Open-loop versus closed-loop identification of Box-Jenkins models: a new variance analysis

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Abstract— **We present some new formulae for the analysis of variance of estimated transfer functions, which are valid for Box-Jenkins (BJ) and Output Error (OE) model structures of finite order, identified in either open-loop or closed-loop, using Prediction Error (PE) Identification. The formulae are based on the asymptotic (in number of data) expression of the parameter covariance. They do not require special assumptions on the generation of the external signals. One of the results of our analysis is to show that, under reasonable assumptions on the signal powers, the variance of the estimated input-output model is smaller with closed-loop than with open-loop identification.**

I. INTRODUCTION

We present new expressions for the variance of estimated transfer functions when these transfer functions are estimated by Prediction Error (PE) Identification using a finite number N of noisy input-output data, for the case where a parametric Box-Jenkins model structure is used and where the true system is contained in the model set. We consider both the case of open-loop and of closed-loop identification. Our formulae are of course also valid for an Output Error model structure, a special case of BJ structure.

Thus, the model structure considered in this paper is

$$\mathcal{M} = \left\{ G(z,\theta) = G(z,\rho),\ H(z,\theta) = H(z,\eta),\ \theta = \begin{pmatrix} \rho \\ \eta \end{pmatrix} \right\} \tag{1}$$

where $G(z,\rho)$ and $H(z,\eta)$ are independently parametrized rational transfer functions[1]. The linear time-invariant single input single output true system $\mathcal{S}$ is assumed to be described in this model structure for a particular value $\theta_0 \in \mathbf{R}^k$ of the parameter vector:

$$y(t) = G(z,\rho_0)u(t) + H(z,\eta_0)e(t) \text{ for some } \theta_0 = \begin{pmatrix} \rho_0 \\ \eta_0 \end{pmatrix} \tag{2}$$

and $e(t)$ is a zero mean white noise of variance σ_e^2.

In open-loop identification, an input signal $u(t)$ with power spectrum $\Phi_u(\omega)$ is applied to (2). In direct closed-loop identification, the system (2) is operated in closed loop with a controller C_{id} and an external signal $r(t)$ is applied

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[1]The OE model structure is a special case with $\theta = \rho$ and $H(z,\eta) \triangleq 1$.

to the loop $[C_{id}\ G_0]$ such that the input signal $u(t)$ is given by:

$$u(t) = r(t) - C_{id}\ y(t) \tag{3}$$

$$= \underbrace{S_{id}r(t)}_{u_r(t)}\ \underbrace{-C_{id}S_{id}H(z,\eta_0)e(t)}_{u_e(t)} \tag{4}$$

where $S_{id} = 1/(1 + C_{id}G(z,\rho_0))$. The input signal is thus made of two uncorrelated components, $u_r(t)$ due to $r(t)$, and $u_e(t)$ due to the noise $e(t)$. Thus $u(t) = u_r(t) + u_e(t)$ and $\Phi_u(\omega) = \Phi_{u_r}(\omega) + \Phi_{u_e}(\omega)$.

The contribution of this paper is to present improved formulae for the variance of the estimated transfer functions $G(e^{j\omega}, \hat\rho_N)$ and $H(e^{j\omega}, \hat\eta_N)$ based on N input-output data. The computation of variance expressions for estimated transfer functions using PE identification methods dates back to [8], where expressions were derived for $var(G(e^{j\omega}, \hat\rho_N))$ and $var(H(e^{j\omega}, \hat\eta_N))$ obtained in open-loop, under an assumption that not only the number of data, but also the model order was tending to infinity. These formulae were later extended to closed-loop identification in [6], again assuming that the model order tends to infinity. For the input-output transfer functions, the following expressions were obtained in [8], [6]:

$$var(G_{OL}(e^{j\omega}, \hat\rho_N)) \simeq \frac{n}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)} \tag{5}$$

$$var(G_{CL}(e^{j\omega}, \hat\rho_N)) \simeq \frac{n}{N} \frac{\Phi_v(\omega)}{\Phi_{u_r}(\omega)} \tag{6}$$

In these expressions, $\Phi_v(\omega)$ is the power spectrum of the noise $v(t) = H(z,\eta_0)e(t)$, while $\Phi_u(\omega)$ and $\Phi_{u_r}(\omega)$ have been defined above. According to the formula (6), only that part $u_r(t)$ of the input signal contributes to the estimation of $\hat\rho_N$ in a closed-loop setup. However, numerous examples have shown that the estimate of the plant model is more accurate in closed-loop identification than in open-loop identification (see e.g. [4], [5]). In addition, recently more accurate variance formulae have been developed, for specific model structures and with some constraints on the input signal properties, that are not asymptotic in model order [12], [10], [11]. In the case of open-loop identification with a BJ model structure, Ninness and Hjalmarsson have derived the following variance expression in [10]:'

$$var(G_{OL}(e^{j\omega}, \hat\theta_N)) = \frac{\kappa_{ol}(\omega)}{N} \frac{\Phi_v(\omega)}{\Phi_u(\omega)} \tag{7}$$

where $\kappa_{ol}(\omega)$ is a complicated function of the poles of $G(z,\rho_0)$ and of the input spectrum Φ_u. In the case of direct

closed-loop identification, the same authors have derived a variance expression for $G_{CL}(e^{j\omega}, \hat{\theta}_N)$ that is presently restricted to an OE model structure and a white noise external excitation $r(t)$ [11]:

$$var(G_{CL}(e^{j\omega}, \hat{\theta}_N)) = \frac{\kappa_{cl}(\omega)}{N} \frac{\sigma_e^2}{\Phi_u(\omega)} \qquad (8)$$

with $\kappa_{cl}(\omega)$ a complicated function of the poles of $G(z, \rho_0)$ and of the sensitivity function S_{id}. One important conclusion from (8) is that the whole spectrum $\Phi_u(\omega) = \Phi_{u_r}(\omega) + \Phi_{u_e}(\omega)$ influences the variance of the closed-loop estimate, and not just $\Phi_{u_r}(\omega)$ as suggested by (6).

Even though the expressions (7)-(8) are valid for finite model order, they do not allow one to compare the accuracy of open-loop and closed-loop estimates in a BJ model structure for general classes of input signal spectra. Indeed, the expression (8) is available only for an OE model structure with a white noise reference excitation. The analysis of $var(G(e^{j\omega}, \hat{\rho}_N))$ and $var(H(e^{j\omega}, \hat{\eta}_N))$ developed in this paper is performed via expressions of the (inverse of the) covariance matrices P_ρ and P_η of the identified parameter vectors $\hat{\rho}_N$ and $\hat{\eta}_N$, under the standard assumption that the data length N tends to infinity. They translate easily into expressions for the variance of the estimated transfer functions, using Gauss' approximation formula: see Section II. The main conclusions derived from our analysis are:

1) $P_{\rho,CL} \prec P_{\rho,OL}$ (and thus $var(G_{CL}(e^{j\omega}, \hat{\rho}_N)) < var(G_{OL}(e^{j\omega}, \hat{\rho}_N))$ $\forall \omega$) when the true system $\mathcal{S}$ is excited with the same amount of external input energy in open-loop as in direct closed-loop identification (i.e. when the spectrum $\Phi_u(\omega)$ used for the open-loop identification is, at each ω, equal to the spectrum $\Phi_{u_r}(\omega)$ in closed-loop identification). Our expressions furthermore demonstrate that this improved accuracy in closed-loop identification is due to the internal noise excitation $\Phi_{u_e}(\omega)$.
2) $P_{\eta,CL} \succ P_{\eta,OL}$ (and thus $var(H_{CL}(e^{j\omega}, \hat{\eta}_N)) > var(H_{OL}(e^{j\omega}, \hat{\eta}_N))$ $\forall \omega$) under any experimental conditions.

The main contribution of this paper are the novel expressions of the variance error in PE Identification of BJ model structures, which can easily be used to compare open-loop and closed-loop identification. The main progress over the results of [8], [6] is that our formulae, just like those of [10], [11], are asymptotic only in the number of data N, but not in the model order n. They lead to Statement 1 above, which is another contribution of this paper. It contradicts the conclusion that can be drawn from the classical expression (6) presented above, which was derived under an asymptotic assumption on the model order n.

The outline of the paper is as follows. In Section II we recall some general expressions for the asymptotic covariance of identified parameters and transfer functions, which exhibit their relation to the information matrix. In Section III we derive the expressions of the submatrices of the information matrix for a BJ model structure. These expressions are then used for the computation of the variance of the input output

model in Section IV, and the noise model in Section V, in each case comparing open-loop and closed-loop identification. A simulation example is presented in Section VI which also highlights the role of the controller on the precision of the estimated transfer functions.

II. PE IDENTIFICATION ASPECTS

Consider the identification of the true system (2) using a Box-Jenkins model from the set $\mathcal{M}$ defined in (1). Once the true system has been excited, a set of input-output data of length N i.e. $\{u(t); y(t) \mid t = 1...N\}$ is collected and the identified parameter vector $\hat{\theta}_N$ is computed via the classical prediction error criterion [9]:

$$\hat{\theta}_N = \begin{pmatrix} \hat{\rho}_N \\ \hat{\eta}_N \end{pmatrix} \triangleq \arg \min_\theta \frac{1}{N} \sum_{t=1}^{N} \epsilon^2(t, \theta) \qquad (9)$$

with $\epsilon(t, \theta) \triangleq H(z, \theta)^{-1} (y(t) - G(z, \theta)u(t))$. In this paper, we will assume that the excitation spectrum $\Phi_u(\omega)$ (resp. $\Phi_r(\omega)$) is nonzero at each frequency in order to avoid the problems of multiple minima of the PE criterion (see [3]).

An important property of the identified parameter vector $\hat{\theta}_N$ is that it is asymptotically normally distributed around the unknown true parameter vector θ_0: $\hat{\theta}_N \sim \mathcal{N}(\theta_0, P_\theta)$. Thus, for large enough N, the covariance matrix P_θ is approximately given by

$$P_\theta = \frac{\sigma_e^2}{N} \left(\bar{E} \left(\psi(t, \theta_0) \psi(t, \theta_0)^T \right) \right)^{-1} \qquad (10)$$

with $\psi(t, \theta) = -\frac{\partial \epsilon(t, \theta)}{\partial \theta}$. The matrix

$$M_\theta \triangleq \frac{N}{\sigma_e^2} \bar{E} \left(\psi(t, \theta_0) \psi(t, \theta_0)^T \right) \qquad (11)$$

is called the *information matrix*. If we now partition the covariance matrix P_θ according to the partition of the parameter vector θ in (1) i.e. $P_\theta = \begin{pmatrix} P_\rho & P_{\rho\eta} \\ P_{\rho\eta}^T & P_\eta \end{pmatrix}$, we obtain, for large N, a distinct distribution for each of the identified parameter vectors $\hat{\rho}_N$ and $\hat{\eta}_N$:

$$\hat{\rho}_N \sim \mathcal{N}(\rho_0, P_\rho) \qquad \hat{\eta}_N \sim \mathcal{N}(\eta_0, P_\eta). \qquad (12)$$

Based on these expressions, we can deduce uncertainty regions for the unknown $G(z, \rho_0)$ and the unknown $H(z, \eta_0)$ centered at the identified model $G(z, \hat{\rho}_N)$ and $H(z, \hat{\eta}_N)$ (see e.g. [1]). The sizes of these uncertainty regions are determined by the covariance matrices P_ρ and P_η. Using Gauss' approximation formula [9], the variances of the identified plant model $G(z, \hat{\rho}_N)$ and the identified noise model $H(z, \hat{\eta}_N)$ are given, approximately for large N, by:

$$var(G(e^{j\omega}, \hat{\rho}_N)) = \Lambda_\rho^*(e^{j\omega}, \rho_0) \, P_\rho \, \Lambda_\rho(e^{j\omega}, \rho_0)$$
$$var(H(e^{j\omega}, \hat{\eta}_N)) = \Lambda_\eta^*(e^{j\omega}, \eta_0) \, P_\eta \, \Lambda_\eta(e^{j\omega}, \eta_0) (13)$$

where $\Lambda_\rho(z, \rho) = \frac{\partial G(z, \rho)}{\partial \rho}$ and $\Lambda_\eta(z, \eta) = \frac{\partial H(z, \eta)}{\partial \eta}$. The covariance matrices P_ρ and P_η are thus perfect tools to compare the accuracy of an identification experiment under different experimental conditions. In particular, in order to compare the variances obtained by open-loop and direct

closed-loop identification, respectively, and to understand the role of the experimental conditions in these variances, we shall derive appropriate expressions for P_ρ and P_η for both identification conditions.

III. INTEGRAL EXPRESSIONS FOR THE INVERSE OF P_θ

In this section we derive various expressions for the information matrix, which reveal the precise way in which it depends on the experimental conditions. Since the covariance matrices P_ρ and P_η are obtained as submatrices of the inverse of the information matrix, these expressions will then enable us to reveal the dependence of these covariances on the excitation signals (u or r) and on the noise ($H(z, \eta_0)e$).

We first rewrite $\psi(t, \theta_0)$ in (10) as a function of the input signal $u(t)$ and of the noise $e(t)$. It follows from the BJ model structure (1) that

$$\psi(t, \theta_0) = \begin{pmatrix} \frac{\Lambda_\rho(z, \rho_0)}{H(z, \eta_0)} \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \frac{\Lambda_\eta(z, \eta_0)}{H(z, \eta_0)} \end{pmatrix} e(t) \quad (14)$$

In the case of **open-loop identification**, the input signal $u(t)$ is uncorrelated with $e(t)$. Therefore, M_θ is given by (see (11) and (14)):

$$M_{\theta, OL} = \begin{pmatrix} P_{\rho, OL} & P_{\rho\eta, OL} \\ P_{\rho\eta, OL}^T & P_{\eta, OL} \end{pmatrix}^{-1} \quad (15)$$

$$= \frac{N}{\sigma_e^2} \left[\begin{pmatrix} R_u & 0 \\ 0 & 0 \end{pmatrix} + \sigma_e^2 \begin{pmatrix} 0 & 0 \\ 0 & R_{v,22} \end{pmatrix} \right]$$

where the following notation is introduced:

$$R_u = \mathcal{I}\left(\frac{\Lambda_\rho(z, \rho_0)}{H(z, \eta_0)}, \Phi_u(\omega) \right) \quad (16)$$

$$R_{v,22} = \mathcal{I}\left(\frac{\Lambda_\eta(z, \eta_0)}{H(z, \eta_0)}, 1 \right) \quad \text{where}$$

$$\mathcal{I}(V(z), \Phi(\omega)) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} V(e^{j\omega}) V(e^{j\omega})^* \Phi(\omega) d\omega$$

It follows directly from the block diagonal structure of expression (15) that, in open-loop identification with BJ model structures, the covariance of ρ, $P_{\rho, OL}$, depends only on the input signal u, while the covariance of η, $P_{\eta, OL}$, depends only on the noise. Thus, $P_{\rho\eta, OL} = 0$ in open-loop identification. In the case of direct **closed-loop identification**, $u(t)$ is correlated with $e(t)$. Consequently, in order to deduce an expression for $P_{\theta, CL}^{-1}$ similar to (15), we first use (4) to transform (14):

$$\psi(t, \theta_0) = \begin{pmatrix} \frac{S_{id}}{H(z, \eta_0)} \Lambda_\rho(z, \rho_0) \\ 0 \end{pmatrix} r(t) + \begin{pmatrix} -C_{id} S_{id} \Lambda_\rho(z, \rho_0) \\ \frac{\Lambda_\eta(z, \eta_0)}{H(z, \eta_0)} \end{pmatrix} e(t) \quad (17)$$

Since $r(t)$ is uncorrelated with $e(t)$, this delivers:

$$M_{\theta, CL} = \begin{pmatrix} P_{\rho, CL} & P_{\rho\eta, CL} \\ P_{\rho\eta, CL}^T & P_{\eta, CL} \end{pmatrix}^{-1} \quad (18)$$

$$= \frac{N}{\sigma_e^2} \left[\begin{pmatrix} R_r & 0 \\ 0 & 0 \end{pmatrix} + \sigma_e^2 \begin{pmatrix} R_{v,11} & R_{v,12} \\ R_{v,12}^T & R_{v,22} \end{pmatrix} \right]$$

with $R_{v,22}$ as in (16) and with the following additional notations:

$$R_r = \mathcal{I}\left(S_{id} \frac{\Lambda_\rho(z, \rho_0)}{H(z, \eta_0)}, \Phi_r(\omega) \right) \quad (19)$$

$$R_{v,11} = \mathcal{I}(C_{id} S_{id} \Lambda_\rho(z, \rho_0), 1)$$

$$R_{v,12} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{-C_{id} S_{id} \Lambda_\rho(e^{j\omega}, \rho_0) \Lambda_\eta^*(e^{j\omega}, \eta_0)}{H^*(z, \eta_0)} d\omega$$

It follows that in closed-loop identification with BJ models, unlike in open-loop identification, both the excitation signal r and the noise e influence the quality of the estimate of ρ.

We observe that, for both open-loop and closed-loop identification, the information matrix can be written as the sum of two contributions: the first contribution is directly related to the external excitation signal ($u(t)$ or $r(t)$) and can therefore be manipulated by the designer, while the second contribution is due to the noise and is thus not under the control of the designer, except of course by acting on the data length. As stated above, in open-loop identification these two contributions are decoupled, i.e. the information matrix is block-diagonal (and of course so is its inverse, the covariance matrix). In closed-loop identification, it can be seen from the expression (17) that the noise contribution to the information matrix in (18) is strictly positive-definite for an estimated full-order BJ model structure:

$$\begin{pmatrix} R_{v,11} & R_{v,12} \\ R_{v,12}^T & R_{v,22} \end{pmatrix} \succ 0 \quad (20)$$

This property has been used in [2] to perform so-called "costless identification experiments", i.e. experiments that do not require any external excitation, which therefore do not perturb the normal operating conditions of the closed-loop system.

IV. VARIANCE OF THE IDENTIFIED PLANT MODEL $G(z, \hat{\rho}_N)$

In the previous section, we have derived expressions for the information matrices in the case of open-loop and closed-loop identification with BJ models. A major benefit of these expressions is that they are affine in the experimental design variables $\Phi_u(\omega)$ and $\Phi_r(\omega)$. In this section, we compute the covariance matrices P_ρ and P_η of the parameters of the input-output and noise model of a BJ model: these are the block-diagonal submatrices of the inverses of the matrices M_θ computed in Section III. We then use these expressions to compute the variance of $G(z, \hat{\rho}_N)$ under specific experimental conditions.

From the expressions (15) and (18) and the matrix inversion Lemma (see e.g. [7]) we obtain immediately the following result.

Theorem 4.1: For a Box Jenkins model structure, the inverse covariance matrices of the parameter estimate $\hat{\rho}_N$ of the input-output model, in open-loop and in closed-loop identification, are given, respectively, by

$$P_{\rho, OL}^{-1} = \frac{N}{\sigma_e^2} R_u = N \mathcal{I}\left(\Lambda_\rho(z, \rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)} \right) \quad (21)$$

$$P_{\rho,CL}^{-1} = N \left[\frac{R_r}{\sigma_e^2} + R_{v,11} - R_{v,12}R_{v,22}^{-1}R_{v,12}^T \right] \quad (22)$$

$$= N \left[\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_{u_r}(\omega)}{\Phi_v(\omega)} \right) + R_{v,11} - R_{v,12}R_{v,22}^{-1}R_{v,12}^T \right]$$

$$= N \left[\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)} \right) - R_{v,12}R_{v,22}^{-1}R_{v,12}^T \right]$$

where

$$\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)} \right) = \mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_{u_r}(\omega)}{\Phi_v(\omega)} \right)$$
$$+ \mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_{u_e}(\omega)}{\Phi_v(\omega)} \right) \quad (23)$$

Proof. The result (21) follows directly from (15) by noting that $\Phi_v(\omega) = |H(e^{j\omega}, \eta_0)|^2 \sigma_e^2$. In order to derive (22), we apply the matrix inversion formula to (18), which delivers:

$$P_{\rho,CL}^{-1} = \frac{N}{\sigma_e^2} R_r + N \left(R_{v,11} - R_{v,12}R_{v,22}^{-1}R_{v,12}^T \right)$$

The other expressions of (22) then follow from the definition (19) of R_r and $R_{v,11}$, and from the fact that $\Phi_{u_r}(\omega) = |S_{id}(e^{j\omega})|^2 \Phi_r(\omega)$ and $\Phi_{ue}(\omega) = |C_{id}S_{id}|^2 \Phi_v(\omega)$. Expression (23) follows simply from $\Phi_u(\omega) = \Phi_{u_r}(\omega) + \Phi_{u_e}(\omega)$. ∎

The Output Error model structure is a special case of the BJ structure in which $H(z,\eta) = 1$. The following result follows immediately from the previous Theorem.

Corollary 4.1: For an OE model structure, the inverse covariance matrices of the parameter estimate $\hat{\rho}_N$, in open-loop and in closed-loop identification, are given by

$$P_{\rho,OL}^{-1} = \frac{N}{\sigma_e^2} R_u = N \, \mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_u(\omega)}{\sigma_e^2} \right) \quad (24)$$

$$P_{\rho,CL}^{-1} = N \left[\frac{R_r}{\sigma_e^2} + R_{v,11} \right] \quad (25)$$

$$= N \left[\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_u(\omega)}{\sigma_e^2} \right) \right]$$

where the quantity $\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_u(\omega)}{\sigma_e^2} \right)$ can again be decomposed as in (23) with the substitution $\Phi_v(\omega) = \sigma_e^2$.

With the results of Theorem 4.1 and Corollary 4.1, we can now make a number of observations concerning the role of the excitation signals on the covariance of the estimated input-output parameter vector ρ, and on the variance of the corresponding transfer function estimates obtained via (13). We can compare our variance expressions with the approximate high order expressions; and we can compare the covariance formulae obtained under open-loop and closed-loop identification.

Remarks

1) In open-loop identification, the covariance of the estimate of ρ is a function of the signal-to-noise ratio $\frac{\Phi_u(\omega)}{\Phi_v(\omega)}$, and of the structure of the input-output model through $\Lambda_\rho(z,\rho_0)$. This is consistent with the

formula (7); however, note that the integral expression of $P_{\rho,OL}^{-1}$ shows that the variance $var(G_{OL}(e^{j\omega}, \hat{\rho}_N))$ at frequency ω depends on the signal-to-noise ratio not just at frequency ω, but at all frequencies. In the expression (7), this dependence is included through the factor $\kappa_{ol}(\omega)$. The same holds for $\kappa_{cl}(\omega)$ in (8).

2) Expressions (22) and (25) show that in closed-loop identification the covariance of $\hat{\rho}_N$ depends not only on the part of the input signal power that comes from the external excitation, but also on the part that comes from the noise, i.e. not just on $\frac{\Phi_{u_r}(\omega)}{\Phi_v(\omega)}$, but also on $\frac{\Phi_{u_e}(\omega)}{\Phi_v(\omega)}$. This is in contradiction with the formula (6) which was derived under the assumptions of a model order tending towards infinity. Thus, the contribution of the noise increases the precision of the parameter estimate, beyond that which is obtained from the reference contribution. This is the object of the next Corollary.

Corollary 4.2: For both a BJ and an OE model structure, the inverse covariance matrix of the parameter estimate $\hat{\rho}_N$ obtained in closed-loop identification obeys the following inequality:

$$P_{\rho,CL}^{-1} \succ N \left[\mathcal{I}\left(\Lambda_\rho(z,\rho_0), \frac{\Phi_{u_r}(\omega)}{\Phi_v(\omega)} \right) \right] = N \frac{R_r}{\sigma_e^2} \quad (26)$$

Proof. The proof follows immediately from (22) and the fact that $R_{v,11} - R_{v,12}R_{v,22}^{-1}R_{v,12}^T \succ 0$, which itself follows from (20). The result in the OE case follows from (25) and the fact that $R_{v,11} \succ 0$. ∎

With the formulae above we can compare the precision of the transfer function estimate of the input-output model, i.e. $var(G(e^{j\omega}, \hat{\rho}_N))$, under a variety of situations and experimental conditions. Here we shall consider just one situation that is representative of a disturbance rejection problem often encountered in practice. We consider a system that operates in closed-loop with a controller C_{id} whose objective is to reduce the effect of the disturbance $v(t)$ on the output. Thus, the normal operating condition is that the external reference is zero, $r(t) = 0$, and hence $\Phi_{u_r}(\omega) = 0 \; \forall \omega$. It is desired to estimate a model of the unknown G in order to replace the present controller C_{id} by one that has better performance. If the identification step is performed in closed-loop with an external excitation signal $r \neq 0$, then this creates an additional component u_r to the normal operating signal u_e; this signal u_r is then a perturbation with respect to the normal operating conditions, that is applied for the purposes of identification. In order to compare this closed-loop identification setup with the alternative of open-loop identification, it then makes sense to compare it with an open loop identification experiment in which $\Phi_{u,OL}(\omega) = \Phi_{u_r,CL}(\omega) \; \forall \omega$. The next Theorem compares the corresponding variances of the estimated $\hat{G}$ for the same number of data, N.

Theorem 4.2: Consider an open-loop and a direct closed-loop identification experiment with the same data length using either a full order OE or a full order BJ model structure, with the true system and the controller described

by (2) to (4). Let the experimental conditions be chosen such that $\Phi_{u,OL}(\omega) = \Phi_{u_r,CL}(\omega) \ \forall \omega$. Then for both the OE and the BJ model structure, we have $P_{\rho,OL} \succ P_{\rho,CL}$ and $var(G_{OL}(e^{j\omega}, \hat{\rho}_N)) > var(G_{CL}(e^{j\omega}, \hat{\rho}_N)) \ \forall \omega$.

Proof. The proof follows immediately from Corollary 4.2 by noting that, with $\Phi_{u,OL}(\omega) = \Phi_{u_r,CL}(\omega) \ \forall \omega$ the quantity on the right hand side of (26) is precisely $P_{\rho,OL}^{-1}$: see (21). ∎

Theorem 4.2 states that, under our assumption that the same amount of external power is put into the input signal, the uncertainty around $G(z, \hat{\rho}_N)$ is always smaller when $G(z, \hat{\rho}_N)$ has been determined via direct closed-loop identification than via open-loop identification. Note that this holds independently of the choice of stabilizing controller C_{id} in direct closed-loop identification. However, the controller C_{id} does play a role, as will be illustrated in Section VI. We observe that, under these experimental conditions ($\Phi_{u,OL} = \Phi_{u_r,CL}$), the output power will be smaller under the closed-loop experimental conditions than in open-loop. Other open-loop versus closed-loop comparisons can of course be made using the covariance formulae of Theorem 4.1 and Corollary 4.1, based on other constraints on the input or output power spectra. The comparison we have made here is a very reasonable one from the practical point of view of the control engineer faced with the task of designing a new controller on the basis of a new model.

V. VARIANCE OF THE IDENTIFIED NOISE MODEL $H(z, \hat{\eta}_N)$

In the previous section, the variance of $G(z, \hat{\rho}_N)$ has been analyzed for BJ and OE models. In this section, we analyze the variance of the identified noise model $H(z, \hat{\eta}_N)$ for BJ models; they can be assessed by the covariance matrix P_η of $\hat{\eta}_N$. From (15) and (18), we directly deduce:

$$P_{\eta,OL}^{-1} = N R_{v,22} = N \ \mathcal{I} \left(\frac{\Lambda_\eta(z, \eta_0)}{H(z, \eta_0)}, 1 \right) \quad (27)$$

$$P_{\eta,CL}^{-1} = N \left[R_{v,22} - R_{v,12}^T \left(\frac{R_r}{\sigma_e^2} + R_{v,11} \right)^{-1} R_{v,12} \right] \quad (28)$$

$$= N \left[\mathcal{I} \left(\frac{\Lambda_\eta(z, \eta_0)}{H(z, \eta_0)}, 1 \right) - R_{v,12}^T \left(\frac{R_r}{\sigma_e^2} + R_{v,11} \right)^{-1} R_{v,12} \right]$$

From those expressions, we observe that $P_{\eta,OL}$ is independent of the input spectrum $\Phi_u(\omega)$ and of the noise level σ_e^2, while $P_{\eta,CL}$ depnds on $\Phi_{u_r}(\omega)$, $\Phi_{u_e}(\omega)$ and $\Phi_v(\omega)$ via the term $\frac{R_r}{\sigma_e^2} + R_{v,11} = \mathcal{I} \left(\Lambda_\rho(z, \rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)} \right)$. Furthermore, we have the following result.

Theorem 5.1: Consider an open-loop and a direct closed-loop identification experiment with the same data length N, using the full-order BJ model structure (1), with the true system and the controller described by (2) to (4). Then, independently of the spectra $\Phi_u(\omega)$ and $\Phi_r(\omega)$ used for the external excitation signals and provided Φ_r remains bounded, we have:

$$P_{\eta,OL} \prec P_{\eta,CL}$$

and hence

$$var(H_{OL}(e^{j\omega}, \hat{\eta}_N)) < var(H_{CL}(e^{j\omega}, \hat{\eta}_N)) \ \forall \omega.$$

Proof. The first inequality is a direct consequence of (27)-(28). The second one then follows from (13). ∎

Thus, unlike the uncertainty of the estimated plant model $G(z, \hat{\rho}_N)$, the uncertainty of the noise model $H(z, \hat{\eta}_N)$ is always smaller when it has been identified via an open-loop identification. Nevertheless, it follows from (27)-(28) that the higher $\mathcal{I} \left(\Lambda_\rho(z, \rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)} \right)$, the closer the variance of $H(z, \hat{\eta}_N)$ in direct closed-loop identification will be to the variance in open-loop identification. Note finally that, as far as the variance of $H(z, \hat{\eta}_N)$ is concerned, our analysis for finite order model structures confirms the conclusions derived in [6] under the "classical" assumption that the model order tends to infinity.

VI. SIMULATION RESULTS

In order to illustrate the results of this paper, we consider the following first-order true system:

$$\mathcal{S}: \quad y(t) = \frac{b_o z^{-1}}{1 + f_o z^{-1}} u(t) + \frac{1 + c_o z^{-1}}{1 + d_o z^{-1}} e(t) \quad (29)$$

with $b_0 = 0.36$, $f_0 = -0.7$, $c_0 = 0.6$, $d_0 = 0.1$ and $\sigma_e^2 = 1$. We compare the variance results of open-loop and direct closed-loop identification on that true system. The closed-loop experiment is performed with a white noise excitation signal $r(t)$ of variance 1 on the loop $[C_{id} \ G_0]$ with $C_{id} = k/(1 - z^{-1})$ (k is a scalar gain). In the open-loop identification experiment the input signal $u(t)$ is chosen as $u(t) = S_{id} r(t)$, so that the assumption $\Phi_{u,OL}(\omega) = \Phi_{u_r,CL}(\omega) \ \forall \omega$ of Theorem 4.2 is satisfied.

To illustrate the influence of the choice of C_{id} on our results, we consider two different choices for the gain k of the controller C_{id}: $k = 0.1$ and $k = 3$. Then, for both values of k and for both identification experiments, we use the expressions (21), (22), (27) and (28) to compute expressions for P_ρ and P_η. The results obtained for open-loop and for closed-loop identification are compared in Figures 1 and 2. In Figure 1, we compare the 95%-confidence region for $\hat{\rho}_N - \rho_0$ and $\hat{\eta}_N - \eta_0$, i.e. $U_\rho = \{\Delta\rho \mid \Delta\rho^T P_\rho^{-1} \Delta\rho \leq 5.99\}$ and $U_\eta = \{\Delta\eta \mid \Delta\eta^T P_\eta^{-1} \Delta\eta \leq 5.99\}$. In Figure 2, we compare the variances of $G(z, \hat{\rho}_N)$ and $H(z, \hat{\eta}_N)$, computed via (13).

The figures confirm the results of this paper. Indeed, they show that the variance of $G(z, \hat{\rho}_N)$ is larger in open-loop than in closed-loop while the variance of $H(z, \hat{\eta}_N)$ is larger in closed-loop than in open-loop identification. Moreover, we observe that the differences between open and closed-loop identification are more important when the gain of C_{id} is equal to $k = 3$ than when $k = 0.1$, i.e. when the controller has less roll-off in high frequencies. The larger difference in variance for $G(z, \hat{\rho}_N)$ can be explained, via (22), by the fact that when $k = 3$ then $\Phi_{u_e}(\omega)$ is, at each ω, larger than $\Phi_{u_e}(\omega)$ when $k = 0.1$: see Figure 3. The larger difference

in variance for $H(z, \hat{\rho}_N)$ can be explained, via (28), by the fact that the quantity $\frac{R_r}{\sigma_e^2} + R_{v,11} = \mathcal{I}\left(\Lambda_\rho(z, \rho_0), \frac{\Phi_u(\omega)}{\Phi_v(\omega)}\right)$ is smaller when $k = 3$ than when $k = 0.1$. Note also that the results derived here with the formulae (21), (22), (27) and (28), which are based on asymptotic distributions (i.e. $N \to \infty$), are confirmed by actual identification experiments on the true system with $N = 1000$ data points.

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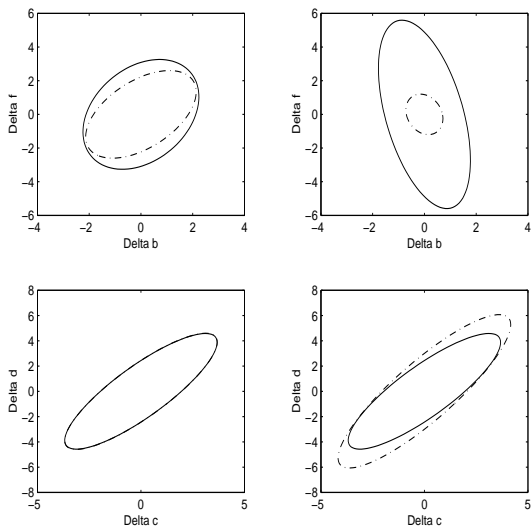Fig. 1. U_ρ (top) and U_η (bottom) for open-loop (solid) and for closed-loop (dashdot). On the left side for $k = 0.1$ and on the right side for $k = 3$. Figure scaled for $N = 1$
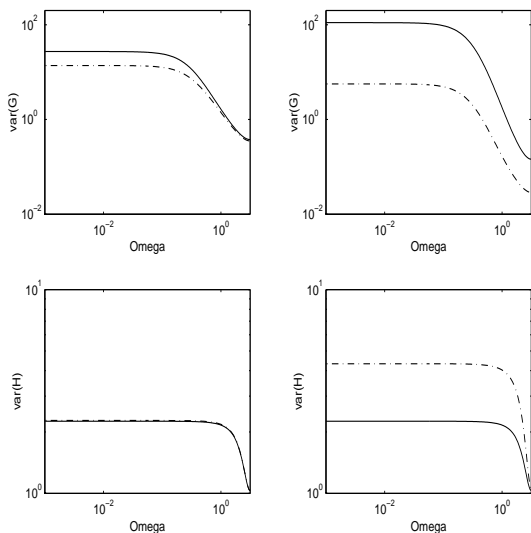

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Fig. 2. $var(G(e^{j\omega}, \hat{\rho}_N))$ (top) and $var(H(e^{j\omega}, \hat{\eta}_N))$ (bottom) for open-loop (solid) and for closed-loop (dashdot). On the left side for $k = 0.1$ and on the right side for $k = 3$. Figure scaled for $N = 1$

VII. Conclusions

We have derived new formulae for the asymptotic (in data length) variance of the parameters of a BJ model, identified by Prediction Error identification, both in open and in closed
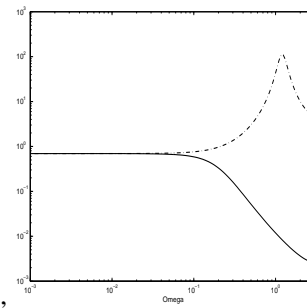

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Fig. 3. $\Phi_{u_e}(\omega)$ when $k = 0.1$ (solid) and when $k = 3$ (dashdot)

loop. These expressions are based on the standard asymptotic Gaussian distribution of parameter estimates obtained by Prediction Error identification methods. These can be used, as is common practice, to derive approximate parameter covariance estimates that are valid for a large number N of data. We have exploited the particular structure of the information matrix of BJ model structures (computed via integral expressions) to derive expressions for the submatrices of the covariance matrix, that clearly exhibit the role of the experimental conditions. This has enabled us to make useful comparisons between the precision of BJ models obtained by open-loop and closed-loop identification.

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