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CONTINUOUS-TIME STOCHASTIC ADAPTIVE CONTROL*

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Abstract. This paper establishes global boundedness for a continuous-time stochastic adaptive control algorithm. It is shown that, with probability one, the system inputs and outputs satisfy a sample mean square boundedness property. The algorithm and method of analysis are not directly analogous to the discrete-time case, since special features are necessary to handle the continuous-time problem.

Key words. global boundedness, stochastic adaptive control

1. Introduction. It is now well known that global boundedness can be established for discrete-time stochastic adaptive control algorithms—see, for example, [1]-[3].

However, to date there has been no corresponding result for continuous-time systems. Preliminary work in this direction appears in [4]-[8]. However, the available adaptive control results rely upon unproven and data dependent conjectures, e.g., the normalized regression vector, $(\varphi^T\varphi/r)$, is assumed to be uniformly bounded. Inspection of the details of the algorithms indicates that this is most likely not true. A further restriction in the work reported in [5] and [6] is that the relative degree of the system is taken to be zero, i.e., there is a direct feedthrough term between input and output. This leads to questions about the validity of the resulting control law as the closed-loop system contains an algebraic loop.

This paper presents a new algorithm for continuous-time stochastic adaptive control in the ideal case (no unmodelled dynamics) together with a proof of global boundedness. The main feature of the algorithm is a new normalization technique that guarantees that the normalized regression vector $(\varphi^T\varphi/r)$ is bounded. Also, the proof of boundedness of the system states has several novel features including a special technique for handling strictly proper systems.

The paper was inspired by an earlier paper [9] which explored the link between discrete- and continuous-time *deterministic* adaptive control theory. The current paper does this for the stochastic case.

Novel aspects of our analysis are the following: We first establish existence and uniqueness for the solutions of the nonlinear stochastic differential equation for the parameter estimation algorithm. We then establish properties of the parameter estimator that hold irrespective of the control law. Finally, we establish a continuous-time key technical lemma that is analogous to that given for continuous-time deterministic systems in [9].

These preliminary results can be combined with a wide class of certainty equivalence control laws to establish global boundedness of all internal variables of the resultant algorithm. We illustrate by using an adaptive pole assignment algorithm as in [10] and [11] for deterministic continuous time systems.

The results are believed to be of importance, in their own right, insofar as they formally establish the boundedness of all system states and parameters for a continuous-time stochastic adaptive control algorithm. However, they also give further insight into

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the discrete-time theory, since they show what happens in the case of rapid sampling as shown in [18]. We have also recently [22] built on the results established here to establish global boundedness for a stochastic model reference adaptive control algorithm which is the continuous-time counterpart of the discrete-time stochastic adaptive minimum variance control results given in [1].

2. The model. In previous papers dealing with continuous-time stochastic adaptive control, an integral operator model has been used (see for example, [4]-[8]). Here, however, we adopt the more conventional continuous-time state space innovations representation, i.e.,

$$dx_t = Ax_t\,dt + Bu_t\,dt + K\,d\omega_t \tag{2.1}$$

$$dy_t = Cx_t\,dt + d\omega_t \tag{2.2}$$

where ω_t is a Wiener process with incremental covariance $\sigma^2\,dt$, x_t is a state vector of dimension n, u_t is a scalar control input, and dy_t is the scalar output of the system. The matrices A, B, K, C contain unknown but fixed parameters. We will denote by $\mathcal{F}_t$ the increasing σ-fields generated by $\{\omega_s, 0 \leqq s \leqq t\}$ and the unknown initial conditions x_0, and we will assume that $\|x_0\|^2$ is bounded. This model is the basis of stochastic control in the nonadaptive literature [12], [15]. It has also been proposed for the adaptive case [7]. Notice that y_t is the integral of the output.

Without loss of generality, we assume that the above model is in observer form where

$$A = \begin{bmatrix} -a_{n-1} & 1 & & \\ -a_{n-2} & & \ddots & \\ \vdots & & & 1 \\ -a_0 & & & 0 \end{bmatrix}; \quad B = \begin{bmatrix} b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}; \quad K = \begin{bmatrix} k_{n-1} \\ \vdots \\ k_0 \end{bmatrix} \tag{2.3}$$

$$C = [1 \quad 0 \quad 0]. \tag{2.4}$$

For the purpose of adaptive control it is convenient to reexpress this model in fractional form [9]. We therefore reparametrize the system as follows. Let

$$E(\rho) = \rho^n + e_{n-1}\rho^{n-1} + \cdots + e_0 \tag{2.5}$$

$$G^T = [g_{n-1}, \cdots, g_0] \quad \text{with } g_i = e_i - a_i; \qquad i = 0, \cdots, n-1 \tag{2.6}$$

and

$$E = \begin{bmatrix} -e_{n-1} & 1 & \\ \vdots & & \ddots \\ \vdots & & 1 \\ -e_0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

where the coefficients are arbitrary subject to $E(\rho)$ having all its zeros in the open left half plane and $K \neq G$.

Then, (2.1), (2.2) can be rewritten as

$$dx_t = Ex_t\,dt + Bu_t\,dt + (K - G)\,d\omega_t + G\,dy_t \tag{2.8}$$

$$dy_t = Cx_t\,dt + d\omega_t. \tag{2.9}$$

Using superposition, (2.8), (2.9) can also be expressed as

$$d\phi_t^1 = E\phi_t^1\,dt + G\,dt_t \tag{2.10}$$

(2.11) $$d\phi_t^2 = E\phi_t^2\,dt + Bu_t\,dt$$

(2.12) $$d\phi_t^3 = E\phi_t^3\,dt + (K-G)\,d\omega_t$$

(2.13) $$dy_t = C[\phi_t^1\,dt + \phi_t^2\,dt + \phi_t^3\,dt] + d\omega_t.$$

Since y_t is a scalar, we have that $C(sI-E)^{-1}B = B^T(sI-E^T)^{-1}C^T$, etc. Hence (2.10) to (2.13) can be rewritten as

(2.14) $$d\phi_t^y = E^T\phi_t^y\,dt + C^T\,dy_t$$

(2.15) $$d\phi_t^u = E^T\phi_t^u\,dt + C^Tu_t\,dt$$

(2.16) $$d\phi_t^\omega = E^T\phi_t^\omega\,dt + C^T\,d\omega_t$$

(2.17) $$dy_t = [G^T\phi_t^y + B^T\phi_t^u + (K-G)^T\phi_t^\omega]\,dt + d\omega_t.$$

In (2.14)–(2.16) we choose $\phi_0^y = 0$, $\phi_0^u = 0$, and ϕ_0^ω such that

(2.18) $$(K-G)^T\phi_0^\omega = Cx_0.$$

Equation (2.17) is in the form of a linear regression, i.e.,

(2.19) $$dy_t = \phi_t^T\theta\,dt + d\omega_t$$

where

(2.20) $$\phi_t^T = [(\phi_t^y)^T, (\phi_t^u)^T, (\phi_t^\omega)^T]$$

(2.21) $$\theta^T = [G^T, B^T, (K-G)^T] = [G^T, B^T, F^T].$$

Note that ϕ_t does not depend on the unknown system parameters since E is known. Thus, (2.14)–(2.16) simply represent a state space form of the usual regression vector as in [2] and [9]. To make the comparison with the regression vector formulations more complete, we might note that with $\rho \triangleq (d/dt)$ we have, with some abuse of notation and ignoring initial conditions,

(2.22) $$(\phi_t^u)^T = \left[\frac{\rho^{n-1}}{E(\rho)}u, \cdots, \frac{1}{E(\rho)}u\right]$$

and similarly for ϕ_t^y, ϕ_t^ω. In the following, however, we will use the rigorous state space formulation as outlined earlier.

3. Pseudolinear regression estimation algorithm. The model (2.14)–(2.17) is not quite in a form that is suitable for parameter estimation. This is because the component ϕ_t^ω depends on the unmeasured noise source ω_t. Thus, as in the discrete case [1]–[3], [13], we define the predicted output by a pseudoregression in which $d\omega_t$ is replaced by the prediction error. Thus we define

(3.1) $$d\hat{y}_t = \psi_t^T\hat{\theta}_t\,dt$$

where $\hat{\theta}_t$ is some bounded $\mathcal{F}_t$-measurable function. Later in this section we will define $\hat{\theta}_t$ as an estimate of θ using a stochastic differential equation driven by the data y_t, u_t. Also, in (3.1) we have

(3.2) $$\psi_t^T = [(\psi_t^y)^T, (\psi_t^u)^T, (\psi_t^e)^T]$$

(3.3) $$d\psi_t^y = E^T\psi_t^y\,dt + C^T\,dy_t$$

(3.4) $$d\psi_t^u = E^T\psi_t^u\,dt + C^Tu_t\,dt$$

(3.5) $$d\psi_t^e = E^T\psi_t^e\,dt + C^T\,de_t$$

(3.6) $$de_t = dy_t - d\hat{y}_t = dy_t - \psi_t^T\hat{\theta}_t\,dt.$$

It is assumed that $\psi_0 = 0$ and that $\hat{\theta}_0$ is $\mathcal{F}_0$-measurable. With this choice of initial conditions, $\psi_t^y = \phi_t^y$ and $\psi_t^u = \phi_t^u$ for $t \geqq 0$.

The effect of using pseudoregressions can be further clarified as follows. We define

(3.7) $$\eta_t = \phi_t^T\theta - \psi_t^T\hat{\theta}_t.$$

We then obtain

(3.8) $$de_t - d\omega_t = \eta_t\,dt.$$

Note that η_t is the "deterministic part" of the prediction error. Now let

(3.9) $$\gamma_t \triangleq \psi_t^e - \phi_t^\omega.$$

Then from (3.5), (3.7), (2.16) we have

(3.10) $$d\gamma_t = E^T\gamma_t\,dt + C^T\eta_t\,dt, \quad \text{with } \gamma_0 = -\phi_0^\omega.$$

Noting that

(3.11) $$\psi_0^y = \phi_0^y = 0 \quad \text{and} \quad \psi_0^u = \phi_0^u = 0,$$

then

(3.12) $$(K-G)^T\gamma_t = -(\phi_t - \psi_t)^T\theta.$$

In particular,

(3.13) $$(K-G)^T\gamma_0 = -Cx_0.$$

Note that γ_0 cannot be made zero if x_0 is unknown, but we will assume that $\|\gamma_0\|^2$ is bounded; this is consistent with our assumption on x_0 and (3.13).

Denoting

(3.14) $$\tilde{\theta}_t \triangleq \hat{\theta}_t - \theta$$

we then have from (3.12), (3.10) that

(3.15a) $$\eta_t = -(K-G)^T\gamma_t - \psi_t^T\tilde{\theta}_t$$

where

(3.15b) $$d\gamma_t = (A-KC)^T\gamma\,dt + C^T(-\psi_t^T\tilde{\theta}_t)\,dt.$$

We thus see that η_t is related to $-\psi_t^T\tilde{\theta}_t$ by the following transfer function equation

(3.16) $$D(\rho)\eta_t = E(\rho)[-\psi_t^T\tilde{\theta}_t]$$

where $D(\rho)$ is the characteristic polynomial of the optimal Kalman filter for the system, i.e.,

(3.17) $$D(\rho) = \rho^n + (k_{n-1} + a_{n-1})\rho^{n-1} + \cdots (k_0 + a_0) = \det(\rho I - A + KC)$$

and where $E(\rho)$ is as in (2.5).

As is standard in pseudoregression algorithms [3], we require that sufficient prior knowledge is available to choose the observer polynomial $E(\rho)$ so as to satisfy the following assumption.

Assumption 1. $D(\rho)$ is strictly Hurwitz and the filter $E(\rho)$ is chosen such that

(1) Re $\sigma_i(E(\rho)) \leqq -\alpha < 0$, $i = 1, \cdots, n$ where $\sigma_i(E)$ are the roots of E.

(2) D/E is input strictly passive, i.e., there exists $\varepsilon > 0$ and $K > 0$ such that

(3.18) $$\forall T > 0, \int_0^T y_\tau u_\tau\,d\tau \geqq \varepsilon \int_0^T u_\tau^2\,d\tau - K\|\gamma_0\|^2$$

where y_t is the output of the filter D/E driven by u_t.

We will later need the following technical result on input strictly passive systems.

LEMMA 3.1. *Let $H(s)$ be input strictly passive with impulse response h, and let $y = h_* u$. Let $r_t > 0$ be monotonically nondecreasing. Then there exists $\varepsilon > 0$ such that for all $T > 0$ and all u_t:*

$$\int_0^T \frac{y_t u_t}{r_t}\,dt \geqq \varepsilon \int_0^T \frac{u_t^2}{r_t}\,dt - \frac{K\|\gamma_0\|^2}{r_0}. \tag{3.19}$$

Proof. The proof follows immediately from (3.18) on using the result in Appendix A. □

Motivated by the discrete-time algorithm given in [1], we next define the estimation algorithm as

$$d\hat{\theta}_t = \frac{\psi_t}{r_t}(dy_t - \psi_t^T \hat{\theta}_t\,dt) \tag{3.20}$$

where

$$r_t \triangleq \sup_{0 \leqq \tau \leqq t} \psi_\tau^T \psi_\tau + \int_0^t \psi_\tau^T \psi_\tau\,d\tau + c_0; \qquad c_0 > 0 \tag{3.21}$$

and c_0 is any positive deterministic number. Note that $\hat{\theta}_t$ is obtained by an Ito integral which makes sense locally since (ψ_t / r_t) is $\mathcal{F}_t$-measurable and continuous.

The definition of r_t given above is not quite the analogue of that used in discrete time. We will discuss the reason for the difference later.

Noting that $d\theta = 0$ and using the definition of e_t and η_t given in (3.6), (3.8), we observe from (3.14) and (3.20) that $\tilde{\theta}_t$ is the solution of the following stochastic differential equation

$$d\tilde{\theta}_t = \frac{\psi_t}{r_t}\eta_t\,dt + \frac{\psi_t}{r_t}\,d\omega_t. \tag{3.22}$$

In the following we will require that $\hat{\theta}_t$ be bounded. We guarantee this by introducing a projection scheme as described below. We first introduce the following assumption.

Assumption 2. There exists a known parameter value θ_c and a positive number R_1 such that the true value θ lies inside $\mathscr{C}_1$ where

$$\mathscr{C}_1 = \{\theta : \|\theta - \theta_c\| \leqq R_1\}.$$

Let R_2 be another positive number larger than R_1. We then modify the parameter estimator to ensure that $\|\hat{\theta}_t - \theta_c\| < R_2$ for all t. We do this by using the following projection scheme.

Parameter estimator with projection. Let τ be a time for which the solution of (3.20) is such that $\|\hat{\theta}_\tau - \theta_c\| = R_2$. Denote the corresponding value of $\hat{\theta}_\tau$ by $\hat{\theta}_{\tau_-}$. At time τ, the estimate $\hat{\theta}_\tau$ is then defined as

$$\hat{\theta}_\tau \triangleq \theta_c + \frac{R_1}{R_2}(\hat{\theta}_{\tau_-} - \theta_c). \tag{3.23}$$

For $t \geqq \tau$, (3.20) is then integrated with initial condition $\hat{\theta}_\tau$ defined by (3.23). This makes $\hat{\theta}_t$ right continuous at the projection times.

4. A general class of feedback control laws. We consider a general class of control laws in state feedback form:

$$u_t = -[l_{n-1}, \cdots, l_0]\psi_t^u - [p_{n-1}, \cdots, p_0]\psi_t^y + y^*. \tag{4.1}$$

This is equivalent to the feedback law

$$Q(\rho)u_t = -P(\rho)y_t + E(\rho)y_t^* \tag{4.2}$$

where

$$Q(\rho) = E(\rho) + L(\rho);\ L(\rho) = l_{n-1}\rho^{n-1} + \cdots + l_0;\ P(\rho) = p_{n-1}\rho^{n-1} + \cdots + p_0. \tag{4.3}$$

Note that the control law transfer function is $-P/Q$, which is strictly proper. Also, y_t^* denotes a bounded reference signal. For the moment, we make no assumptions about L, P stabilizing the system or whether the control law depends on the estimated parameters $\hat{\theta}$. To allow for the latter possibility, we express the control law as:

$$u_t = -[\hat{l}_{n-1}, \cdots, \hat{l}_0]\psi_t^u - [\hat{p}_{n-1}, \cdots, \hat{p}_0]\psi_t^y + y_t^*. \tag{4.4}$$

For the moment, the only restriction we place on the above general control law is that $\hat{l}_{n-1}, \cdots, \hat{l}_0, \hat{p}_{n-1}, \cdots, \hat{p}_0$ by Lipschitz functions of $\tilde{\theta}$.

From the model (2.1), (2.2), the general controller (4.4), the definition of ψ_t ((3.2)-(3.5)), and the definition of the errors ((3.6)-(3.8)), we can write:

$$d\psi_t = A_t \psi_t\,dt + B_1(\eta_t\,dt + d\omega_t) + B_2 y_t^*\,dt \tag{4.5}$$

where

$$A_t = \left[\begin{array}{ccc|ccc|ccc}
-\hat{a}_{n-1} & \cdots & -\hat{a}_0 & \hat{b}_{n-1} & \cdots & \hat{b}_0 & \hat{f}_{n-1} & \cdots & \hat{f}_0 \\
 & & 0 & & & & & & \\
 & I_{n-1} & \vdots & & & & & & \\
\cdots & & 0 & \cdots & & \cdots & \cdots & & \cdots \\
\hline
-\hat{p}_{n-1} & \cdots & -\hat{p}_0 & -\hat{q}_{n-1} & \cdots & -\hat{q}_0 & 0 & \cdots & 0 \\
 & & & & & 0 & & & \\
 & & & & I_{n-1} & \vdots & & & \\
\cdots & & \cdots & \cdots & & 0 & \cdots & & \cdots \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & -e_{n-1} & \cdots & -e_0 \\
 & & & & & & & & 0 \\
 & & & & & & & I_{n-1} & \vdots \\
\cdots & & \cdots & \cdots & & \cdots & \cdots & & 0
\end{array}\right] \tag{4.6a}$$

$$B_1^T = [10 \cdots 010 \cdots 0] \tag{4.6b}$$

$$B_2^T = [0 \cdots 010 \cdots 0] \tag{4.6c}$$

where B_1^T has 1's in the first and $(2n+1)$st positions and B_2^T has 1 in the $(n+1)$st position.

A key point about (4.5) is that A_t is a Lipschitz function of $\tilde{\theta}_t$ provided the general control law (4.2) is chosen in which $\hat{l}_{n-1}, \cdots, \hat{l}_0, \hat{p}_{n-1}, \cdots, \hat{p}_0$ are Lipschitz in $\tilde{\theta}$.

5. General properties of the estimation algorithm with feedback. We first address the question of existence and uniqueness [14] of the solution of the full set of equations describing the system and estimation algorithm. Combining (3.14), (3.22), and (4.5), the full set of equations is

$$d\psi_t=\left[A_t(\tilde{\theta}_t)\psi_t-B_1\psi_t^T\tilde{\theta}_t-B_1\int_0^t h_{t-\tau}\psi_\tau^T\tilde{\theta}_\tau\,d\tau\right]dt+B_1\,d\omega_t+B_2y_t^*\,dt \tag{5.1a}$$

$$d\tilde{\theta}_t=\left[-\frac{\psi_t\psi_t^T}{r_t(\psi_\cdot)}\tilde{\theta}_t-\int_0^t h_{t-\tau}\frac{\psi_\tau\psi_\tau^T}{r_\tau(\psi_\cdot)}\tilde{\theta}_\tau\,d\tau\right]dt+\frac{\psi_t}{r_t(\psi_\cdot)}\,d\omega_t \tag{5.1b}$$

where $A_t(\tilde{\theta}_t)\equiv A_t(\hat{\theta}_t)$ and h_t is the impulse response of the strictly proper part of the transfer function E/D in (3.16).

We then have the following result.

LEMMA 5.1. *The composite set of* (5.1) *has a unique solution with continuous sample paths almost surely up to the random time T of the first explosion.* (*That is, T is the first time that either a component of ψ or $\tilde{\theta}$ becomes infinite or $T=\infty$.*)

Proof. We first note from (4.6a) that $A_t(\tilde{\theta}_t)$ is Lipschitz in $\tilde{\theta}_t$ due to the assumed form of the dependence of $\hat{l}_{n-1},\cdots,\hat{l}_0,\hat{p}_{n-1},\cdots,\hat{p}_0$ on $\hat{\theta}$. This implies that the coefficient vectors multiplying dt and $d\omega_t$ in (5.1) are locally Lipschitz with respect to the supremum norm on sample paths—see Appendix B (that is, given a compact set in the space of ψ_t, $\tilde{\theta}_t$, the functions are Lipschitz with constant depending on the choice of the set).

The result then follows from Theorems (14.18) and (14.20) of [20]. □

Note that Lemma 5.1 does not use the projection of the parameter estimates as described at the end of § 3. When this additional facet of the algorithm is included, we can strengthen Lemma 5.1 as follows.

LEMMA 5.2. *With the addition of the projection scheme* (3.23) *to* (5.1b), *then $\hat{\theta}_t$ remains in $\mathscr{C}_2$ for all t and the composite set of equations* (5.1) *has a unique solution almost surely with sample paths* $(\psi_t,\tilde{\theta}_t)$ *which are continuous except at the projection times.*

Proof. Up to the time of the first projection, θ_t is bounded and thus (5.1a) is a linear time varying equation with bounded coefficients and hence ψ_t cannot become unbounded in a finite time. Hence the first projection occurs strictly before the explosion time T of Lemma 5.1 (unless both T and the first projection time are infinite).

After projection, we can apply Lemma 5.1 again and repeat the same argument. Hence due to the linearity of (5.1b) for given $\psi_\cdot$, $\tilde{\theta}_t$ exists, is unique and is bounded by the projection. Then (5.1a) is a linear equation with bounded coefficients, and hence ψ_t exists almost surely for all t. □

Lemma 5.2 provides the basic existence and uniqueness result necessary to establish the following result giving properties of the parameter estimator. This result does not depend on a priori boundedness of the system states. Note, however, that we have already established that $\tilde{\theta}_t$ and ψ_t/r_t are bounded.

THEOREM 5.1. *For the general class of feedback control laws described in* § 4 *and under Assumptions* 1 *and* 2, *the following properties hold for the model* ((2.1), (2.2)) *and the estimator* (3.20)–(3.21) *with the projection scheme* (3.23):

(5.2) (i) $\limsup_{t\to\infty}\int_0^t\frac{\eta_\tau^2}{r_\tau}\,d\tau\le K_1<\infty$ *a.s.*

where K_1 is a random variable (*realisation dependent*).

(5.3) (ii) *For all finite* Δ, $\lim_{t\to\infty}\sup_{0\le T\le\Delta}\|\hat{\theta}_{t+T}-\hat{\theta}_t\|=0$ *a.s.*

(iii) *There exists a finite random time t_R beyond which no further parameter projections occur.*

Proof. (a) Starting from (3.22) and using Ito's rule (see, e.g., [15]), we have that, between projections:

$$\begin{aligned}d(\tilde{\theta}_t^T\tilde{\theta}_t)&=2\tilde{\theta}_t^T\frac{\psi_t}{r_t}(\eta_t\,dt+d\omega_t)+\sigma^2\frac{\|\psi_t\|^2}{r_t^2}\,dt\\&=-\frac{2}{r_t}(-\tilde{\theta}_t^T\psi_t\eta_t-\varepsilon\eta_t^2)\,dt-2\varepsilon\frac{\eta_t^2}{r_t}\,dt+2\tilde{\theta}_t^T\frac{\psi_t}{r_t}\,d\omega_t+\sigma^2\frac{\|\psi_t\|^2}{r_t^2}\,dt\end{aligned}\tag{5.4}$$

for some $\varepsilon>0$.

Note that from (3.23) and using Assumption 2, at the times of projection we have $\|\tilde{\theta}_t\|^2\le\|\tilde{\theta}_{t_-}\|^2-(R_2-R_1)^2$.

Defining $\xi_t\triangleq-\tilde{\theta}_t^T\psi_t\eta_t-\varepsilon\eta_t^2$ where ε is as in Assumption 1, integrating (5.4) and accounting for projections yields

$$\begin{aligned}\tilde{\theta}_t^T\tilde{\theta}_t\le\tilde{\theta}_s^T\tilde{\theta}_s-2\int_s^t\frac{\xi_\lambda}{r_\lambda}\,d\lambda-2\varepsilon\int_s^t\frac{\eta_\lambda^2}{r_\lambda}\,d\lambda+2\int_s^t\tilde{\theta}_\lambda^T\frac{\psi_\lambda}{r_\lambda}\,d\omega_\lambda\\+\int_s^t\sigma^2\frac{\|\psi_\lambda\|^2}{r_\lambda^2}\,d\lambda-N_{t,s}(R_2-R_1)^2\end{aligned}\tag{5.5}$$

where $N_{t,s}$ is the number of times that projections occur between times s and t. Consider now the integral

$$\int_0^t\frac{\|\psi_\lambda\|^2}{r_\lambda^2}\,d\lambda.$$

We have, using (3.21),

$$\int_0^t\frac{\|\psi_\lambda\|^2}{r_\lambda^2}\,d\lambda\le\int_0^t\frac{dr_\lambda}{r_\lambda^2}=\frac{1}{r_0}-\frac{1}{r_t}\le\frac{1}{r_0}=\frac{1}{c_0}.\tag{5.6}$$

(This operation makes makes sense due to Bonnet's and Du Bois-Reymond's formulae [21] which allow the integral to be considered as a Riemann integral.) We now define $\mathscr{X}_t$ as the solution of the Ito integral

$$d\mathscr{X}_t=\frac{2\tilde{\theta}_t^T\psi_t}{r_t}\,d\omega_t,\qquad\mathscr{X}_0=\frac{\sigma^2}{c_0}+\frac{2K\|\gamma_0\|^2}{r_0}+\tilde{\theta}_0^T\tilde{\theta}_0.\tag{5.7}$$

This integral makes sense thanks to (5.6) and Lemma 5.2. Since c_0, r_0, γ_0, and $\tilde{\theta}_0$ are $\mathscr{F}_0$-measurable, $(\mathscr{X}_t,\mathscr{F}_t)$ is a martingale. Moreover it follows from (5.5) and (5.6) that $\mathscr{X}_t$ satisfies

$$\begin{aligned}\mathscr{X}_t\ge\tilde{\theta}_t^T\tilde{\theta}_t+2\int_0^t\frac{\xi_\lambda}{r_\lambda}\,d\lambda+2\varepsilon\int_0^t\frac{\eta_\lambda^2}{r_\lambda}\,d\lambda+\frac{\sigma^2}{c_0}-\int_0^t\sigma^2\frac{\|\psi_\lambda\|^2}{r_\lambda^2}\,d\lambda+\frac{2K\|\gamma_0\|^2}{r_0}\\+N_{t,0}[R_2-R_1]^2\end{aligned}\tag{5.8}$$

and is positive in view of Lemma 3.1 and (5.6). Thus $(\mathscr{X}_t,\mathscr{F}_t)$ is a positive martingale and hence

$$\lim_{t\to\infty}\mathscr{X}_t=\mathscr{X}<\infty\qquad\text{a.s.}\tag{5.9}$$

Using (5.9) and noting (5.6) and Lemma 3.1, we conclude that (5.2) holds for some finite random variable K_1.

This establishes (i). Also, since $R_2 > R_1$, then from (5.8), (5.9) $N_{t,0}$ is bounded almost surely. Hence (iii) follows.

(b) Now let $t \geqq t_R$. Then from (3.22) we can write, using $\|x+y\|^2 \leqq 2\|x\|^2 + 2\|y\|^2$,

$$\text{(5.10)}\quad \begin{aligned}\|\tilde{\theta}_{t+T} - \tilde{\theta}_t\|^2 &\leqq 2\left\{\left\|\int_t^{t+T} \frac{\psi_\tau \eta_\tau}{r_\tau} d\tau\right\|^2 + \left\|\int_t^{t+T} \frac{\psi_\tau}{r_\tau} d\omega_\tau\right\|^2\right\} \\ &\leqq 2T\int_t^{t+T} \frac{\|\psi_\tau\|^2}{r_\tau^2}\eta_\tau^2\, d\tau + 2\left\|\int_t^{t+T}\frac{\psi_\tau}{r_\tau} d\omega_\tau\right\|^2.\end{aligned}$$

The last inequality is obtained by applying the Schwartz inequality. We consider the two terms of the right-hand side of (5.10) separately.

Consider a realization for which (i) holds. Thus, given $\varepsilon > 0$, $\Delta > 0$ there exists a $t_0(\Delta, \varepsilon)$ such that for all $t \geqq t_0$

$$\text{(5.11)}\quad \int_t^{t+T} \frac{\eta_\tau^2}{r_\tau} d\tau \leqq \frac{1}{4}\frac{\varepsilon^2}{\Delta} \quad \forall t \geqq t_0 \quad \text{and} \quad 0 \leqq T \leqq \Delta.$$

Since $(\|\psi_\tau\|^2/r_\tau) \leqq 1$ by our definition (3.21) of r_τ, it follows that

$$\text{(5.12)}\quad 2T\int_t^{t+T} \frac{\|\psi_\tau\|^2}{r_\tau^2}\eta_\tau^2\, d\tau \leqq \frac{1}{2}\varepsilon^2 \quad \forall t \geqq t_0 \quad \text{and} \quad \forall\, T \in [0, \Delta]$$

and hence

$$\text{(5.13)}\quad \lim_{t\to\infty}\sup_{0\leqq T\leqq\Delta} 2T\int_t^{t+T}\frac{\|\psi_\tau\|^2}{r_\tau^2}\eta_\tau^2\, d\tau = 0.$$

Since (i) holds almost surely, so does (5.13). The result then follows using part (a) of Lemma C.1 (Appendix C) for the second term on the right-hand side of (5.10). □

To make use of the result in Theorem 5.1, we will need the following continuous-time Kronecker lemma. Since we have been unable to find a proof in the literature for this continuous-time version, we supply one in Appendix D. It parallels the discrete-time proof given in [2].

LEMMA 5.3 (Continuous-time Kronecker Lemma). *Assume that*

(A.1) $S_t = \int_0^t x_\tau\, d\tau$ *converges to* $S < \infty$ *as* $t \to \infty$.

(A.2) $b_t > 0$ *is monotone nondecreasing and* $\lim_{t\to\infty} b_t = \infty$.

Then

$$\lim_{t\to\infty}\frac{1}{b_t}\int_0^t b_\tau x_\tau\, d\tau = 0.$$

Proof. For the proof see Appendix D.

We now prove a continuous time equivalent of the discrete-time key technical lemma given in [1], [2].

LEMMA 5.4. *Consider a realisation produced by the model* (2.1)–(2.2). *Suppose the estimator is such that*

$$\text{(5.14)}\quad \lim_{t\to\infty}\sup\int_0^t \frac{\eta_\tau^2}{r_\tau} d\tau \leqq K_1 < \infty$$

and the controller is such that the following growth condition is satisfied:

$$\text{(5.15)}\quad \frac{r_t}{t} \leqq C + \frac{K_2}{t}\int_0^t \eta_\tau^2\, d\tau$$

where r_t, η_t *are as defined before and* C, K_2 *are finite positive constants. Then*

$$\text{(5.16)}\quad \text{(i)}\ \lim_{t\to\infty}\sup\frac{r_t}{t} \leqq K_3 < \infty$$

$$\text{(5.17)}\quad \text{(ii)}\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_\tau^2\, d\tau = 0.$$

Proof. (i) From (5.15) and the nonnegativity of η_τ^2 and r_τ it follows that

$$\frac{r_t}{t} \leqq C + K_2\int_0^t \frac{\eta_\tau^2}{r_\tau}\frac{r_\tau}{\tau} d\tau.$$

The result then follows from the Bellman-Gronwall lemma (see e.g., [17]) using (5.14):

(ii) Suppose first that $\lim_{t\to\infty} r_t = \infty$. Then, by the Kronecker Lemma 5.3:

$$\text{(5.18)}\quad \lim_{t\to\infty}\frac{1}{r_t}\int_0^t \eta_\tau^2\, d\tau = 0.$$

Hence, using (5.16) and (5.18)

$$\text{(5.19)}\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_\tau^2\, d\tau = \lim_{t\to\infty}\frac{r_t}{t}\frac{1}{r_t}\int_0^t \eta_\tau^2\, d\tau = 0.$$

Alternatively, if $\lim_{t\to\infty} r_t \leqq K_4 < \infty$ for some K_4, then

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_\tau^2\, d\tau &\leqq \lim_{t\to\infty}\frac{1}{t}\frac{K_4}{r_t}\int_0^t \eta_\tau^2\, d\tau \\ &\leqq \lim_{t\to\infty}\frac{K_4}{t}\int_0^t \frac{\eta_\tau^2}{r_\tau} d\tau = 0\end{aligned}$$

using (5.2). The last inequality follows from the monotonicity of r_τ. □

Comment 5.1. A key feature of our estimator is our choice of r_t (see (3.21)). It guarantees that $(\|\psi_t\|^2/r_t) \leqq 1$ for all t, and this is crucial in establishing (5.2). In [5], [6] the equivalent term to r_t is defined as $r_t = \int_0^t \psi_\tau^T \psi_\tau\, d\tau$ and, as a consequence, all the convergence proofs require the assumption that $(\|\psi_t\|^2/r_t)$ is almost surely bounded. This assumption is rather unrealistic given that ψ_t contains signals driven by white noise.

Comment 5.2. Comparing the properties of our estimator with those proved for the continuous-time deterministic algorithms in [9], we observe that we have not proved the uniform boundedness of $(\eta_t/r_t^{1/2})$ or of something like $\dot{\theta}$; neither have we proved $\dot{\theta} \in L_2$. The uniform boundedness of $(\eta_t/r_t^{1/2})$ is not needed in the subsequent analysis ($(\eta_t/r_t^{1/2}) \in L_2$ suffices). As for $\dot{\theta}$, it does not exist in our stochastic framework, but the conditions on $\dot{\theta}$ in [9] are replaced by the weaker condition of (5.3).

Comment 5.3. The remaining step in the development is to verify the growth condition (5.15). This will require us to impose additional constraints on the general feedback law described in § 4. In particular, we will choose the feedback so as to stabilize the (frozen) estimated model.

6. Boundedness of the system states. Finally we show that, provided the feedback control law is appropriately chosen as a function of $\hat{\theta}$, then the adaptive law stabilizes the system in the sense that all system states and the input are mean square bounded almost surely.

The proofs given below depend upon the fact that the certainty equivalence control law stabilize the frozen *estimated* model. This is true of a wide class of algorithms. However, to be specific, we will illustrate the analysis procedure by considering adaptive pole assignment. In this algorithm the polynomials $\hat{L}$ and $\hat{P}$ defining the control law are computed from the estimated $\hat{A}(\hat{\theta}_t)$ and $\hat{B}(\hat{\theta}_t)$ as follows.

Let

$$\hat{\theta}=[\hat{g}_{n-1},\cdots,\hat{g}_0,\hat{b}_{n-1},\cdots,\hat{b}_0,\hat{f}_{n-1},\cdots,\hat{f}_0]$$

and define (cf. (2.6), (2.21))

$$\hat{a}_i=e_i-\hat{g}_i;\qquad i=0,\cdots,n-1$$
$$\hat{A}(\rho)=\rho^n+\hat{a}_{n-1}\rho^{n-1}+\cdots+\hat{a}_0$$
$$\hat{B}(\rho)=\hat{b}_{n-1}\rho^{n-1}+\cdots+\hat{b}_0$$
$$\hat{F}(\rho)=\hat{f}_{n-1}\rho^{n-1}+\cdots+\hat{f}_0.$$

Then, for a given possibly time varying A^* of degree $2n$, solve the following equation for $\hat{Q}$ and $\hat{P}$:

$$\hat{Q}\hat{A}+\hat{P}\hat{B}=A^* \quad \text{with} \quad \text{Re}\,\lambda_i(A^*)\leqq -\beta<0, \qquad i=1,\cdots,2n \tag{6.1}$$

where $\lambda_i(A^*)$ are the eigenvalues of the polynomial A^*. Finally compute

$$\hat{L}(\rho)=\hat{Q}(\rho)-E(\rho).$$

Equation (6.1) can also be written

$$M(\hat{\theta})\begin{bmatrix}\hat{q}_0\\ \vdots\\ \hat{q}_{n-1}\\ \hat{p}_0\\ \vdots\\ \hat{p}_{n-1}\end{bmatrix}=\begin{bmatrix}a_0^*\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ a_{2n-1}^*\end{bmatrix}. \tag{6.2}$$

The polynomial A^* is a design polynomial available to the user. The only restriction we require on the polynomial $A^*(t)$ is seen in the following assumption.

Assumption 3. For all t, the polynomial $A^*(t)$ is continuous and bounded, has a uniform stability margin (i.e., $\text{Re}\,\lambda_i(A^*(t))\leqq -\beta<0$, $i=1,\cdots,2n$), and $\lim_{t\to\infty}\sup_{0\leqq T\leqq\Delta}\|a^*(t+T)-a^*(t)\|=0$ for some $\Delta>0$, where a^* is the vector of coefficients of A^*.

As in § 4, we also require that $\hat{L}$, $\hat{P}$ be Lipschitz functions of $\hat{\theta}$ for all time. We note that the projection facility (3.23) ensures that $\hat{\theta}\in\mathscr{C}_2$ for all time. Then, the Lipschitz condition is guaranteed provided all models corresponding to $\hat{\theta}\in\mathscr{C}_2$ are uniformly stabilizably. This assumption appears in all contemporary treatments of indirect adaptive control (see, for example, [2], [10], [19]). We remark that this assumption can be eliminated in the case of direct adaptive control of stably invertible systems. However, this is only achieved at the expense of a much more complex stability proof (see [9], [22]) which necessarily builds on the result presented here. For the case of indirect adaptive pole assignment we introduce the following additional assumption.

Assumption 4. Assumption 2 is satisfied and there exists a known positive constant ε such that for all θ_t in $\mathscr{C}_2$,

$$\det M(\theta_t)\geqq\varepsilon.$$

Subject to Assumption 4, the projection scheme ensures that $\det M(\theta_t)\geqq\varepsilon$ for all t. In view of (6.2) this will ensure that $\hat{L}$, $\hat{P}$ are Lipschitz functions of $\hat{\theta}$ as required.

We next establish that, for the above adaptive control law, the homogeneous part of (4.5) is exponentially stable.

LEMMA 6.1. *Consider the differential equation*

$$\frac{d}{dt}\psi_t=A_t\psi_t$$

with A_t given by (4.6a). *Assume that the $\hat{a}_i$, $\hat{b}_i$, $\hat{f}_i$ are estimated using the parameter estimator of* § 3 *including the projection scheme* (3.23) *and that Assumptions* 1–4 *hold. Then* (6.1) *is exponentially stable almost surely.*

Proof. We have shown in Theorem 5.1 that there exists a random time t_R beyond which no further projections occur. Thus, for $t>t_R$ $\hat{\theta}_t$ is sample continuous from Lemma 5.2. Therefore $\hat{A}$, $\hat{B}$, $\hat{F}$ are sample continuous and, by Assumption 4, $\hat{Q}$ and $\hat{P}$ are sample continuous given that $\hat{q}_i$ and $\hat{p}_i$ can be written as the solution of the linear system (6.2) or $M(\hat{\theta})\hat{c}=a^*$, where the elements of M are either zero or $\hat{a}_i$, $\hat{b}_j$. For the same reason, A_t is uniformly bounded. The eigenvalues of A_t are the roots of $E(s)$ and the $2n$ roots of $\hat{A}^*$; therefore, by Assumptions 1 and 3 $\text{Re}\,\lambda_i(A_t)\leqq-\delta<0$, $i=1,\cdots,3n$, for all t, where $\delta\triangleq\min(\alpha,\beta)$. Finally,

$$M(\hat{\theta}_t)[\hat{c}_{t+T}-\hat{c}_t]+[M(\hat{\theta}_{t+T})-M(\hat{\theta}_t)]\hat{c}_{t+T}=a^*_{t+T}-a^*_t.$$

Therefore, by the triangle inequality,

$$\|\hat{c}_{t+T}-\hat{c}_t\|\leqq\|M^{-1}(\hat{\theta}_t)\|\{\|M(\hat{\theta}_{t+T})-M(\hat{\theta}_t)\|\cdot\|\hat{c}_{t+T}\|+\|a^*_{t+T}-a^*_t\|\}.$$

Hence, using (5.3) and Assumption 3 it follows that

$$\lim_{t\to\infty}\sup_{0\leqq T\leqq\Delta}\|A_{t+T}-A_t\|=0 \quad \text{a.s. for some } \Delta>0. \tag{6.3}$$

The result then follows from Lemma 3 of [10]. □

We now establish the main result of this paper.

THEOREM 6.1. *Consider the system* ((2.1), (2.2)), *with the parameter vector θ satisfying Assumption* 2, *the parameter estimator of* § 3 *with projection, an observer polynomial E satisfying Assumption* 1, *and an adaptive pole assignment control law of the form* (4.2), (6.1) *satisfying Assumptions* 3 *and* 4. *Then, for arbitrary finite initial conditions and an arbitrary, piecewise continuous, uniformly bounded reference input y_t^*, the following results hold*:

(i) $\displaystyle\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|\psi_\tau\|^2\,d\tau<\infty$ *a.s.*

(ii) $\displaystyle\limsup_{t\to\infty}\frac{1}{t}\int_0^t u_\tau^2\,d\tau<\infty$ *a.s.*

(iii) $\displaystyle\limsup_{t\to\infty}\frac{1}{t}\int_0^t |Cx_\tau|^2\,d\tau<\infty$ *a.s.*

Proof. (i) We first establish (5.15), i.e., that there exist finite random variables C and K such that:

$$\frac{r_t}{t}\leqq C+\frac{K}{t}\int_0^t\eta_\tau^2 \quad \text{a.s.}$$

Now by definition

$$r_t\triangleq\sup_{0\leqq\tau\leqq t}\|\psi_\tau\|^2+\int_0^t\|\psi_\tau\|^2\,d\tau+c_0. \tag{6.4}$$

From (4.5)

$$\psi_t = \phi(t,0)\psi_0 + \int_0^t \phi(t,\tau)B_1\eta_\tau\,d\tau + \int_0^t \phi(t,\tau)B_2 y_\tau^*\,d\tau + \int_0^t \phi(t,\tau)B_1\,d\omega_\tau$$

$$(6.5)\qquad = \psi_t^{(1)} + \psi_t^{(2)}$$

where $\psi_t^{(1)}$ is the sum of the first three terms, $\psi_t^{(2)}$ is the last term and $\phi(t,\tau)$ denotes the state transition matrix for (4.5). Because ϕ is exponentially stable by Lemma 6.1 and y_τ^* is uniformly bounded, it follows using the Cauchy–Schwartz inequality that there exist finite constants K_1 and K_2 such that

$$(6.6)\qquad \|\psi_t^{(1)}\|^2 \leqq K_1 + K_2\int_0^t \eta_\tau^2\,d\tau$$

for all t. Also, by Lemma C.2

$$(6.7)\qquad \frac{1}{t}\|\psi_t^{(2)}\|^2 \to 0 \quad \text{a.s.}$$

as $t\to\infty$.

Similarly, using a continuous-time version of the proof of Lemma B.3.3 in [2], it follows that there exist finite constants K_3 and K_4 such that

$$(6.8)\qquad \int_0^t \|\psi_\tau^{(1)}\|^2\,d\tau \leqq K_3 + K_4\int_0^t \eta_\tau^2\,d\tau$$

for all t. Finally, by Lemma C.2,

$$(6.9)\qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t \|\psi_\tau^{(2)}\|^2\,d\tau \leqq K_5 < \infty \quad \text{a.s.}$$

Combining (6.6) to (6.9) we obtain

$$\frac{r_t}{t} = \frac{1}{t}\sup_{0\leqq\tau\leqq t}\|\psi_\tau\|^2 + \frac{1}{t}\int_0^t\|\psi_\tau\|^2\,d\tau + \frac{c_0}{t}$$

$$(6.10)\qquad \leqq \frac{1}{t}\max\left[\sup_{0\leqq\tau\leqq 1}\|\psi_\tau\|^2,\ \sup_{1\leqq\tau\leqq t}\|\psi_\tau\|^2\right] + \frac{1}{t}\int_0^t\|\psi_\tau\|^2\,d\tau + \frac{c_0}{t}$$

$$\leqq K_6 + \frac{K_7}{t}\int_0^t \eta_\tau^2\,d\tau \quad \text{a.s.}$$

Thus, we have established (5.15) so that, by Lemma 5.4, there exists a finite K_8 such that

$$(6.11)\qquad \frac{r_t}{t} \leqq K_8 \quad \text{a.s.}$$

It follows from the definition of r_t that

$$(6.12)\qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t\|\psi_\tau\|^2\,dt < \infty \quad \text{a.s.,}$$

which establishes part (i) of the theorem.

(ii) Since $u_t = \hat{\xi}_t\psi_t + y_t^*$, where $\hat{\xi}_t$ is a vector whose components are $\{\hat{p}_i\}$ and $\{\hat{l}_i\}$ (see (4.4) and the definition of ψ_t) it follows from (6.10) and the boundedness of $\{\hat{p}_i\}$ and $\{\hat{l}_i\}$ that

$$(6.13)\qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t u_\tau^2\,d\tau < \infty \quad \text{a.s.}$$

which establishes part (ii) of the theorem.

(iii) From (2.9), (2.19), (3.7), and (3.8) we have

$$(6.14)\qquad Cx_t = \psi_t^T\hat{\theta}_t + \eta_t.$$

Therefore, since $\hat{\theta}_t$ is bounded, $\lim_{t\to\infty}\sup(1/t)\int_0^t \eta_t^2\,d\tau = 0$, (Lemma 5.4) and $\lim_{t\to\infty}\sup(1/t)\int_0^t\|\psi_\tau\|^2\,d\tau < \infty$ almost surely (see (6.12)), it follows that

$$(6.15)\qquad \limsup_{t\to\infty}\frac{1}{t}\int_0^t [Cx_t]^2\,dt < \infty \quad \text{a.s.}$$

which establishes part (iii) of the theorem. □

Comment 6.1. Note that Cx_t is the "deterministic part" of the output. Thus, (6.15) establishes that this part of the output is sample mean square bounded. This is all that can be said about the output since it contains a Wiener process.

Comment 6.2. The proof of Theorem 6.1 relies on the now standard argument on robustness of bounded solutions of exponentially stable linear time varying systems.

7. Conclusions. This paper has analyzed a class of continuous-time stochastic adaptive control algorithms and has shown that, under suitable conditions, they will almost surely ensure global boundedness of all the internal variables, in a sample mean square sense. Previous results in the same direction relied upon an assumption on the normalized regression vector which, in fact, meant that the noise driving the system was wide band but bounded. Moreover, the systems described in previous works were of relative degree zero.

The results described in this paper are thus believed to constitute the first complete and rigorous analysis of a realistic continuous time stochastic adaptive control algorithm.

The main result is a boundedness result that states that the deterministic part of the output remains almost surely bounded in the sample mean square sense. No tracking property is achieved, but the analysis described here has recently been extended to a model reference control algorithm yielding a result on the asymptotic tracking error [22], which is analogous to the discrete-time tracking error result established in [1].

Appendix A.

LEMMA A.1. *Let r_τ be a nondecreasing function with $r_0 > 0$. Then*

$$(A.1)\qquad \int_0^T f_\tau\,d\tau \geqq -K; \qquad K>0, \quad \forall T\geqq 0$$

implies

$$(A.2)\qquad \int_0^T \frac{f_\tau}{r_\tau}\,d\tau \geqq -\frac{K}{r_0}; \quad \forall T \geqq 0.$$

Proof. Let $\rho_\tau = (1/r_\tau)$, $\tau\geqq 0$, and denote $F_t = \int_0^t f_\tau\,d\tau + K$. Then, by integration by parts:

$$\int_0^T f_\tau\rho_\tau\,d\tau = \rho_T F_T + \left\{-\int_0^T F_\tau\,d\rho_\tau\right\} - \rho_0 K.$$

The first term is nonnegative because $\rho_T > 0$ and $F_T \geqq 0$. The second term is also nonnegative because $F_\tau \geqq 0$, for all $\tau \geqq 0$ by (A.1), and $d\rho_\tau \leqq 0$ since ρ_τ is nonincreasing. □

Appendix B.

LEMMA B.1. *The coefficient vectors in* (5.1) *are locally Lipschitz with respect to the supremum norm on the sample paths ψ_t, $\hat{\theta}_t$.*

Proof. We consider (5.1a), (5.1b) separately. Equation (5.1a) has the form:

(B.1) $$d\psi_t = f_1(t, \tilde{\theta}_\cdot, \psi_\cdot)\, dt + B_1\, d\omega_t + B_2 y_t^*\, dt$$

where

(B.2) $$f_1(t, \tilde{\theta}_\cdot, \psi_\cdot) = \left[A_t(\tilde{\theta}_t)\psi_t - B_1\psi_t^T\tilde{\theta}_t - B_1 \int_0^t h_{t-\tau}\psi_\tau^T\tilde{\theta}_\tau\, d\tau \right].$$

We then have, by the triangle inequality,

(B.3) $$\begin{aligned} &\|f_1(t, \tilde{\theta}^1_\cdot, \psi^1_\cdot) - f_1(t, \tilde{\theta}^2_\cdot, \psi^2_\cdot)\| \\ &\leqq \|A_t(\tilde{\theta}^1_t)\psi^1_t - A_t(\tilde{\theta}^2_t)\psi^2_t\| + \|-B_1(\psi^1_t)^T\tilde{\theta}^1_t + B_1(\psi^2_t)^T\tilde{\theta}^2_t\| \\ &\quad + \left\| -B_1 \int_0^t h_{t-\tau}[(\psi^1_\tau)^T\tilde{\theta}^1_\tau - (\tilde{\psi}^2_\tau)^T\tilde{\theta}^2_\tau]\, d\tau \right\| \\ &\leqq \|A_t(\tilde{\theta}^1_t) - A_t(\tilde{\theta}^2_t)\| \operatorname{Max}(\|\psi^1_t\|, \|\psi^2_t\|) \\ &\quad + \operatorname{Max}(\|A_t(\tilde{\theta}^1_t)\|, \|A_t(\tilde{\theta}^2_t)\|)\|\psi^1_t - \psi^2_t\| + \|B_1\|\, \|\psi^1_t - \psi^2_t\| \operatorname{Max}(\|\tilde{\theta}^1_t\|, \|\tilde{\theta}^2_t\|) \\ &\quad + \|B_1\| \operatorname{Max}(\|\psi^1_t\|, \|\psi^2_t\|)\|\tilde{\theta}^1_t - \tilde{\theta}^2_t\| + \|B_1\| \left(\int_0^t |h_{t-\tau}|\, d\tau \right) \end{aligned}$$

(B.4) $$\begin{aligned} &\left\{ \sup_{0\leqq\tau\leqq t} \|\psi^1_\tau - \psi^2_\tau\| \sup_{0\leqq\tau\leqq t} \operatorname{Max}(\|\tilde{\theta}^1_\tau\| \cdot \tilde{\theta}^2_\tau\|) \right. \\ &\quad \left. + \sup_{0\leqq\tau\leqq t} \operatorname{Max}(\|\psi^1_\tau\|, \|\psi^2_\tau\|) \sup_{0\leqq\tau\leqq t} \|\tilde{\theta}^1_\tau - \tilde{\theta}^2_\tau\| \right\} \\ &\leqq K_n \left\{ \sup_{0\leqq\tau\leqq t} \|\psi^1_\tau - \psi^2_\tau\| + \sup_{0\leqq\tau\leqq t} \|\tilde{\theta}^1_\tau - \tilde{\theta}^2_\tau\| \right\} \end{aligned}$$

where the last line follows since $A_t(\tilde{\theta}_t)$ is Lipschitz in $\tilde{\theta}_t$ and since $\int |h_{t-\tau}|\, d\tau < \infty$ by the exponential stability of $(E-D)/D$. In (B.4), the notation K_n indicates that the Lipschitz constant depends on the maximum values of ψ, $\tilde{\theta}$ in the compact set defining the local conditions.

For (5.1b) the proof is similar on noting that, by the Cauchy-Schwartz inequality,

(B.5) $$\frac{|\psi_t^T\psi_{t-\tau}|}{r_t} \leqq \frac{|\psi_t^T\psi_t|^{1/2}|\psi_{t-\tau}^T\psi_{t-\tau}|^{1/2}}{r_t}$$

and that

$$\frac{|\psi_t^T\psi_t|}{r_t} \leqq 1 \quad \text{and} \quad \frac{|\psi_{t-\tau}^T\psi_{t-\tau}|}{r_t} \leqq 1 \quad \text{by definition of } r_t. \qquad \square$$

Appendix C.

LEMMA C.1 ("In-flight Lemma"). *Let* $S_t = \int_0^t h(\tau)\, d\omega_\tau$. *Assume that*

(C.1)
(i) $h(\tau)$ *is* $\mathscr{F}_s$ *measurable for* $\tau \leqq s$, *and*
(ii) $\int_0^\infty \|h(\tau)\|^2\, d\tau \leqq K < \infty$ a.s. *where* K *is* $\mathscr{F}_0$-*measurable.*

Then

(C.2)
(a) *given* $\Delta > 0$, $\lim_{t\to\infty} \sup_{0\leqq T\leqq\Delta} \|S_{t+T} - S_t\|^2 = 0$ a.s.
(b) $\|S_t\|^2$ *converges to a finite limit* a.s.

Proof. (a) Take any $t_0 \in [0, \Delta)$ and partition the positive real line in intervals of length Δ: $0, t_0, t_0+\Delta, t_0+2\Delta, \cdots$. Define

(C.3) $$L_j(t_0) = \sup_{T\in[0,\Delta]} \|S_{t_0+(j-1)\Delta+T} - S_{t_0+(j-1)\Delta}\|^2; \qquad j = 1, 2, \cdots$$

and let

(C.4) $$T_j(t_0) = \arg\left\{ \sup_{T\in[0,\Delta]} \|S_{t_0+(j+1)\Delta+T} - S_{t_0+(j-1)\Delta}\|^2 \right\}.$$

Also define

(C.5) $$J_n(t_0) = \sum_1^n L_j + K - \sum_{j=1}^n \int_{t_0+(j-1)\Delta}^{t_0+(j-1)\Delta+T_j} \|h(\tau)\|^2\, d\tau; \qquad n = 1, 2, \cdots.$$

Then, by assumption (i), we have

$$\begin{aligned} E\{J_{n+1}(t_0) \mid \mathscr{F}_n\} &= J_n(t_0) + E\{L_{n+1}(t_0) \mid \mathscr{F}_n\} - E\left\{ \int_{t_0+n\Delta}^{t_0+n\Delta+T_n} \|h(\tau)\|^2\, d\tau \mid \mathscr{F}_n \right\} \\ &= J_n(t_0) \quad \text{a.s.} \end{aligned}$$

Therefore, using assumption (ii), $J_n(t_0)$ is a nonnegative martingale, and hence converges almost surely to a finite limit [16]. Hence, since the sum of the last two terms in (C.5) decreases monotonically, $L_j(t_0)$ converges to zero almost surely. Now recalling (C.3), since $L_j(t_0)$ goes to zero (almost surely) for any t_0, we conclude that (a) holds.

(b) By the Ito rule [13]

(C.6) $$\|S_t\|^2 = \|S_s\|^2 + 2S_s \int_s^t h(\tau)\, d\omega_\tau + \sigma^2 \int_s^t |h(\tau)\|^2\, d\tau.$$

Now define

(C.7) $$X_t = \|S_t\|^2 + K\sigma^2 - \sigma^2 \int_0^t \|h(\tau)\|^2\, d\tau.$$

We note that X_t is positive and $\mathscr{F}_s$-measurable for $t \leqq s$. Substituting (C.6) into (C.7)

(C.8) $$\begin{aligned} X_t &= \|S_s\|^2 + 2S_s \int_s^t h(\tau)\, d\omega_\tau + \sigma^2 \int_s^t \|h(\tau)\|^2\, d\tau + K\sigma^2 - \sigma^2 \int_0^t \|h(\tau)\|^2\, d\tau \\ &= X_s + 2S_s \int_s^t h(\tau)\, d\omega_\tau. \end{aligned}$$

Taking conditional expectations yields

(C.9) $$E[X_t \mid \mathscr{F}_s] = X_s.$$

Therefore, using (C.1) in (C.7), X_t is a nonnegative martingale and hence converges [16] to a finite limit almost surely. This limit is a random variable. The last term in (C.7) is monotone nondecreasing and bounded, thus it converges also. Thus $\|S_t\|^2$ converges also to a finite random variable. $\square$

LEMMA C.2. *Let $S_t \triangleq \int_0^t \phi(t,\tau)a_\tau\, d\omega_\tau$ where $\phi(t,\tau)$ is the state transition matrix of an exponentially stable system and a_τ is uniformly bounded. Then, there exist finite random variables K_1 and K_2, such that*

(C.10) (i) $\limsup_{t\to\infty} \|S_t\|^2 < K_1$ a.s.

(C.11) (ii) $\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|S_\tau\|^2\, d\tau < K_2$ a.s.

Proof. (i) $S_t = S_s + \int_s^t \phi(t,\tau)a_\tau\, d\omega_\tau$. Hence using the Ito rule we have

$$S_t^2 = S_s^2 + 2S_s\int_s^t \phi(t,\tau)a_\tau\, d\omega_\tau + \int_s^t \sigma^2\phi(t,\tau)^2 a_\tau^2\, d\tau.$$

Let $\mathcal{F}_t$ denote the increasing σ-fields generated by $\{\omega_s, 0 \leqq s \leqq t\}$. Then

$$E\{S_t^2 \mid \mathcal{F}_s\} = S_s^2 + \sigma^2\int_s^t \phi(t,\tau)^2 a_\tau^2\, d\tau.$$

We note that, by the assumption on ϕ and a, there exists a constant $K<\infty$ such that

$$\int_0^T \phi(T,\tau)^2 a_\tau^2\, d\tau < K \text{ for all } T.$$

Let X_t be defined by

$$X_t \triangleq S_t^2 + \sigma^2 K - \sigma^2\int_0^t \phi(t,\tau)^2 a_\tau^2\, d\tau.$$

Clearly $X_t \geqq 0$ for all t and

$$E[X_t \mid \mathcal{F}_s] = S_s^2 + \sigma^2 K - \sigma^2\int_0^s \phi(s,\tau)^2 a_\tau^2\, d\tau$$
$$= X_s.$$

It follows that $(X_t, \mathcal{F}_t)$ is a positive martingale so that there exists a finite random variable X_∞ such that

$$X_t \to X_\infty \quad \text{a.s.}$$

as $t\to\infty$. Hence

$$\limsup_{t\to\infty} S_t^2 \leqq X_\infty \quad \text{a.s.}$$

(ii) This part follows as in part (i). □

Appendix D.

CONTINUOUS TIME KRONECKER LEMMA. *Assume that*

(D.1) $S_t = \int_0^t x_\tau\, d\tau$ *converges to* $S<\infty$ *as* $t\to\infty$

(D.2) $b_t \geqq 0$ *is monotone nondecreasing and* $\lim_{t\to\infty} b_t = \infty$.

Then

$$\lim_{t\to\infty}\frac{1}{b_t}\int_0^t b_\tau x_\tau\, d\tau = 0.$$

Proof. (1) We first establish that, under the same assumptions:

$$\lim_{t\to\infty} y_t = 0, \text{ where } y_t \triangleq \frac{1}{b_t}\int_0^t (S_\tau - S)\, db_\tau.$$

By (D.1), for any $\varepsilon>0$, $\exists t(\varepsilon)$ s.t. $\forall\tau \geqq t(\varepsilon)$, $|S_\tau - S| < \varepsilon$. Therefore

$$|y_t| \leqq \frac{1}{b_t}\left|\int_0^{t(\varepsilon)} (S_\tau - S)\, db_\tau\right| + \frac{1}{b_t}\int_{t(\varepsilon)}^t |S_\tau - S|\, db_\tau$$
$$\leqq \frac{1}{b_t}\cdot C(t(\varepsilon)) + \varepsilon\left[1 - \frac{b(t(\varepsilon))}{b_t}\right]$$

where

$$\frac{b(t(\varepsilon))}{b_t} < 1, \lim_{t\to\infty}\frac{b(t(\varepsilon))}{b_t} = 0$$

and $C(t(\varepsilon))$ is a constant. Therefore by (D.2), $\lim_{t\to\infty}|y_t| \leqq \varepsilon$ with ε arbitrarily small, and hence $\lim_{t\to\infty} y_t = 0$.

(2) Integrating by parts,

$$\frac{1}{b_t}\int_0^t b_\tau x_\tau\, d\tau = \frac{1}{b_t}[b_\tau S_\tau]_0^t - \frac{1}{b_t}\int_0^t S_\tau\, db_\tau$$
$$= S_t - \frac{1}{b_t}\int_0^t S_\tau\, db_\tau \quad (\text{using } S_0 = 0)$$
$$= S_t - S - \frac{1}{b_t}\int_0^t (S_\tau - S)\, db_\tau.$$

Using (D.1) and part (1) of the proof establishes the result. □

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