phi}_{\bar{v}}]$, or $[\tilde{T}(z), \tilde{N}(z), Q]$, where $[\tilde{N}(z), Q]$ is any factorisation of $\tilde{\Phi}_{\bar{v}}$. Different methods can be used for this open-loop identification; one possible method would be to identify $T(z)$ by an instrumental variable method, compute the residuals $\bar{v} = w - \tilde{T}(z)r$, and then model $\bar{v}$ using a stochastic realization method. From this identification one gets a $\tilde{T}(z)$ that is uniquely defined and converges asymptotically[2] to $T^0(z)$, while $\tilde{N}(z)$ and Q are non-unique but have the property that, asymptotically, $\tilde{N}(z)Q\tilde{N}^T(z^{-1}) = \Phi_{\bar{v}}$, where $\Phi_{\bar{v}}$ was defined in (9).

We then adopt the following definition for the identifiability of the true network model.

Definition: *Identifiability of the network*. The "true" dynamical network (1) is identifiable if there exists no other network $[\tilde{G}(z), \tilde{K}(z), \tilde{\Phi}_v(z)] \neq [G^0(z), K^0(z), \Phi_v(z)]$, with

[1]Not all $H(z)$ matrices for Factorisation 2 are in this form.

[2]as the number of data tends to infinity

$\tilde{G}(z)$ admissible, such that $(I-\tilde{G}(z))^{-1}\tilde{K}(z)=T^0(z)$ and $(I-\tilde{G}(z))^{-1}\tilde{\Phi}_v(z)(I-\tilde{G}(z^{-1}))^{-T}=\Phi_{\bar{v}}(z)$.

This definition is an extension to the case of networks with reduced rank noise of Definition 2 in [3], which relates to the identifiability of the true network. For networks with rank reduced noise vectors we can now extend Theorem 5.1 of [3] and characterize the set of all network models that produce the same I/O model.

Theorem 4.1: The set of all network models that produce an I/O model $M_{io}=[T(z)\ \ \Phi_{\bar{v}}(z)]$ is given by $\{[\tilde{G}(z),\tilde{K}(z),\tilde{H}(z),\tilde{\Sigma}]\}$ where

- $\tilde{G}(z)$ is any admissible network matrix of size $L\times L$
- $\tilde{K}(z)=(I-\tilde{G}(z))T(z)$
- $\tilde{H}(z),\tilde{\Sigma}$ is any pair such that $\tilde{\Sigma}$ is real symmetric and nonnegative definite with

$$\tilde{H}(z)\tilde{\Sigma}\tilde{H}^T(z^{-1})=(I-\tilde{G}(z))\Phi_{\bar{v}}(z)(I-\tilde{G}(z^{-1}))^T \quad (10)$$

Proof: We first show that the set of network matrices described above produces the correct I/O model $M_{io}=[T,\Phi_{\bar{v}}]$. Indeed, the corresponding I/O model is defined by $\tilde{T}=(I-\tilde{G})^{-1}\tilde{K}=T$ and $\tilde{\Phi}_{\bar{v}}$, or by $\tilde{T}=(I-\tilde{G})^{-1}\tilde{K}=T$, $\tilde{N}=(I-\tilde{G})^{-1}\tilde{H}$ and $\tilde{Q}=\tilde{\Sigma}$. From this it follows, using (10) that

$$\tilde{\Phi}_{\bar{v}}=\tilde{N}\tilde{Q}\tilde{N}^*=(I-\tilde{G})^{-1}\tilde{H}\tilde{\Sigma}\tilde{H}^*(I-\tilde{G})^{-*}=\Phi_{\bar{v}}$$

where $A^*(z)\triangleq A^T(z^{-1})$ and $A^{-*}(z)\triangleq[A^T(z^{-1})]^{-1}$. Conversely, let $[\tilde{G},\tilde{K},\tilde{H},\tilde{\Sigma}]$ be any network that produces the correct T and $\Phi_{\bar{v}}$ with $\tilde{G}$ admissible. Then we must have $(I-\tilde{G})^{-1}\tilde{K}=T$ and $(I-\tilde{G})^{-1}\tilde{H}\tilde{\Sigma}\tilde{H}^*(I-\tilde{G})^{-*}=\Phi_{\bar{v}}$. Pre- and post-multiplying by $(I-\tilde{G})$ yields the desired result. ■

With the definition of network identifiability in our hands, we now use the spectral factorization results of section III to show that a network with a singular noise spectrum is identifiable under a same set of sufficient conditions that applies for a network with full rank noise.

A. Network identifiability with reduced rank noise using Factorization 1

Consider a true network with $L-p$ noise-free nodes defined by (2)-(3), and consider that its noise vector is modeled using Factorization 1:

$$v=\begin{pmatrix} v_1\\ 0\end{pmatrix}=H^0e\triangleq\begin{pmatrix} H_1^0\\ 0\end{pmatrix}e \quad (11)$$

with $\dim(H_1^0)=p\times q$ and $\mathrm{cov}(e)=I_q$ with $q\le p$. A corresponding true I/O model is then

$$\begin{pmatrix} w_1\\ w_2\end{pmatrix}=\begin{pmatrix} T_{11}^0 & T_{12}^0\\ T_{21}^0 & T_{22}^0\end{pmatrix}\begin{pmatrix} r_1\\ r_2\end{pmatrix}+\begin{pmatrix} N_1^0\\ N_2^0\end{pmatrix}e \quad (12)$$

where

$$[T^0\ \ N^0]=(I-G^0)^{-1}[K^0\ \ H^0]. \quad (13)$$

The respective noise models are:

$$v=\begin{pmatrix} v_1\\ 0\end{pmatrix}\triangleq\begin{pmatrix} H_1^0\\ 0\end{pmatrix}e \ \text{ and }\ \bar{v}=\begin{pmatrix} \bar{v}_1\\ \bar{v}_2\end{pmatrix}\triangleq\begin{pmatrix} N_1^0\\ N_2^0\end{pmatrix}e \quad (14)$$

with $\mathrm{cov}(e)=I_q$. Now the network identifiability problem is as follows. We identify an I/O model $[T(z),N(z)]$ from data using Factorization 1 for the modeling of the noise $\bar{v}$. The question then is: under what prior knowledge conditions on $[G^0(z),K^0(z),H^0(z)]$ can we recover a noise model for the network that will have the correct form

$$v=\begin{pmatrix} H_1\\ 0\end{pmatrix}e, \quad (15)$$

that is with the property that $v_2=0$ and $H_1(z)H_1^T(z^{-1})=H_1^0(z)(H_1^0(z^{-1}))^T=\Phi_{v_1}(z)$, where $\Phi_{v_1}(z)$ was defined in (3). In other words, under what conditions on the excitation structure do we recover the information of which nodes are noise-free and the correct spectrum for the noise vector on the other nodes? We establish the following sufficient condition for network identifiability, which are identical to those established in Theorem 6.1 of [3] for the full rank noise case.

Theorem 4.2: Let the true network be (1)-(3) and let $H^0(z)$ be any factorization $\Phi_v(z)=H^0(z)[H^0(z^{-1})]^T$. The true network is identifiable if the matrix $[K^0(z)\ \ H^0(z)]$ contains L known and linearly independent columns.

Proof: We first compute the noise model of (12) as a function of the noise model (11) of the true network (2)-(3):

$$\begin{aligned} N^0 &= \begin{pmatrix} N_1^0\\ N_2^0\end{pmatrix}=\begin{pmatrix} A^{-1}H_1^0\\ (I-G_{22}^0)^{-1}G_{21}^0A^{-1}H_1^0\end{pmatrix} \quad \text{with}\\ A &\triangleq I-G_{11}^0-G_{12}^0(I-G_{22}^0)^{-1}G_{21}^0 \end{aligned} \quad (16)$$

Since the noise vector $\bar{v}$ of (14) has rank $q\le p\le L$, it can be modeled using a noise realization resulting from Factorization 1 and it is possible to identify from (w,r) data an I/O model of the following form:

$$\begin{pmatrix} w_1\\ w_2\end{pmatrix}=\begin{pmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}\begin{pmatrix} r_1\\ r_2\end{pmatrix}+\begin{pmatrix} N_1\\ N_2\end{pmatrix}e \quad (17)$$

where $\dim(N_1)=p\times q$, $\dim(N_2)=(L-p)\times q$ and $\mathrm{cov}(e)=I_q$. From the properties of Factorization 1, we know that such noise model is not unique, but that they all have the same covariance $\mathrm{cov}(e)=I_q$, and that any two noise models of dimension $L\times q$ are related by an orthogonal matrix $P\in\Re^q$. This means that the identification of the I/O model (17) using Factorization 1 will asymptotically yield a $L\times q$ noise model that is related to the noise model (16) of the true I/O model by

$$\begin{pmatrix} N_1\\ N_2\end{pmatrix}=\begin{pmatrix} N_1^0\\ N_2^0\end{pmatrix}P=\begin{pmatrix} A^{-1}H_1^0P\\ (I-G_{22}^0)^{-1}G_{21}^0A^{-1}H_1^0P\end{pmatrix} \quad (18)$$

for some P such that $PP^T=I_q$. We show that there is no other network model $[\bar{G},\bar{K},\bar{H}]\neq[G^0,K^0,H^0]$, using Factorization 1 for the noise model, that can produce the I/O model (17) with a noise model belonging to the set (18).

Suppose such $[\bar{G},\bar{K},\bar{H}]$ exists. Then, necessarily:

$$\begin{gathered}(I-\bar{G})^{-1}\bar{K}=(I-G^0)^{-1}K^0 \ \text{ and}\\ (I-\bar{G})^{-1}\bar{H}\bar{H}^*(I-\bar{G})^{-*}=(I-G^0)^{-1}H^0(H^0)^*(I-G^0)^{-*}\end{gathered} \quad (19)$$

By assumption, there exists a $L\times L$ submatrix W of $[K^0 \;\; H^0]$ that contains L known and linearly independent columns, and the same columns are therefore known in $[\bar{K} \;\; \bar{H}]$. Extracting the corresponding equations from (19)-(19) yields the following set of equations:

$$(I-\bar{G})^{-1}W = (I-G^0)^{-1}W \tag{20}$$

We now write $\bar{G} = G^0 + \Delta G$, observing that ΔG has zero diagonal elements since $\bar{G}$ and G^0 are both admissible. Substituting in (20) shows that this expression is equivalent with $\Delta G(I-G^0)^{-1}W = 0$, where W and 0 have size $L\times L$. This represents a set of $L\times L$ linearly independent equations for the $L\times L$ unknown elements of ΔG, from which it follows that $\Delta G = 0$, and hence $\bar{G} = G^0$. It then follows from (19) that $\bar{K} = K^0$. It remains to show that any network noise model $\bar{H}$ that is reconstructed from any one of the I/O noise models (18) has the correct spectrum, i.e. $\bar{H}\bar{H}^* = \Phi_v$. Since $\bar{G} = G^0$ we have

$$\begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \end{pmatrix} = (I-G^0)\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \tag{21}$$

This leads to the following set of equations:

$$\bar{H}_1 = (I-G^0_{11})A^{-1}H^0_1P - G^0_{12}(I-G^0_{22})^{-1}G^0_{21}A^{-1}H^0_1P$$
$$\bar{H}_2 = -G^0_{21}A^{-1}H^0_1P + (I-G^0_{22})(I-G^0_{22})^{-1}G^0_{21}A^{-1}H^0_1P$$

from which it follows that $\bar{H}_1 = H^0_1P$ and $\bar{H}_2 = 0$. Thus, the network noise model $\bar{H}$ has been reconstructed with the correct noise spectrum, showing that $v_2 = 0$ and $\Phi_{v_1} = \bar{H}_1\bar{H}_1^* = H^0_1(H^0_1)^*$. ■

We illustrate the theorem with the following example. Consider the 3-node network with rank reduced noise:

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & G_{32} & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} K_1r_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{22}$$

with $\text{cov}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \triangleq \Sigma$, so that $q = \text{rank}(v) = 1$. By factorizing Σ as $\Sigma = LL^T$ as in (8), we can alternatively express the noise model of (22) as

$$v = \begin{pmatrix} 0 \\ H_1 \\ H_2 \end{pmatrix} e \quad \text{with } \text{cov}(e) = 1 \tag{23}$$

It then follows from Theorem 4.2 that the network is identifiable if K_1, H_1 and H_2 are known and not identically zero. It is easy to verify that this is indeed the case.

B. Network identifiability with reduced rank noise using Factorization 2

We now show that the same result holds true when we use Factorization 2 for the modelling of the rank-reduced noise in the I/O model. This has the advantage that we can then use a Prediction Error method for the identification of the corresponding I/O model since it has a noise model $N(z)$ that is stable, stably invertible and with the property that $N(\infty) = I_L$.

Thus suppose that we model the true I/O model (4) using factorization 2 for its noise model:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{24}$$

We define by $\bar{v}$ the noise vector of this true I/O model, and we note that

$$\Phi_{\bar{v}}(z) = N(z)Q[N(z^{-1})]^T \tag{25}$$

with $N(\infty) = I_L$, $\text{cov}\begin{pmatrix} e_1, \\ e_2 \end{pmatrix} = Q$, and rank $(Q) = q \leq p$. We can then factorize $Q = LL^T$ where L is as in (8), and define $\tilde{N}(z) \triangleq N(z)L$. Thus $\Phi_{\bar{v}}(z)$ can be refactored as

$$\Phi_{\bar{v}}(z) = \tilde{N}(z)\tilde{N}^T(z^{-1}) = N_1(z)N_1^T(z^{-1}) \tag{26}$$

where $N_1(z)$ is the submatrix formed from the q first columns of $\tilde{N}(z)$.

The identification of an I/O model of the form (24) converges to a model with $T = T^0$ and a noise model for $\bar{v}$ that has the correct spectrum $\Phi_{\bar{v}}$ with the properties (25). We have shown that it can be transformed to a noise model $\bar{v} = N_1(z)e$ with $\dim(N_1) = L\times q$ and $\text{cov}(e) = I_q$. Therefore the network identifiability result of Theorem 4.2 applies to this case as well.

C. Network identifiability with reduced rank noise based on the spectrum of the I/O model

We have defined identifiability of the true dynamical network as the possibility of recovering the true network $[G^0(z), K^0(z), \Phi_v(z)]$ uniquely from the transfer matrix $T^0(z)$ and the spectrum $\Phi_{\bar{v}}(z)$ of its I/O model. In this subsection, we show that a standard network identifiability result for networks with full rank noise can be applied to the rank reduced noise case by reconstructing $[G^0(z), K^0(z), \Phi_v(z)]$ directly from $T^0(z)$ and $\Phi_{\bar{v}}(z)$, without the use of a noise model for either the network or the I/O description.

Theorem 4.3: Let the true network be described by (1) with a noise spectrum described by (3). Then the true network is identifiable if the matrix $K^0(z)$ is known to be diagonal and of full rank.

Proof: Consider that we have obtained $T^0(z)$ and $\Phi_{\bar{v}}(z)$ from the identification of the I/O model. From $K(z) = (I - G(z))T(z)$, and knowing that $K(z) = diag(k_i), i = 1, \ldots, L$ and that the diagonal elements of $G(z)$ are zero for any admissible $G(z)$, we first compute the diagonal elements of $K(z)$ as $k_i = \frac{1}{t^-_{ii}}$ where t^-_{ii} are the diagonal elements of $[T^0(z)]^{-1}$. $G(z)$ is then obtained as $G(z) = I - K(z)[T^0(z)]^{-1}$. Finally, we compute $\Phi_v(z)$ from

$$\Phi_v(z) = (I-G(z))\Phi_{\bar{v}}(z)(I-G(z^{-1}))^T. \tag{27}$$

■

V. A CASE STUDY

In this section, we present a case study that illustrates the result of Theorem 4.3 and the procedure proposed in its proof. Consider the following 3-node true network with rank reduced noise.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & -0.3z^{-1}+0.8z^{-2} & -0.5z^{-1} \\ z^{-1} & 0 & 0 \\ 0 & 0.5z^{-1} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (28)$$

with

$$cov \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \triangleq \Phi_v \quad (29)$$

Thus, there is no noise on the third node and the same white noise with variance 0.1 enters the first two nodes. The input matrix of the true network is the identity matrix, i.e. $K^0(z) = I_3$, but we assume that the only prior knowledge available to the user is that $K^0(z)$ is diagonal with nonzero diagonal elements. Identifying the diagonal elements of $K^0(z)$ will thus be part of the identification of the network, along with the identification of the network matrix $G^0(z)$ and of the noise spectrum Φ_v.

The corresponding true I/O model is given by:

$$T^0(z) = \begin{pmatrix} \frac{1}{1+0.3z^{-2}-0.55z^{-3}} & \frac{-0.3z^{-1}+0.55z^{-2}}{1+0.3z^{-2}-0.55z^{-3}} & \frac{-0.5z^{-1}}{1+0.3z^{-2}-0.55z^{-3}} \\ \frac{z^{-1}}{1+0.3z^{-2}-0.55z^{-3}} & \frac{1}{1+0.3z^{-2}-0.55z^{-3}} & \frac{-0.5z^{-2}}{1+0.3z^{-2}-0.55z^{-3}} \\ \frac{0.5z^{-2}}{1+0.3z^{-2}-0.55z^{-3}} & \frac{0.5z^{-1}}{1+0.3z^{-2}-0.55z^{-3}} & \frac{1+0.3z^{-3}-0.8z^{-3}}{1+0.3z^{-2}-0.55z^{-3}} \end{pmatrix}$$

We perform a simulation experiment based on $N = 4 \times 10^4$ data to check whether we can recover the network model (28)-(29) from an I/O model that is identified first. Since the white noise generation in Matlab is not perfect, the realization of a noise with spectrum given by (29) yields a signal whose spectrum is presented in Figure 1. It is clear from the figure that the rank is indeed one, but it is not exactly white noise, for the spectra are not perfectly flat. So, what we can expect to recover from the identification experiment is the spectrum in Figure 1, and not the one in equation (29).

The experiment consists of the application of uncorrelated white noises with variance 1 to the three inputs r_i and of white noises v_i with covariance given in Figure 1, and the collection of the corresponding w_i data. From these data, the estimate $\hat{T}(z)$ (Eq. (30) on top of the next page) was obtained by an instrumental variable method for the transfer matrix $T^0(z)$ from $r(t)$ to $w(t)$.

We then compute the predictions $\hat{w}(t) = \hat{T}(z)r(t)$ and estimate the noise $\bar{v}(t)$ by the prediction residuals, that is $\hat{\bar{v}}(t) = w(t) - \hat{w}(t)$. The spectrum of this vector is calculated by FFT, giving the estimate $\hat{\Phi}_{\bar{v}}$ for the spectrum of $\bar{v}(t)$ shown in Figure 2.

Before proceeding to the recovery of the network model from this identified I/O model, we verify the quality of the identified I/O model by computing the spectrum $\hat{\Phi}_v$ of the network model that would be obtained from the estimated $\hat{\Phi}_{\bar{v}}$ if the true network matrix were known. From (5) we compute

$$\hat{\Phi}_v(z) = (I - G^0(z))\hat{\Phi}_{\bar{v}}(z)(I - G^0(z^{-1}))^T \quad (31)$$

This estimated $\hat{\Phi}_v$ is presented in Figure 3, where it is seen that we have indeed recovered the original noise spectrum up to a very good precision. In order to quantify this assessment, we compare the estimate with the actual noise by forming the error $\Phi_{error}(z) = \Phi_v - \Phi_{v,true}$, computing its energy $E_{error} = \int_0^{2\pi} \Phi_{error}(e^{j\omega})d\omega$ and normalizing it to the energy of the noise $E_v = \int_0^{2\pi} \Phi_{v,true}(e^{j\omega})d\omega$; the normalization is done by dividing each element by $tr(E_v)$. This yields the following normalized error matrix:

$$E_{error,norm} = \begin{pmatrix} 1.3 \times 10^{-4} & 3.1 \times 10^{-5} & 3.4 \times 10^{-3} \\ 3.1 \times 10^{-5} & 8.5 \times 10^{-5} & 3.4 \times 10^{-3} \\ 3.4 \times 10^{-3} & 3.4 \times 10^{-3} & 3.7 \times 10^{-5} \end{pmatrix}$$

We now proceed to the recovery of the network quantities $G^0(z)$, $K^0(z)$ and Φ_v^0 from the estimated $\hat{T}(z)$ and $\hat{\Phi}_{\bar{v}}$ using the procedure described in the proof of Theorem 4.3. This

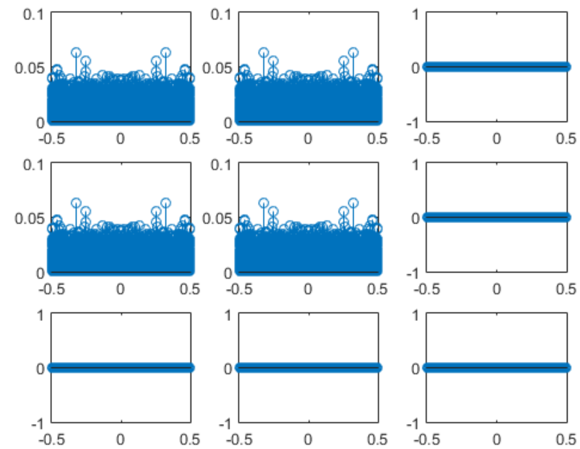

Fig. 1. $\Phi_{v,true}$ obtained by simulation with $N = 40,000$ data

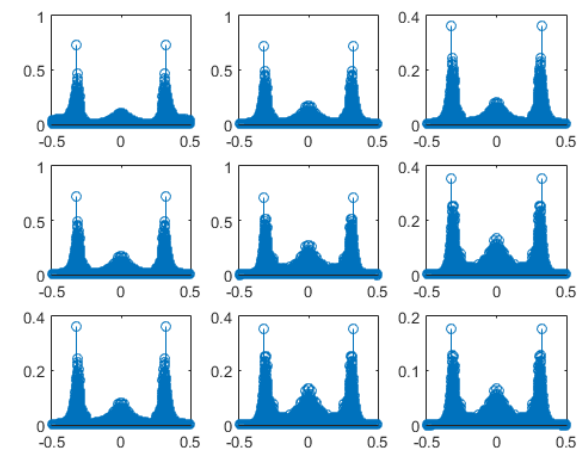

Fig. 2. $\hat{\Phi}_{\bar{v}}$: spectrum of the noise of the estimated I/O model

$$\hat{T}(z) = \begin{pmatrix} \frac{1}{1-0.0003626z^{-1}+0.2996z^{-2}-0.5502z^{-3}} & \frac{-0.2997z^{-1}+0.5503z^{-2}}{1-0.0006292z^{-1}+0.2996z^{-2}-0.5501z^{-3}} & \frac{-0.5005z^{-1}}{1-0.000895z^{-1}+0.2989z^{-2}-0.551z^{-3}} \\ \frac{1z^{-1}}{1-0.0008495z^{-1}+0.2992z^{-2}-0.5505z^{-3}} & \frac{1}{1+0.000545z^{-1}+0.2996z^{-2}-0.5503z^{-3}} & \frac{-0.5006z^{-2}}{1+0.001842z^{-1}+0.2988z^{-2}-0.5508z^{-3}} \\ \frac{0.5001z^{-2}}{1-0.00085z^{-1}+0.2993z^{-2}-0.5506z^{-3}} & \frac{0.5z^{-1}}{1-0.0005506z^{-1}+0.2995z^{-2}-0.5503z^{-3}} & \frac{0.9999-0.001888z^{-1}+0.2984z^{-2}-0.8013z^{-3}}{1-0.001735z^{-1}+0.2988z^{-2}-0.5507z^{-3}} \end{pmatrix} \quad (30)$$

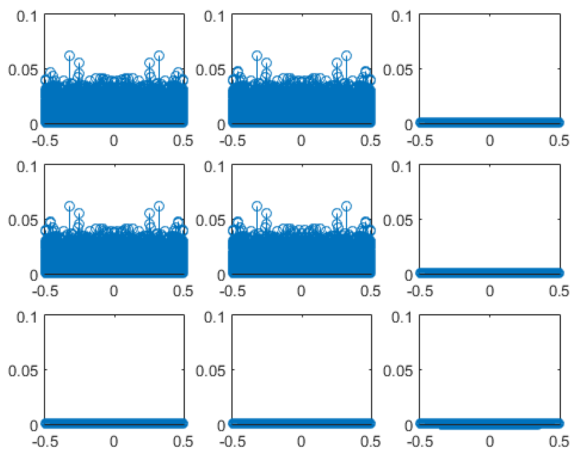

Fig. 3. $\hat{\Phi}_v$ spectrum of the network model, reconstructed from the estimated I/O model using the true network matrix $G^0(z)$

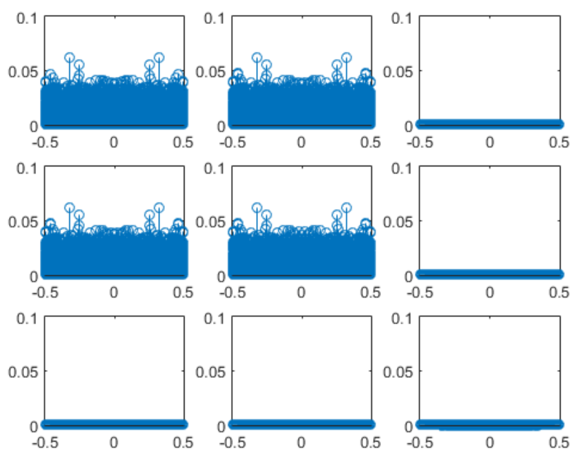

Fig. 4. $\hat{\Phi}_v$ spectrum of the network model, reconstructed from the estimated I/O model using the estimated network matrix $\hat{G}(z)$

results, after all coefficients of order 10^{-5} or smaller have been rounded up to zero, in the following estimates:

$$\hat{K}(z) = \begin{pmatrix} 1.0001 & 0 & 0 \\ 0 & 1.0001 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix}$$

$$\hat{G}(z) = \begin{pmatrix} 0 & -0.2996z^{-1} + 0.8006z^{-2} & -0.5005z^{-1} \\ 1z^{-1} & 0 & 0 \\ 0 & 0.5000z^{-1} & 0 \end{pmatrix}$$

We see that only errors of the order of 10^{-4} or smaller appear in all elements.

Finally, from the estimated $\hat{\Phi}_{\bar{v}}$ and $\hat{G}$ we can now compute the estimated $\hat{\Phi}_v$ using expression (31) with $G^0(z)$ replaced by $\hat{G}(z)$. This yields the spectrum presented in Figure 4, which is an excellent estimate of the real spectrum shown in Figure 1.

VI. Conclusions

We have shown in this paper how to extend the network identifiability results for networks with full rank noise vectors to networks with rank reduced noise, which includes networks that may have no noise on some nodes. Our methods and results have been based on properties of the spectral factorization of singular noise spectra. Essentially two different forms of spectral factorization have been proposed in the statistics literature for the factorization of such spectra. We have established the relationship between these two types of spectral factors.

On the basis of these spectral factorization results, we have shown how the definition of network identifiability can be adapted to the case of networks with rank reduced noise. Our main contribution has been to show that standard network identifiability results for networks with full rank noise apply unchanged to the case of networks with rank reduced noise.

Acknowledgment

This work is supported by the Program Science Without Borders, CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil, by the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office, and by Wallonie-Bruxelles International.

References

[1] J. Gonçalves and S. Warnick, “Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks,” *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1670–1674, 2008.

[2] H. Weerts, A. Dankers, and P. Van den Hof, “Identifiability in dynamic network identification,” in *USB Proc. 17th IFAC Symp. on System Identification*, Beijing, P.R. China, 2015, pp. 1409–1414.

[3] M. Gevers, A. Bazanella, and A. Parraga, “On the identifiability of dynamical networks,” in *USB Proc. IFAC World Congress 2017*. Toulouse, France: IFAC, July 2017, pp. 11 069–11 074.

[4] H. Weerts, P. Van den Hof, and A. Dankers, “Identifiability of dynamic networks with part of the nodes noise-free,” in *12th IFAC Workshop on Adaptation and Learning (ALCOSP)*, vol. 49, no. 13. IFAC PapersOnLine, 2016, pp. 19–24.

[5] ——, “Identification of dynamic networks with rank-reduced process noise,” in *Proc. IFAC World Congress*, 10562-10567, Ed. Toulouse, France: IFAC, July 2017, pp. 10 562–10 567.

[6] ——, “Identifiability of linear dynamic networks,” *to appear in Automatica. Also arxiv.org/abs/1609.00864*, 2017.

[7] M. Gevers, A. Bazanella, X. Bombois, and L. Mišković, “Identification and the information matrix: how to get just sufficiently rich?” *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2828–2840, December 2009.

[8] D. Youla, “On the factorization of rational matrices,” *IRE Transactions on Information Theory*, vol. 7, pp. 172–189, July 1961.

[9] Y. A. Rozanov, *Stationary random processes*. San Francisco: Holden-Day, 1967.
[10] E. J. Hannan, *Multiple Time Series*. New York: Wiley, 1970.
[11] M. Gevers and B. Anderson, "Representation of jointly stationary stochastic feedback processes," *International Journal of Control*, vol. 33, pp. 777–809, 1981.
[12] P. E. Caines, *Linear stochastic systems*. New York, USA: John Wiley, 1988.
[13] E. Hannan and M. Deistler, *The Statistical Theory of Linear Systems*. John Wiley & Sons, 1988.

Michel Gevers Michel Gevers obtained an Electrical Engineering degree from the University of Louvain, Belgium, in 1968, and a Ph.D. degree from Stanford University, California, in 1972, under the supervision of Tom Kailath. He is an IFAC Fellow, a Fellow of the IEEE, a Distinguished Member of the IEEE Control Systems Society. He holds Doctor Honoris Causa degrees from the University of Brussels and from Linköping University, Sweden. He has been President of the European Union Control Association (EUCA) from 1997 to 1999, and Vice President of the IEEE Control Systems Society in 2000 and 2001.

Michel Gevers is Professor Emeritus at the Department of Mathematical Engineering of the University of Louvain, Louvain la Neuve, and scientific collaborator at the Free University Brussels (VUB). He has been for 20 years the coordinator of the Belgian Interuniversity Network DYSCO (Dynamical Systems, Control, and Optimization) funded by the Federal Ministry of Science. He has spent long-term visits at the University of Newcastle, Australia, and the Technical University of Vienna, and was a Senior Research Fellow at the Australian National University from 1983 to 1986. His present research interests are in system identification, experiment design for identification of linear and nonlinear systems, and identifiability and informativity issues in networks of linear systems.

Michel Gevers has been Associate Editor of Automatica, of the IEEE Transactions on Automatic Control, of Mathematics of Control, Signals, and Systems (MCSS) and Associate Editor at Large of the European Journal of Control. He has published about 280 papers and conference papers, and two books: "Adaptive Optimal Control - The Thinking Man's GPC", by R.R. Bitmead, M. Gevers and V. Wertz (Prentice Hall, 1990), and "Parametrizations in Control, Estimation and Filtering Problems: Accuracy Aspects", by M. Gevers and G. Li (Springer-Verlag, 1993).

Alexandre Bazanella Alex Bazanella received his PhD degree in Electrical Engineering in 1997. He is currently a Full Professor with the Department of Automation and Energy of Universidade Federal do Rio Grande do Sul, in Porto Alegre, Brazil. His main research interests are presently in system identification and data-driven control design, but he has also authored a number of papers in nonlinear systems theory, particularly its application to electric power systems and electrical machines. He is the author of two books: "Control Systems - Fundamentals and Design Methods" (in portuguese), and "Data-driven Controller Design: the H2 Approach" (Springer, 2011). Dr. Bazanella has served as associate editor of the IEEE Transactions on Control Systems Technology from 2002 to 2008 and as an Editor of the Journal of Control, Automation and Electrical Systems from 2008 to 2012. He has held visiting professor appointments at Universidade Federal da Paraíba in 2001 and at Université catholique de Louvain, where he spent a sabbatical year in 2006. Dr. Bazanella is a Senior Member of the IEEE, and is currently vice-president of the Brazilian Automation Society and vice-coordinator of the area of Electrical Engineering for CAPES, with the Brazilian Ministry of Education.

Guilherme A. Pimentel Guilherme Araujo Pimentel received his Control and Automation Engineering degree and Master degree in Electrical Engineering from the Pontifícia Universidade Católica do Rio Grande do Sul (Brazil) in 2007 and 2010, respectively. He obtained a joint PhD degree in Mathematics & Modeling and Science of Engineering in 2015. The former from the University of Montpellier (France) with the Project Group INRA-INRIA MODEMIC and UMR INRA-SupAgro MISTEA, and the latter from the University of Mons (Belgium) with the Automatic Control Laboratory. Between 2015 and 2017, he had a postdoctoral position in the Department of Automation and Energy of Universidade Federal do Rio Grande do Sul, in Porto Alegre, Brazil. Since 2015, he has a position at Faculty of Engineering from Pontifícia Universidade Católica do Rio Grande do Sul (PUCRS) where he teaches control theory and is a member of the Automation and System Control Group (GACS). His main research interests are in system identification, modeling and control of bio-systems.