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Identifiability of Linear Stochastic Systems Operating Under Linear Feedback*

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New results on the identification of plants, including those having nonminimum phase operating with closed-loop controllers, rely on unravelling spectral data.

Key Words—Identification; closed-loop systems; stochastic systems.

Abstract—The identifiability of multiple input-multiple output stochastic systems operating in closed loop is considered for the case where the plant and the regulator are both linear and time-invariant. Two basic identification methods have been proposed for such systems: the joint input–output method, in which the input and output processes are modelled jointly as the output of a white noise driven system; and the direct method, in which a prediction error method is used on the input–output data as if the system were in open loop. Previously obtained identifiability results for the joint input–output method are extended to a number of new situations, including but extending beyond the identifiability results obtained with the direct method.

1. INTRODUCTION

THE IDENTIFIABILITY of multiple input–multiple output (MIMO) linear dynamic systems operating in closed loop has been the subject of much research in recent years. See Gustavsson, Ljung and Söderström (1977) for an excellent survey on this subject and also Akaike (1968); Bohlin (1971); Vorchik, Fetisov and Shteinberg (1973); Phadke (1973); Phadke and Wu (1974); Ljung, Gustavsson and Söderström (1974); Vorchik (1975); Wellstead and Edmunds (1975); Caines and Chan (1975); Gevers (1976); Söderström, Ljung and Gustavsson (1976); Ng, Goodwin and Anderson (1977); Anderson and Gevers (1979). The question at hand is whether the forward path dynamics (i.e. the plant or process dynamics) can be identified from input and output measurements despite the presence of feedback whilst the measurements are taken. We shall restrict our attention here to the case where the feedback dynamics are unknown (e.g. the feedback is a manual operator), and where no measurable external input perturbation signal can be applied for identification purposes.

For such a case two major identification methods have been proposed:

(1) The direct identification method I_1: an open loop model is identified using a prediction error method on the plant input–output data just as if the system were in open loop. This method has been proposed and extensively studied in Ljung, Gustavsson and Söderström (1974), and also in Vorchik (1975). It is based on the fact that under suitable conditions, the predicted plant model output is independent of the white noise generating the process noise, despite the presence of the feedback path.

(2) The joint input–output identification method I_2: the input–output process is first modelled jointly as the output of a system driven by white noise. The plant dynamics are subsequently derived from the joint model by matrix operations. The method has been proposed by Phadke (1973, 1974) and independently by Caines and Chan (1975). It is based on the fact that, under suitable conditions, the forward path and the feedback path can be obtained from a factorization of the joint (input–output) spectral density matrix $\phi_{yu}(z)$.

The method I_1 has the major advantage over I_2 that it allows for a wider variety of possible structures for the unknown regulator (namely the regulator can be time varying), whereas the application of I_2 is limited to systems with a linear and time-invariant regulator corrupted by noise. Actually, the feedback dynamics are not even identified with I_1, whereas I_2 identifies both the plant and the feedback dynamics, even

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though this is in most practical cases unnecessary. On the other hand, when no *a priori* knowledge is available about the structure of the system, I_2 has the advantage that it allows the use of nonparametric methods, such as spectral or covariance factorization methods, whereas I_1 requires that a parametric structure be chosen *a priori*.

The most general identifiability conditions for closed-loop systems have been obtained for the direct method by Söderström, Ljung and Gustavsson (1976) and for the joint input–output method by Ng, Goodwin and Anderson (1977). It turns out that, in the case of a time-invariant linear system with a time-invariant linear feedback (i.e. the only case where I_1 and I_2 can be compared), the sufficient conditions for identifiability derived in the above references were different. For this case the plant and the regulator can be described by (see Fig. 1)

$$y_i = F(z)u_i + m_i, \quad m_i = G(z)w_i \tag{1a}$$

$$u_i = H(z)y_i + n_i \tag{1b}$$

$u_i \in \mathbf{R}^q$ and $y_i \in \mathbf{R}^p$ are, respectively, the input and the output of the process, $F(z)$, $G(z)$ and $H(z)$ are causal real transfer function matrices, m_i and n_i are the forward path noise and the feedback path noise, respectively. Various assumptions can be made about the relations between n_i and m_i. In all cases these noises will be assumed stationary.*

We shall use the standard definitions for causal, stable, minimum phase and strictly minimum phase transfer function matrices: $G(z)$ is causal if $G(\infty) < \infty$, stable if all entries have poles inside $|z| < 1$, minimum phase if $G(z)$ is square and $G^{-1}(z)$ is causal and analytic in $|z| > 1$, strictly minimum phase if in addition $G^{-1}(z)$ is analytic in $|z| \geq 1$.

For this set-up the identifiability conditions using I_2, derived in Ng, Goodwin and Anderson (1977), require that there be no correlation at all between the noise processes $\{m_i\}$ and $\{n_i\}$, whereas for I_1 a one-sided correlation is allowed, see Söderström, Ljung and Gustavsson (1976), namely a model of the form

$$n_i = L(z)w_i + K(z)v_i \tag{2}$$

where $L(z)$ and $K(z)$ are causal stable filters, and $\{v_i\}$ is orthogonal to $\{w_i\}$. On the other hand, $G(z)$ is restricted to being minimum phase, a nontrivial restriction when $L(z)$ is nonzero.

*Since the entire theory is a second-order theory, wide sense stationarity is actually sufficient.

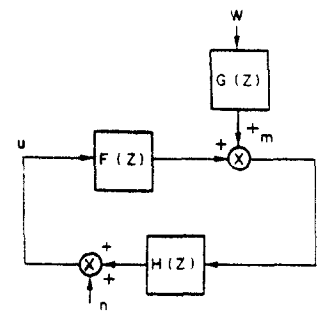

Fig. 1. Closed-loop system to which identification methods I_1 and I_2 may be applied.

In this paper we study the identifiability of MIMO systems using method I_2, motivated by the discrepancy between the results of Ng, Goodwin and Anderson (1977) and the conditions obtained for I_1 in Söderström, Ljung and Gustavsson (1976). We extend the results of Ng, Goodwin and Anderson (1977) in two directions. First we show that the unique stable minimum phase spectral factor of the joint spectrum $\phi_{yu}(z)$ used in Ng, Goodwin and Anderson (1977) may not be consistent with the *a priori* knowledge on the delay structure of the system. We therefore introduce the notion of an admissible spectral factor, namely one that is consistent with *a priori* knowledge of the delay structure. This enables us to extend the sufficient conditions obtained in Ng, Goodwin and Anderson (1977) to a broader class of situations in the case where $\{m_i\}$ and $\{n_i\}$ are uncorrelated [i.e. when $L(z) \equiv 0$ in (2)]. Next we observe that, if only a model of the plant is required and not of the regulator (as is most often the case), identification method I_2 can often still be used in the case where there is a one-sided correlation between n_i and m_i [i.e. when $L(z) \neq 0$ in (2)]. We shall show that there is an equivalence class of spectral factors of $\phi_{yu}(z)$ that is uniquely related to the same forward path dynamics [i.e. $F(z)$ and $G(z)$ in Fig. 1], but that produces different feedback dynamics. We shall derive conditions under which F and G can be uniquely identified from the joint spectrum of the (y, u) process.

In extending the results of Ng, Goodwin and Anderson (1977) we show that a linear system with a linear feedback is identifiable with I_2 whenever it is identifiable with I_1 [i.e. whenever the conditions derived in Söderström, Ljung and Gustavsson (1976) are satisfied]. But in addition we extend the applicability of method I_2 to a case where G is nonminimum phase with $L(z) = 0$, a situation that was not considered in Söderström, Ljung and Gustavsson (1976) and for which I_1 would apparently not work; we shall explain why systems with nonminimum phase G cannot be ruled out in the case of correlated noise sources.

In Section 2 we shall summarize the major results of Gevers and Anderson (1981) that will be needed in this paper. In Section 3 we present new identifiability results for the case where there is no correlation between the forward path and the feedback noises. In Section 4 we consider closed-loop systems with one-sided correlation between the regulator noise n_i and the process noise m_i. We derive conditions under which F and G can be identified for this case using I_2. In Section 5 we compare the identifiability results obtained with I_1 and I_2. The identifiability using I_2 is based on obtaining a uniquely defined spectral factor $W(z)$ from the joint spectrum $\phi_{yu}(z)$. In any practical identification experiment only estimates of $\phi_{yu}(z)$ are available. It is important to know, therefore, whether estimates $\hat{\phi}_{yu}(z)$ that are close to the true $\phi_{yu}(z)$ will yield estimates $\hat{W}(z)$ that are close to the true $W(z)$. We show in Section 6 that the estimates $\hat{W}(z)$ are a continuous function of $\hat{\phi}_{yu}(z)$. As a consequence, $W(z)$ can be estimated from measured data, and so can the transfer functions of the closed-loop system in all cases where the identifiability conditions are satisfied.

2. STATEMENT OF THE PROBLEM

We shall consider (y, u) processes, $y \in \mathbf{R}^p$, $u \in \mathbf{R}^q$, generated by a linear *closed-loop system*

$$y_i = F(z)u_i + m_i \tag{3a}$$

$$u_i = H(z)y_i + n_i \tag{3b}$$

where $F(z)$ and $H(z)$ are causal real rational transfer function matrices, and (m, n) is a stationary noise process. Here (3a) represents the forward path and (3b) the feedback path of the closed-loop system. The following two assumptions will be made throughout

A.1: There exists a delay somewhere in the loop, i.e. $F(\infty)H(\infty) = 0$, where $F(\infty) = \lim_{z\to\infty} F(z)$.

A.2: The joint process (y, u) is a stationary, full rank, mean square bounded stochastic process with rational spectrum.†

By assumption A.2 (m, n) is also a full rank process. Without loss of generality it can be represented as

$$\begin{bmatrix} m_i \\ n_i \end{bmatrix} = \begin{bmatrix} G(z) & J(z) \\ L(z) & K(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} \tag{4}$$

where G, J, L, K are causal real rational transfer function matrices, $\dim w \geq \dim m = p$, $\dim v \geq \dim n = q$, and (w, v) is a white noise process with a nonnegative definite covariance matrix Q

$$E\left\{\begin{bmatrix} w_i \\ v_i \end{bmatrix} [w_j^T \, v_j^T]\right\} = Q\delta_{ij}, \text{ with } Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0. \tag{5}$$

We shall often consider the case where $\dim w = p$ and $\dim v = q$, in which case Q is positive definite by the full rank assumption on (m, n).

From the 6-block model (3) and (4) we can derive a matrix transfer function model for the joint process (y, u)

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} = W(z)\begin{bmatrix} w_i \\ v_i \end{bmatrix}. \tag{6}$$

This model will be called a *joint model* for (y, u). Clearly the 4 blocks W_{ij} can be uniquely computed from the 6 blocks F, G, J, H, K, L, but the forward path model F, G, J, and *a fortiori* F, G, J, H, K, L, cannot be recovered from the W_{ij}s. Therefore *we shall in the following assume that* $J(z) \equiv 0$. Then‡

$$W_{11} = (I - FH)^{-1}(G + FL) \quad W_{12} = (I - FH)^{-1}FK$$
$$W_{21} = (I - HF)^{-1}(HG + L) \quad W_{22} = (I - HF)^{-1}K. \tag{7}$$

The joint spectral density matrix $\phi_{yu}(z)$ of (y, u) is

$$\phi_{yu}(z) = W(z)QW^*(z) \tag{8}$$

where $W^*(z) = W^T(z^{-1})$.

For future reference we rewrite the equations of the closed-loop system (Fig. 2).

$$y_i = F(z)u_i + G(z)w_i \tag{9a}$$

$$u_i = H(z)y_i + L(z)w_i + K(z)v_i. \tag{9b}$$

We shall frequently need the inverses of W_{11} and W_{22}, defined by (7). If $W(z)$ is not square these will be understood to be right inverses. In a first lemma we examine when these inverses exist.

Lemma 2.1. If $L = 0$, W_{11}^{-1} and W_{22}^{-1} exist by assumptions A.1 and A.2. If $L \neq 0$, W_{22}^{-1} exists if any one of the following conditions holds: (1) $W(z)$ is square, (2) $K(\infty)$ has full row rank, (3) $L(z) = L_0(z)G(z)$.

Proof: See Appendix A.

In addition to the standing assumptions A.1

†Many of the results will turn out to be valid without the restriction of rationality of the spectrum.

‡The argument z in the transfer function matrices will most often be omitted in the sequel.

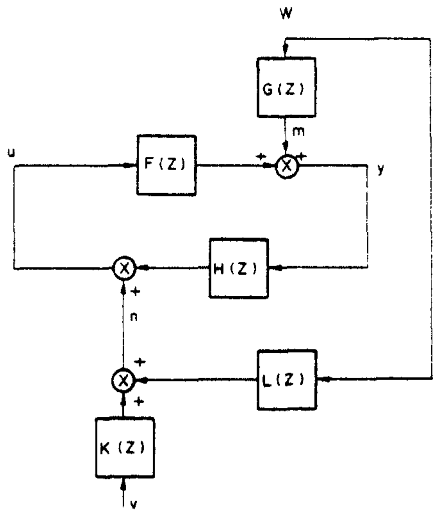

Fig. 2. More detail of closed-loop system considered; w and v are independent.

and A.2 we shall often need to make the following assumption for which sufficient conditions have just been given.

A.3: W_{22} has full normal row rank.

Lemma 2.2. Consider the closed-loop system (9) with assumptions A.1–A.3, and the corresponding $W(z)$ given by (7). The transfer function matrices F and G are uniquely expressible in terms of W as follows:

$$F = W_{12}W_{22}^{-1} \quad G = W_{11} - W_{12}W_{22}^{-1}W_{21}. \tag{10a}$$

Furthermore, if $L = 0$, then H and K are also uniquely expressed in terms of W

$$H = W_{21}W_{11}^{-1} \quad K = W_{22} - W_{21}W_{11}^{-1}W_{12}. \tag{10b}$$

The proof is trivial, using Lemma 2.1.

Comment: Lemma 2.2 is an extension of Lemma 3.1 in Ng, Goodwin and Anderson (1977) where only the case $L = 0$ was considered and where $W(z)$ was assumed square.

Since there is a one-to-one relation from $W(z)$ to $\{F, G\}$ [and to $\{F, G, H, K\}$ when $L(z) = 0$], the investigation of whether a closed-loop system is identifiable must center on an analysis of the spectral factorization of $\phi_{yu}(z)$ into $W(z)$ and Q [see (10)]. Therefore we briefly recall some important facts from spectral factorization theory. These are discrete-time extensions of continuous-time results of Youla (1961).

Spectral factorization theorem

Let $\phi(z)$ be a $n \times n$ real rational full rank spectral density matrix.

(a) There exists a unique factorization of the form $\phi(z) = \bar{W}(z)\bar{Q}\bar{W}^*(z)$, in which $\bar{W}(z)$ is $n \times n$ real, rational, stable, minimum phase and such that $\bar{W}(\infty) = I$, with $\bar{Q}$ positive definite symmetric.

(b) Any other factorization of the form $\phi(z) = W(z)QW^*(z)$ in which $W(z)$ is real rational, and Q is nonnegative definite symmetric, is such that $W(z) = \bar{W}(z)V(z)$, where $V(z)$ is a real rational scaled paraunitary matrix, i.e. $V(z)QV^*(z) = \bar{Q}$. Moreover $V(z)$ is stable if and only if $W(z)$ is stable.

(c) Any other factorization of the form $\phi(z) = W(z)QW^*(z)$ in which $W(\infty)$ is finite and nonsingular, $W(z)$ is $n \times n$ real rational, stable and minimum phase, and Q is positive definite symmetric is such that $W(z) = \bar{W}(z)T$, where T is a real nonsingular constant matrix, with $TQT^T = \bar{Q}$.

From the spectral factorization theorem it follows that the spectral factors $\{W(z), Q\}$ obtained from factoring the spectrum $\phi_{yu}(z)$ of a closed-loop process are highly nonunique. They are all related by scaled paraunitary transformations, i.e. if $\{W_1, Q_1\}$ and $\{W_2, Q_2\}$ are two spectral factors of $\phi_{yu}(z)$, then $W_2(z) = W_1(z)V(z)$ and $V(z)Q_2V^*(z) = Q_1$. Different spectral factors will normally lead to different F and G matrices via (10a), which is precisely why there is an identifiability problem.

Definition 1. We call $\bar{W}(z)$, defined in part (a) of the spectral factorization theorem, the ***normalized minimum phase spectral factor*** (NMSF) of $\phi(z)$.

Notice that the NMSF is canonical, i.e. uniquely defined by $\phi(z)$. However there are other ways of defining canonical spectral factors (Phadke, 1973). Now the delay structure of the closed-loop system and the nature of the correlation between the forward path noise and the feedback path noise imposes certain constraints on the corresponding $W(z)$ [computed from (7)] and Q. For example, if there is a delay in F, H and L, then (7) shows that $W(\infty)$ is block-diagonal. If only F has a delay, then $W(\infty)$ is lower block-triangular. If $\{w\}$ and $\{v\}$ are uncorrelated, then Q is block-diagonal. *A priori* knowledge about the delay structure or the correlation between the noises is captured in the notion of admissible spectral factors of $\phi_{yu}(z)$.

Definition 2. Given the spectrum $\phi_{yu}(z)$ of a joint process (y, u) generated by the closed-loop system (9) we shall say that the spectral factorization $\phi_{yu}(z) = W(z)QW^*(z)$ is ***admissible*** if $W(z)$ and Q are consistent with the *a priori* knowledge on the delay structure and the noise correlation of the closed-loop system.

To any set of *a priori* conditions on the closed-loop system there shall correspond a class of admissible spectral factorizations $\{W(z), Q\}$. It is always possible to define a canonical (i.e. uniquely defined) element in this class. This element will be square and will normally be minimum phase and nonsingular at $z=\infty$. For example, if it is known that $F(\infty) = H(\infty) = L(\infty) = 0$, then the NMSF will be chosen because it is consistent with this *a priori* knowledge [see (7)]. But if only F has a delay, then a canonical factorization $\tilde{W}(z)$, $\tilde{Q}$ is obtained from the NMSF $\bar{W}(z)$, $\bar{Q}$ by choosing $\tilde{W}(z) = \bar{W}(z)L$, $\tilde{Q} = I$, where L is a lower triangular matrix with positive diagonal elements, uniquely defined by $\bar{Q} = LL^T$. In the sequel we shall denote canonical admissible spectral factors by $\{\tilde{W}(z), \tilde{Q}\}$. The identifiability of a closed-loop system can now be defined precisely as follows.

Definition 3. Given the closed loop system (9) and (5), and the joint spectrum $\phi_{yu}(z)$ generated by (y, u), let $\{\tilde{W}(z), \tilde{Q}\}$ be a canonical admissible spectral factor of $\phi_{yu}(z)$, and let $\tilde{F}$ and $\tilde{G}$ be obtained from $\tilde{W}$ by (10a).

(a) We shall say that the forward path is identifiable if

$$F = \tilde{F},\ G = \tilde{G}V_1 \text{ and } V_1(z)Q_{11}V_1^\dagger(z) = \tilde{Q}_{11} \tag{11}$$

where $\tilde{Q}_{11}$ is the (1, 1)-block of $\tilde{Q}$ and $V_1(z)$ is a scaled paraunitary matrix.

(b) In the case where it is known that $L(z) \equiv 0$, we shall say that $\{F, G, H, K\}$ are identifiable if (11) holds and if

$$H = \tilde{H},\ K = \tilde{K}V_2,\ V_2(z)Q_{22}V_2^\dagger(z) = \tilde{Q}_{22}, \tag{12}$$

where $\tilde{H}$, $\tilde{K}$, $\tilde{Q}_{22}$ and $V_2(z)$ are similarly defined.

Definition 3 is a precise statement for the identifiability of a closed-loop system (the forward path, or the global model) using the joint input–output identification method I_2. We have considered that the matrices $\tilde{G}$ and $\tilde{K}$ may differ from the true values by right multiplication by a scaled paraunitary matrix. This ambiguity occurs even in the identification of open-loop systems (cf. the spectral factorization theorem) and does not influence the input–output characteristics of the model.

Comments

(1) It might happen that the class of admissible spectral factors contains only one element, or that the pairs $\{F, G\}$ obtained from all admissible spectral factors are all related by (11), in which case the system is obviously identifiable.

(2) In Ng, Goodwin and Anderson (1977), where only the case $L \equiv 0$ was considered, a closed-loop system was called identifiable if (11) and (12) hold with $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{K}$ obtained from the NMSF, rather than from an admissible canonical factorization. This definition was unduly restrictive, since a system that has no delay in either F or H would never be identifiable under this definition; this follows from the following trivial result.

Lemma 2.3. Consider the closed-loop system described by Fig. 3 and let $W(z)$ be the corresponding joint process transfer function model defined by (7) with $L = 0$. There is a delay in F and H if and only if $W(\infty)$ is block-diagonal.

Proof: Follows immediately from (7) (with $L = 0$) and (10) evaluated at $z = \infty$.

Since the NMSF has $\bar{W}(\infty) = I$, it follows immediately that the corresponding 4-block realization $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{K}$, which we shall call the *normalized minimum-phase realization* (NMR), has a delay in $\bar{F}$ and $\bar{H}$. With our definition the NMSF is not an admissible factorization if either F or H had no delay.

In Section 4 we shall present identifiability conditions for 4-block models $\{F, G, H, K\}$ (for the case where $L = 0$), and in Section 5 for the forward path model $\{F, G\}$ in the case where a one-sided correlation is allowed between the feedback noise n_i and the generating noise of the forward path w_i [i.e. $L(z) \neq 0$].

3. SOME FACTS ABOUT CLOSED-LOOP STOCHASTIC SYSTEMS

In this section we summarize without proof a number of results about closed-loop stochastic systems described by (9) and their relation to the transfer function matrix $W(z)$ of the joint

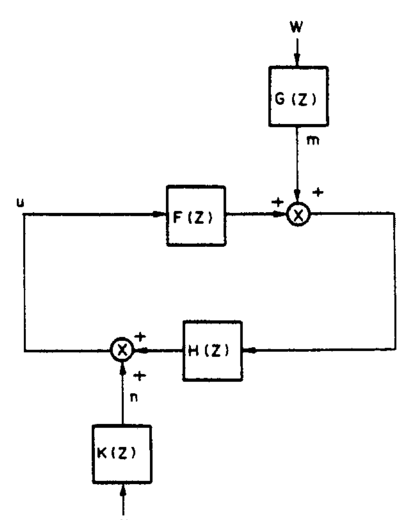

Fig. 3. Specialized version of Fig. 2 arrangement; *m* and *n* are now independent.

(y, u) process. These results have all been established in Gevers and Anderson (1981) where the proofs can be found.

We first introduce some notations and definitions.

Definition 4. Let $H(z)$ be a proper rational transfer function matrix. The *unstable part of* $H(z)$, written $H_+(z)$, is the sum of those terms in a partial fraction expansion of $H(z)$ that have poles in $|z| \geq 1$. The *stable part* of $H(z)$ is then $H_-(z) \triangleq H(z) - H_+(z)$.

We denote by $\delta[H]$ the McMillan degree of $H(z)$; see (McMillan, 1952; Kalman, 1965). Since H_+ and H_- have no common poles it is clear that $\delta[H] = \delta[H_+] + \delta[H_-]$. We shall also use the following definition.

Definition 5. (Anderson and Gevers, 1981). Let F and H be two proper transfer function matrices. There is no unstable pole-zero cancellation in the product FH if $\delta[(FH)_+] = \delta[F_+] + \delta[H_+]$.

We shall frequently make use of left or right polynomial matrix fraction descriptions of transfer function matrices (Rosenbrock, 1970; Wolovich, 1974; Kailath, 1980). If $A^{-1}B$ is a left coprime polynomial matrix fraction description of H (i.e. $H = A^{-1}B$), we shall refer to it simply as a coprime MFD. We shall also consider that the transfer function matrices G, K and L in (9) may have been obtained by referring to the outputs of the plant and/or the regulator, respectively, noise sources that enter the plant and/or the regulator at some internal part. For example; the actual physical system may be as in Fig. 4; then in (9) $F = F_2F_1$, $G = F_2G_1$, $H = H_2H_1$, $K = H_2K_1$.

After these preliminaries we now give alternate sets of necessary and sufficient conditions for stability of closed-loop systems. Theorem 3.1 is a collection of results established in Gevers and Anderson (1981) (with very minor extension) and results in Anderson and Gevers (1981).

Theorem 3.1. Consider the joint process (y, u) generated by the closed-loop system (9). Let $A^{-1}[B : C]$ be a coprime MFD of $[F : G]$, and $D^{-1}[M : N : R]$ be a coprime MFD of $[H : K : L]$. Then the joint process (y, u) is stationary if and only if either one of the following two conditions hold

(i) $\det\begin{bmatrix} A & -B \\ -M & D \end{bmatrix} \neq 0$ for $|z| \geq 1$. (13)

(ii) $\delta[F : G]_+ = \delta[F_+]$, $\delta[H : K : L]_+ = \delta[H_+]$ (14)

and the closed loop is stable.†

The closed loop is stable if and only if either

(a) $(I - FH)^{-1}$, $(I - FH)^{-1}F$, $(I - HF)^{-1}$ and $(I - HF)^{-1}H$ have all their poles in $|z| < 1$; or (15)

(b) $(I - FH)^{-1}$ has all its poles in $|z| < 1$ and there is no unstable pole-zero cancellation in FH. (16)

†By $[F : G]_+$ we denote $[F_+ : G_+]$, and similarly for other matrices.

Theorem 3.2. Consider the closed-loop system (9) with assumptions A.1 and A.2, and the corresponding $W(z)$ defined by (7). Assume that $W(z)$ is square and

(i) $\delta[F : G]_+ = \delta[G_+]$ and $\delta[H : K : L]_+ = \delta[K : L]_+$ (17)

(ii) G and K are minimum phase (18)

Then $W(z)$ is minimum phase.

Proof: From the sufficiency part of the proof of Theorem 5 in Gevers and Anderson (1981) it follows that there exist proper transfer function matrices F_1, F_2, G_1, H_1, H_2, K_1 and L_1 such that

$$F = F_2F_1,\ G = F_2G_1,\ H = H_2H_1,\ K = H_2K_1,\ L = H_2L_1 \quad (19)$$

$$\delta(F_1]_+ = \delta[G_1]_+ = \delta[H_1]_+ = \delta[K_1]_+ = \delta[L_1]_+ = 0 \quad (20)$$

$$\delta[F_2]_+ = \delta[F_2] = \delta[F : G]_+,\ \delta[H_2]_+ = \delta[H_2] = \delta[H : K : L]_+ \quad (21)$$

with no unstable zero cancellations in the products appearing in (19). Now from (7) it follows that $W(z)$ can be written

$$W(z) = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ L & K \end{bmatrix} \quad (22)$$

$W(z)$ is stable by the stationarity of (y, u). Evidently $W^{-1}(z)$ exists precisely when G^{-1} and K^{-1} exist. From (22) and (19) it follows that

$$W^{-1}(z) = \begin{bmatrix} G^{-1} & -G_1^{-1}F_1 \\ -K_1^{-1}(L_1G^{-1} + H_1) & K_1^{-1}L_1G_1^{-1}F_1 + K^{-1} \end{bmatrix}. \quad (23)$$

The result follows from (18) and (22).

Corollary 3.1. Under assumptions A.1 and A.2, $W(z)$ defined by (7) is minimum phase if F and H are stable and G and K are minimum phase.

Proof: If F and H are stable, condition (17) is satisfied.

The 'stability' assumption (17) is crucial, since otherwise minimum phase $G(z)$, $K(z)$ can yield nonminimum phase $W(z)$. Consider, for example, $F = (1/z-2)$, $G = 1$, $H = -1.5$, $K = 1$, $L = 0$ which yields the nonminimum phase

$$W(z) = \begin{bmatrix} z-2 & -1 \\ 1.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z-2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice that assumption (17), together with the closed-loop stability of the system, corresponds to a physical closed-loop system that has the form of Fig. 4, in which all the instabilities of the forward path are in F_2 and all the unstable poles of the feedback path are in H_2, with F_1, G_1, H_1 and K_1 all stable.

We shall also need the following result (Gevers and Anderson, 1981).

Theorem 3.3. Consider the closed-loop system of Fig. 3 [i.e. $L = 0$ in (9)]. Let $[F : G] = A^{-1}[B : C]$ and $[H : K] = D^{-1}[M : N]$, left coprime factorizations, and let r = highest power of z^{-1} in det CC^* and s = highest power of z^{-1} in det NN^*. Let $W(z)$ be the associated joint model and let $\phi_{yu}(z) = W(z)QW^*(z)$. Assume that

(i) $W(z)$ has minimal degree, i.e. $\delta[W(z)] = \frac{1}{2}\delta[\phi_{yu}(z)]$ (24)

(ii) $z^r \det C(z)C^*(z)$ and $z^s \det N(z)N^*(z)$ have no common zeros (25)

(iii) $F(\infty) = H(\infty) = 0$, $G(\infty)$ and $K(\infty)$ have full rank (26)

(iv) Q, defined in (5), is block-diagonal, i.e. $w_i \perp v_j$ for all i, j. (27)

Then any other square minimal degree spectral factor $\{\hat{W}(z), \hat{Q}\}$, with $\hat{W}(\infty)$ block diagonal and nonsingular, $\phi_{yu}(z)$ has the following properties:

(a) $\hat{Q}$ is block-diagonal;
(b) the scaled paraunitary transformation $V(z) = \hat{W}^{-1}(z)W(z)$ is block diagonal;
(c) the 4-block realization $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{K}$ corresponding to $\hat{W}(z)$ via (10) is such that $\hat{F} = F$, $\hat{G} = A^{-1}\hat{C}$, $\hat{H} = H$, $\hat{K} = D^{-1}\hat{N}$ with $z^r \det \hat{C} \det \hat{C}^*$ and $z^s \det \hat{N} \det \hat{N}^*$ having no common zeros.

In Gevers and Anderson (1981) we have shown that almost all feedback systems satisfy the conditions (24) and (25), which were therefore called *generic*. We shall state this as a definition for further use.

Definition 6. Consider the closed loop system of Fig. 3. Let A, B, C, D, M, N, r, s, $W(z)$ and $\phi_{yu}(z)$ be defined as in Theorem 3.3. The system

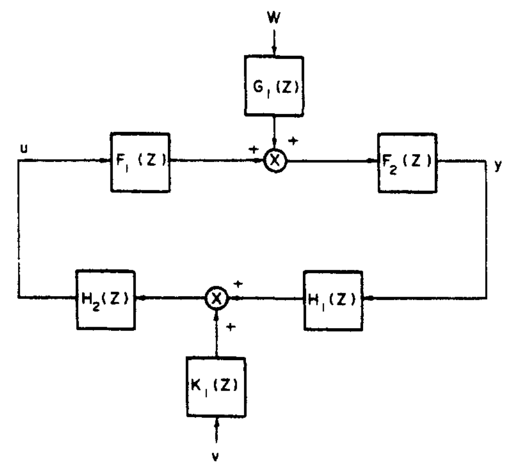

Fig. 4. Possible arrangement of noise inputs in actual physical system.

$\{F, G, H, K\}$ is called *generic* if conditions (24) and (25) hold.

In Gevers and Anderson (1981) the conditions (24) and (25) have also been expressed in terms of the poles and zeros of F, G, H, K.

4. IDENTIFIABILITY OF 4-BLOCK CLOSED-LOOP SYSTEMS

In this Section we restrict our attention to closed-loop systems in which the forward path noise m_i and the feedback path noise n_i are generated by two separate white noise sources that can have only instantaneous correlation, i.e. $L(z) = J(z) = 0$ in (4). Such a 4-block model is now represented by (Fig. 3)

$$y_i = F(z)u_i + G(z)w_i \tag{28a}$$

$$u_i = H(z)y_i + K(z)v_i \tag{28b}$$

$$E\left\{\begin{bmatrix} w_i \\ v_i \end{bmatrix}[w_j^T v_j^T]\right\} = Q\delta_{ij},\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0. \tag{28c}$$

The corresponding joint process representation is

$$W(z) = \begin{bmatrix} (I-FH)^{-1}G & (I-FH)^{-1}FK \\ (I-HF)^{-1}HG & (I-HF)^{-1}K \end{bmatrix}. \tag{29}$$

We consider here that the model (28) originates from a physical feedback system. In Gevers and Anderson (1981) we have also shown that almost all full rank stationary joint processes with a rational spectrum $\phi_{yu}(z)$ can be represented by a closed-loop model (28), a sufficient condition being that $\phi_{yu}(z)$ is positive definite (rather than just nonnegative definite) on $|z| = 1$.

We recall from Lemma 2.2 that the relationship between $\{F, G, H, K\}$ and $\{W_{ij}\}$ is one-to-one and onto. In this section we shall derive a number of conditions for the identifiability of the quadruple $\{F, G, H, K\}$; recall Definition 3 in Section 2.

Lemma 4.1. Consider the closed-loop system (28) with assumptions A.1 and A.2, and $W(z)$ defined by (29). Let $\phi_{yu}(z)$ be the joint spectrum and let $\{\bar{W}(z), \bar{Q}\}$ be a canonical admissible spectral factor of $\phi_{yu}(z)$. Then $\{F, G\}$ is identifiable if and only if the scaled paraunitary transformation $V(z)$ from $W(z)$ to $\bar{W}(z)$ is lower block-triangular, i.e.

$$W(z) = \bar{W}(z)\begin{bmatrix} V_1(z) & 0 \\ V_3(z) & V_2(z) \end{bmatrix} = \bar{W}(z)V(z). \tag{30}$$

The quadruple $\{F, G, H, K\}$ is identifiable if and only if $V(z)$ is block-diagonal, i.e.

$$W(z) = \bar{W}(z)\begin{bmatrix} V_1(z) & 0 \\ 0 & V_2(z) \end{bmatrix} \tag{31}$$

Proof

(1) ***Sufficiency***: By (30) $W_{12} = \bar{W}_{12}V_2$, $W_{22} = \bar{W}_{22}V_2$. Since $\bar{W}(z)$ is admissible and square, $\bar{W}_{22}$ is nonsingular by Lemma 2.1. Since $L = 0$, W_{22} has full row rank and so V_2 also has a right inverse. Therefore by (30)

$$\begin{aligned} F &= W_{12}W_{22}^{-1} = \bar{W}_{12}\bar{W}_{22}^{-1} = \bar{F} \\ G &= W_{11} - W_{12}W_{22}^{-1}W_{21} \\ &= (\bar{W}_{11} - \bar{W}_{12}\bar{W}_{22}^{-1}\bar{W}_{21})V_1 = \bar{G}V_1. \end{aligned}$$

Also by (8) $V_1Q_{11}V_1^\dagger = \bar{Q}_1$. Similarly, by (31), $H = \bar{H}$, $K = \bar{K}V_2$, $V_2Q_{22}V_2^\dagger = \bar{Q}_2$.

(2) ***Necessity***: Assume first that $\{F, G\}$ are identifiable. By the spectral factorization theorem $W(z) = \bar{W}(z)V(z)$. Suppose $V_{12}(z)$ is not zero. Then

$$\begin{bmatrix} I & -\bar{W}_{12}\bar{W}_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} \bar{W}_{11} - \bar{W}_{12}\bar{W}_{22}^{-1}\bar{W}_{21} & 0 \\ \bar{W}_{21} & \bar{W}_{22} \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Consider the $(1-2)$-block terms on each side, and use the fact that $\bar{W}_{12}\bar{W}_{22}^{-1} = F$, $W_{12} = (I - FH)^{-1}FK$, $W_{22} = (I - HF)^{-1}K$, and $\bar{G} = \bar{W}_{11} - \bar{W}_{12}\bar{W}_{22}^{-1}\bar{W}_{21}$. This term yields

$$(I - FH)^{-1}FK - F(I - HF)^{-1}K = \bar{G}V_{12}.$$

Using $(I - FH)^{-1}F = F(I - HF)^{-1}$, this gives $\bar{G}V_{12} = 0$. Now $\bar{G}$ is nonsingular because $\bar{W}$ is nonsingular by assumption A.2. (See the proof of Lemma 2.1.) Therefore $V_{12}(z) = 0$. Suppose now that $\{F, G, H, K\}$ are identifiable. Then (31) follows immediately from (11), (12) and (29).

The first part of Lemma 4.1 is new; the last part is a slight generalization of a result of Ng, Goodwin and Anderson (1977) where $\bar{W}(z)$ was restricted to being the NMSF of $\phi_{yu}(z)$ (cf. the discussion at the end of Section 2). The following theorems gives alternative sets of sufficient conditions for identifiability of $\{F, G, H, K\}$ under two different sets of assumptions on the structure of the closed loop system.

Theorem 4.1. Consider the closed-loop system (28) with assumptions A.1 and A.2. Then $\{F, G, H, K\}$ is identifiable if it is known *a priori* that either one of the following conditions hold:

(i) w and v are uncorrelated i.e. Q is block-diagonal. (32)

(ii) $\delta[F : G]_+ = \delta[G_+]$, $\delta[H : K]_+ = \delta[K_+]$; there is a delay in both F and H, G and K are minimum phase while $G(\infty)$ and $K(\infty)$ are nonsingular.

Proof: (i) By A.1 there is a delay in either F or H. We consider the case $F(\infty) = 0$ [The case $H(\infty) = 0$ proceeds similarly.] The proof proceeds in three steps. First we construct a lower triangular canonical admissible spectral factor $\bar{W}(z)$. Then, with the true $W(z)$, obtained from the physical $\{F, G, H, K\}$ by (29), we associate a minimum phase $\hat{W}(z)$ such that the transformation from $W(z)$ to $\hat{W}(z)$ is block-diagonal. Then we show that $\hat{W}(z) = \bar{W}(z)$. The result will then follow from Lemma 4.1.

(*a*) *Construction of a canonical admissible* $\bar{W}(z)$. Since $F(\infty) = 0$, any admissible $\bar{W}(z)$ must be lower block-triangular at $z = \infty$. We construct $\bar{W}(z)$ to be stable minimum phase with lower triangular $\bar{W}(\infty)$. Let $\{\tilde{W}(z), \tilde{Q}\}$ be the NMSF of $\phi_{yu}(z)$, and let $\tilde{Q} = LL^T$ with L lower triangular with positive diagonal elements. This defines L uniquely. Define

$$\bar{W}(z) = \tilde{W}(z)L, \qquad \bar{Q} = I. \tag{34}$$

Then $\bar{W}(z)$ is stable, minimum phase and $\bar{W}(\infty) = L$, nonsingular, $\bar{Q}$ is diagonal, and $\phi_{yu}(z) = \bar{W}(z)\bar{W}^*(z)$. Therefore $\{\bar{W}, \bar{Q}\}$ is an admissible factorization, and since it is uniquely defined it is canonical. Notice that the NMSF is in general not admissible, because of the failure of $\tilde{Q}$ to be block-diagonal.

(*b*) *Construction of* $\hat{W}(z)$. We follow a construction similar to that used in Ng, Goodwin and Anderson (1977). Let $[F : G] = A^{-1}[B : C]$, a left coprime MFD, and similarly $[H : K] = D^{-1}[M : N]$. Then there exist square polynomial matrices V_A and V_D such that det V_A and det V_D

have all zeros at the origin, and that $V_A A$ and $V_D D$ are polynomial matrices of the form $z^{n_A}I +$ lower order terms and $z^{n_D}I +$ lower order terms, respectively (Ng, Goodwin and Anderson, 1977). Then

$$W(z) = \begin{bmatrix} A & -B \\ -M & D \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$$
$$= \begin{bmatrix} V_A A & -V_A B \\ -V_D M & V_D D \end{bmatrix}^{-1} \begin{bmatrix} V_A C & 0 \\ 0 & V_D N \end{bmatrix}. \quad (35)$$

By Theorem 3.1 the denominator matrix in (35) has all zeros in $|z| < 1$, because

$$\det \begin{bmatrix} V_A A & -V_A B \\ -V_D M & V_D D \end{bmatrix} = \det V_A \det V_D \det \begin{bmatrix} A & -B \\ -M & D \end{bmatrix}.$$

Also $\lim_{z\to\infty} (V_A A)^{-1}(V_A C) = G(\infty) = \lim_{z\to\infty} z^{-n_A} V_A C.$ Hence $\lim_{z\to\infty} z^{-n_A} V_A C$ is finite, and similarly $\lim_{z\to\infty} z^{-n_D} V_D N = K(\infty)$ is finite. Furthermore $V_A C$ and $V_D N$ have full row rank by Assumption A.2. Therefore [see Lemma 2 of Appendix in Ng, Goodwin and Anderson (1977)] there exist square matrix polynomials $\bar{C}$ and $\bar{N}$ such that

$$(V_A C) Q_{11} (V_A C)^* = \bar{C}\bar{C}^*, \quad (V_D N) Q_{22} (V_D N)^* = \bar{N}\bar{N}^* \quad (36)$$

with $\det \bar{C} \det \bar{N} \neq 0$ in $|z| > 1$, and $\lim_{z\to\infty} z^{-n_A}\bar{C}$ and $\lim_{z\to\infty} z^{-n_D}\bar{N}$ lower triangular with positive diagonal elements. Now define

$$\hat{W}(z) \triangleq \begin{bmatrix} V_A A & -V_A B \\ -V_D M & V_D D \end{bmatrix}^{-1} \begin{bmatrix} \bar{C} & 0 \\ 0 & \bar{N} \end{bmatrix} \quad (37)$$

$\hat{W}(z)$ has the following properties.

(1) $\hat{W}(z)$ is square, stable, and is minimum phase by construction.

(2) $$\hat{W}(z)\hat{W}^*(z) = W(z) \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} W^*(z) = \phi_{yu}(z).$$

Recall that Q is block-diagonal by assumption.

(3) $$\hat{W}(\infty) = \lim_{z\to\infty} \begin{bmatrix} I & -A^{-1}B \\ -D^{-1}M & I \end{bmatrix}^{-1} \begin{bmatrix} (V_A A)^{-1}\bar{C} & 0 \\ 0 & (V_D D)^{-1}\bar{N} \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ -H(\infty) & I \end{bmatrix}^{-1} \begin{bmatrix} \lim_{z\to\infty} z^{-n_a}\bar{C} & 0 \\ 0 & \lim_{z\to\infty} z^{-n_D}\bar{N} \end{bmatrix}.$$

Hence $\hat{W}(\infty)$ is finite and lower triangular with positive diagonal elements.

(4) With $V_1(z) = \bar{C}^{-1} V_A C$, $V_2(z) = \bar{N}^{-1} V_D N$, one has $V_1(z)$, $V_2(z)$ proper with $V_1(z) Q_{11} V_1^*(z) = I$, $V_2(z) Q_{22} V_2^*(z) = I$, and

$$W(z) = \hat{W}(z) \begin{bmatrix} V_1(z) & 0 \\ 0 & V_2(z) \end{bmatrix}. \quad (38)$$

(c) $\hat{W}(z) = \bar{W}(z)$. By the spectral factorization theorem, since $\hat{W}(z)$ is square, stable, minimum phase, $\hat{W}(\infty)$ is finite and nonsingular and $\hat{Q} = I$, there exists a constant nonsingular T such that

$$\hat{W}(z) = \bar{W}(z) T, \qquad TT^T = \bar{Q}. \quad (39)$$

Now since $\bar{W}(\infty) = I$, $T = \hat{W}(\infty)$ and hence T is lower triangular with positive diagonal elements; since $TT^T = \bar{Q}$, therefore $T = L$ [see part (b) above], because this factorization is unique. Hence $\hat{W}(z) = \hat{W}(z)$ by (34) and (39). The result then follows by (38) and Lemma 4.1.

(ii) By Theorem 3.2, $W(z)$ is minimum phase. By (29) and the assumptions, $W(\infty)$ is finite and nonsingular

$$W(\infty) = \begin{bmatrix} G(\infty) & 0 \\ 0 & K(\infty) \end{bmatrix}. \quad (40)$$

Therefore, by the spectral factorization theorem, there exists a constant nonsingular T such that $W(z) = \bar{W}(z)T$. By (40) T is block-diagonal. Hence by Lemma 4.1 $\{F, G, H, K\}$ are identifiable.

Comment: For the construction of $W(z)$ it is clear that one must know whether the delay is in F or in H (cf. Assumption A.1).

Corollary 4.1. Consider the closed-loop system (28) with assumption A.2. Then $\{F, G\}$ is identifiable if it is known *a priori* that $\delta[F:G]_+ = \delta[G_+]$, $\delta[H:K]_+ = \delta[H_+]$, G and K are minimum phase with $G(\infty)$ and $K(\infty)$ nonsingular and there is a delay in F.

Proof: Same proof as part (ii) of Theorem 3.2, save that

$$W(\infty) = \begin{bmatrix} G(\infty) & 0 \\ H(\infty)G(\infty) & K(\infty) \end{bmatrix}.$$

Therefore T is lower block-triangular, and the result follows from (30) in Lemma 4.1.

Comment: Part (i) of Theorem 4.1 represents a strict improvement over the results of Ng, Goodwin and Anderson (1977) where a delay was required in both F and H, rather than in one of both. Part (ii) and Corollary 4.1 are new results. It could be argued that the absence of correlation between the process noise and the regulator noise [i.e. condition (32)] is difficult to check *a priori*. The following theorem shows that, under certain conditions, the absence of correlation can be checked from the spectral density itself.

Theorem 4.1. Consider the closed-loop system (28) with assumption A.2. Assume also that $F(\infty) = H(\infty) = 0$, $G(\infty)$ and $K(\infty)$ are nonsingular, and that the associated $W(z)$ has minimal degree, i.e. $\delta[W] = \frac{1}{2}\delta[\phi_{yu}]$, a generic situation. Let $\bar{W}(z)$, $\bar{Q}$ be the NMSF of $\phi_{yu}(z)$, and $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{K}$ the corresponding NMR. Let $[\bar{F} : \bar{G}] = \bar{A}^{-1}[\bar{B} : \bar{C}]$ and $[\bar{H} : \bar{K}] = \bar{D}^{-1}[\bar{M} : \bar{N}]$, coprime factorizations, and let $r = \deg|\bar{C}(z)|$ and $s = \deg|\bar{N}(z)|$. If

(i) $\bar{Q}$ is block-diagonal (41)

(ii) $z^r|\bar{C}(z)\bar{C}^*(z)|$ and $z^s|\bar{N}(z)\bar{N}^*(z)|$ have no common zeros. (42)

then Q is block-diagonal, $\{F, G, H, K\}$ is identifiable and is generic. (See Definition 6.)

Proof: The NMSF has $\bar{W}(\infty) = I$. Therefore by Lemma 2.2 $\bar{F}(\infty) = \bar{H}(\infty) = 0$, and $\bar{G}(\infty) = \bar{K}(\infty) = I$. By the assumptions $W(\infty)$ is block diagonal and nonsingular. Since W is also of minimal degree, the result follows from Theorem 3.3 and Lemma 4.1.

Comments

(1) A result similar to Theorem 4.2 was first obtained by Sin and Goodwin (1980). However, they implicitly assume that $W(z)$ has minimal degree, and the genericity condition (42) (see also the comment at the end of Section 3) is not explicitly stated. Instead it is assumed that a state-variable representation, obtained by combining a minimal state-variable model of the forward path and a minimal state-variable model of the feedback path, is itself minimal. Condition (42) is the required condition that will guarantee this.

(2) An important consequence of Theorem 4.2 is that the absence of correlation between the process noise m_i and the regulator noise n_i (or equivalently the block-diagonality of Q) can be checked *a posteriori* from the computed factorization $\{\bar{W}(z), \bar{Q}\}$ of $\phi_{yu}(z)$ provided G and K are square, hence the term "checkable condition" used by Sin and Goodwin (1980). However, one should bear in mind that the theorem imposes other conditions which are not checkable from the data, most importantly the presence of a delay in F and H.

(3) In any real-life identification context, where $\phi_{yu}(z)$ is not known but estimated from the (y, u) data, the question arises as to whether the NMSF $\{\hat{\bar{W}}(z), \hat{\bar{Q}}\}$, computed from the estimated $\hat{\phi}_{yu}(z)$, will have the desired property that $\hat{\bar{Q}}$ is almost block-diagonal when n_i and m_i are uncorrelated. We shall show in Section 4 that this is in fact so, because $\bar{W}$, $\bar{Q}$ are continuous functions of $\phi_{yu}(z)$; hence if $\hat{\phi}_{yu}(z)$ is close to the true $\phi_{yu}(z)$, then $\{\hat{\bar{W}}, \hat{\bar{Q}}\}$ will be close to the true $\{\bar{W}, \bar{Q}\}$.

5. IDENTIFIABILITY OF CLOSED-LOOP SYSTEMS WITH ONE-SIDED NOISE CORRELATION

We now examine closed-loop systems with a one-sided correlation between the regulator noise n and the process driving noise w, i.e.

$$n_i = L(z)w_i + K(z)v_i \tag{43}$$

where L and K are causal, and w_i and v_j are uncorrelated, except possibly for $i = j$.

The closed-loop system is then given by (9). To every such 5-block model there corresponds a unique $W(z)$ through (7), and therefore a unique $\phi_{yu}(z)$ by (8). By Lemma 2.2, to a given $W(z)$ there corresponds a unique pair $\{F, G\}$. The question we examine in this section is what *a priori* knowledge on the structure of the system is required to guarantee that $\bar{W}(z)$, $\bar{Q}$ will be such that F, G and Q_{11} can be identified [cf. (11)].

One might think that the easy way of solving this problem is to transform the 5-block model of Fig. 2 into an equivalent 4-block model like Fig. 3 and to apply the results of Section 4 to this 4-block model. By applying (10) to the expression (7) of $W(z)$ we obtain the following equivalent 4-block model

$$\hat{F} = F, \qquad \hat{G} = G \tag{44a}$$

$$\hat{H} = (I + LG^{-1}F)^{-1}(H + LG^{-1}), \qquad \hat{K} = (I + LG^{-1}F)^{-1}K. \tag{44b}$$

However, the 4-block model $\{F, \hat{G}, \hat{H}, \hat{K}\}$ may not be stable even though the corresponding 5-block model $\{F, G, H, K, L\}$ is stable.

Example 5.1

$$F = \frac{0.9}{z-2}, \qquad H = \frac{-2}{z+0.5}, \qquad G = \frac{z}{z-0.5},$$
$$K = \frac{z}{z+0.5}, \qquad L = \frac{1}{z+0.5}.$$

The 5-block model is stable because G, K, L, $(I - FH)^{-1}$, $(I - FH)^{-1}F$ and $(I - HF)^{-1}H$ are all stable (see Theorem 3.1).

The equivalent 4-block models has $\hat{F} = F$, $\hat{G} = G$

$$\hat{H} = \frac{(z-2)(z+0.5)}{d(z)}, \qquad \hat{K} = \frac{z^2(z-2)}{d(z)}$$

with $d(z) = z^3 - 1.5z^2 - 0.1z - 0.45$. Now

$$(I - \hat{F}\hat{H})^{-1}\hat{F} = \frac{0.9d(z)}{z(z-2)(z^2 - 1.5z + 0.8)}.$$

It has an unstable pole at $z = 2$. Therefore the closed-loop system is unstable by Theorem 3.1.

Example 5.1 shows that 5-block closed-loop systems require a separate treatment. For such systems, Söderström, Ljung and Gustavsson (1976) have obtained the following identifiability conditions using I_1.

Theorem 5.1. Consider the closed-loop system (9) with the assumptions A.1 to A.3. The forward path $\{F, G\}$ is identifiable using a prediction error method (I_1) if

(1) L is stable. (45a)

(2) Q is block-diagonal, i.e. $w_i \perp v_i$. (45b)

(3) $G(z)$ is square, strictly minimum phase and $G(\infty)$ is nonsingular. (45c)

(4) There is a delay either in F or in H and L. (45d)

(5) $G^{-1}F$ is stable. (45e)

(6) The parameterized model $F_\theta(z)$, $G_\theta(z)$ used in the prediction error method contains the true $F(z)$, $G(z)$ for at least one value of the parameter vector θ. (45f)

Comments

(1) Theorem 5.1 is a special case of the result proved in Söderström, Ljung and Gustavsson (1976) which applies also to the case of time-varying regulators.

(2) Assumption (6) applies also to all the results obtained for method I_2 insofar as a parametric method is used. It has not been explicitly stated because I_2 can also be used with correlation methods which do not require an *a priori* parameterization.

(3) Assumption (5) was not explicitly stated in Söderström, Ljung and Gustavsson (1976), but as confirmed in an exchange of letters with Söderström (personal communication) the condition is actually used in the proof of the theorem. This assumption is necessary for the stability of the predictor: $\hat{w}_i = G^{-1}y_i - G^{-1}Fu_i$. The important point is that the product $G^{-1}F$ is stable, but unstable pole-zero cancellations in $G^{-1}F$ are acceptable. This allows for situations as in Fig. 4 where F_1 and G_1 are stable, and all instabilities are in F_2: see the comment after Corollary 3.1. Consider, for example the model $A(z)y_i = B(z)u_i + w_i$, in which $A(z)$, $B(z)$ are polynomials and where $|A(z)|$ has an unstable zero (Söderström, personal communication).

(4) The assumption that $G(z)$ is minimum phase is a restrictive one in the case where $L(z) \neq 0$. It might be argued that if the physical $G(z)$ is nonminimum phase, one can replace it by the equivalent minimum phase $\bar{G}(z)$ producing the same spectrum. However consider that the physical noises m_i, n_i have the following structure with stable G, L, K and nonminimum phase G

$$\begin{bmatrix} m_i \\ n_i \end{bmatrix} = \begin{bmatrix} G(z) & 0 \\ L(z) & K(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix}, \quad Q = I. \tag{46}$$

Let $\phi_{mn}(z)$ be the spectrum of the joint process (m, n). We show that, in general, one cannot find a stable triple $\{\bar{G}, \bar{L}, \bar{K}\}$ producing the same $\phi_{mn}(z)$ with $\bar{G}$ minimum phase

$$\phi_{mn} = \begin{bmatrix} GG^* & GL^* \\ LG^* & LL^* + KK^* \end{bmatrix} = \begin{bmatrix} \bar{G}\bar{G}^* & \bar{G}\bar{L}^* \\ \bar{L}\bar{G}^* & \bar{L}\bar{L}^* + \bar{K}\bar{K} \end{bmatrix}. \tag{47}$$

By (47) $GG^* = \bar{G}\bar{G}^*$. Hence $G = \bar{G}V$ for some paraunitary $V(z)$. Since G and $\bar{G}$ are stable, and $\bar{G}$ is minimum phase, V has all its poles in $|z| \leq 1$. Again by (47) $\bar{G}\bar{L}^* = GL^* = \bar{G}VL^*$. Therefore $\bar{L} = LV^*$. Since V has all its poles in $|z| \leq 1$, V^* has all its poles in $|z| \geq 1$, and hence $\bar{L}$ is unstable if L is arbitrary.

This shows that, when $L \neq 0$ but otherwise arbitrary, a nonminimum phase $G(z)$ in the physical plant noise cannot be replaced by its minimum phase counterpart. Such a physical nonminimum phase $G(z)$ could arise, for example, if the process noise enters the plant in front of some nonminimum phase part. Consider Fig. 4 when F_2 is nonminimum phase.

Lemma 5.1. Consider the closed-loop system (9) with the assumptions A.1–A.3, and $W(z)$ defined by (7). Let $\phi_{yu}(z)$ be the joint spectrum and let $\{\tilde{W}(z), \tilde{Q}\}$ be a canonical admissible spectral factor of $\phi_{yu}(z)$. Then $\{F, G\}$ is identifiable if and only if the scaled paraunitary transformation $V(z)$ from $W(z)$ to $\tilde{W}(z)$ is lower-triangular, i.e.

$$W(z) = \tilde{W}(z)\begin{bmatrix} V_1(z) & 0 \\ V_3(z) & V_2(z) \end{bmatrix}. \tag{48}$$

Proof: By assumption A.3 W_{22} has a right inverse, and therefore $V_{22}(z)$ has a right inverse. The remainder of the proof is identical to that of Lemma 4.1, this proof being unaffected by the presence of $L(z)$.

The next Theorem gives a first set of sufficient conditions for identifiability of $\{F, G\}$.

Theorem 5.2. Consider the closed-loop system (9) with assumptions A.1 and A.2. The forward path $\{F, G\}$ is identifiable using I_2 if

(1) $L(z) = L_0(z)G(z)$, i.e. $n_i = L_0(z)m_i + K(z)v_i$, with L_0 causal. (49a)

(2) Q is block-diagonal, i.e. $w_i \perp v_j$ for all i, j. (49b)

(3) $\delta[H : K : L_0]_+ = \delta[H_+]$. (49c)

(4) There is a delay either in F or in H and L_0. (49d)

(5) F is stable.

Note: Equation (49c) is implied by assumption A.2; it has been included here only for comparison purposes [cf. (45a) in Theorem 5.1].

Proof: First notice that by Theorem 3.1, stability of F implies that of G. The proof follows a line similar to that of Theorem 4.1.

Let $A^{-1}[B : C]$ and $D^{-1}[M : N : R]$ be left coprime MFDs of $[F : G]$ and $[H : K : L_0]$, respectively. Let $V_A(z)$ and $V_D(z)$ be square polynomial matrices with all determinantal zeros at the origin, and such that $V_AA = z^{n_A}I +$ lower order terms, and $V_DD = Z^{n_D}I +$ lower order terms. Let $W(z)$ be associated with F, G, H, K, L through (7). Then

$$W(z) = \begin{bmatrix} V_AA & -V_AB \\ -V_DM & V_DD \end{bmatrix}^{-1} \begin{bmatrix} V_AC & 0 \\ V_DRG & V_DN \end{bmatrix}. \tag{50}$$

(*a*) *Construction of a minimum phase* $\hat{W}(z)$. By the same argument as in the proof of Theorem 4.1 there exist square polynomial matrices $\bar{C}$ and $\bar{N}$ with $\lim_{z\to\infty} z^{-n_A}\bar{C}$ and $\lim_{z\to\infty} z^{-n_D}\bar{N}$ finite and nonsingular, with $\det \bar{C} \det \bar{N} \neq 0$ in $|z| > 1$, and such that

$$V_ACQ_{11}(V_AC)^* = \bar{C}\bar{C}^*, \qquad V_DNQ_{22}(V_DN)^* = \bar{N}\bar{N}^*. \tag{51}$$

Now define

$$\hat{W}(z) = \begin{bmatrix} V_AA & -V_AB \\ -V_DM & V_DD \end{bmatrix}^{-1} \begin{bmatrix} \bar{C} & 0 \\ V_DR(V_AA)^{-1}\bar{C} & \bar{N} \end{bmatrix} \tag{52}$$

$\hat{W}(z)$ has the following properties, as arguments similar to those used in earlier theorems show.

(i) $\hat{W}(z)$ is square, stable and minimum phase

(ii) $$\hat{W}(z)\hat{W}^*(z) = W(z)\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} W^*(z) = \phi_{yu}(z). \tag{53}$$

(iii) If $F(\infty) = H(\infty) = L_0(\infty) = 0$, $\hat{W}(\infty)$ is block-diagonal. It is lower block-triangular if $F(\infty) = 0$, and upper block-triangular if $H(\infty) = L_0(\infty) = 0$. In all cases it is finite and nonsingular.

(iv) With $V_1(z) = \bar{C}^{-1}V_AC$, $V_2(z) = \bar{N}^{-1}V_DN$, one has $V_1(z)$, $V_2(z)$ proper with $V_1(z)Q_{11}V_1^*(z) = I$, $V_2(z)Q_{22}V_2^*(z) = I$, and

$$W(z) = \hat{W}(z)\begin{bmatrix} V_1(z) & 0 \\ 0 & V_2(z) \end{bmatrix}. \tag{54}$$

(*b*) *Construction of a canonical admissible* $\bar{W}(z)$. Let $\bar{W}(z)$, $\bar{Q}$ be the NMSF of $\phi_{yu}(z)$. If $F(\infty) = H(\infty) = L_0(\infty) = 0$, take $\bar{W}(z) = \bar{W}(z)$, $\bar{Q} = \bar{Q}$. If $F(\infty) = 0$, take $\bar{W}(z) = \bar{W}(z)L$, $\bar{Q} = I$, where $LL^T = \bar{Q}$ and L is the unique lower triangular factor of Q with positive diagonal elements. If $H(\infty) = L_0(\infty) = 0$, take $\bar{W}(z) = \bar{W}(z)U$, $\bar{Q} = I$, where $UU^T = \bar{Q}$, U being upper triangular with positive diagonal elements. In all three cases $\bar{W}(\infty)$ is finite and nonsingular. $\{\bar{W}(z), \bar{Q}\}$ are clearly admissible in the second and third case, since $\bar{W}(\infty)$ is consistent with the delay structure and $\bar{Q} = I$. As for the first case, by the spectral factorization theorem, $\hat{W}(z) = \bar{W}(z)T$, for some constant T for which $TT^T = \bar{Q}$. Letting $z \to \infty$, it follows that T, and hence $\bar{Q}$, is block-diagonal. This establishes admissibility.

(c) *Relation between* $\hat{W}(z)$ *and* $\bar{W}(z)$. By arguments like those in the proof of Theorem 4.1, $W(z)$ is related to a canonical admissible spectral $\bar{W}(z)$ by a block-diagonal transformation; hence $\{F, G\}$ is identifiable by Lemma 5.1.

Comparison between Theorem 5.1 (*method* I_1) *and Theorem* 5.2 (*method* I_2). The relation (49a) is a special case of the model (43). It establishes a relation directly between the physical noises n_i and m_i. With I_1 the more general model (43) is allowed, but $G(z)$ is assumed minimum phase and $L(z)$ stable; this is a more restrictive assumption, because then one can always define $L_0(z) = L(z)G^{-1}(z)$, with $L_0(z)$ causal and stable; hence (49a) and (49c) are always satisfied under assumptions of Theorem 5.1. The structural assumptions (delays and block-diagonal Q) are identical for both methods. The major difference therefore is that Theorem 5.2 requires a stable F, while Theorem 5.1 requires G minimum phase and $G^{-1}F$ stable. We shall show in Theorem 5.3 that, if G is minimum phase, $\{F, G\}$ is identifiable with I_2 under the same assumptions as those required for I_1 in Theorem 5.1.

We show now by a counterexample that if G is nonminimum phase, $\{F, G\}$ is not identifiable with I_2 if the conditions (49a, c) are violated.

Example 5.2

$F = z^{-1}$, $G = (1 - 2z^{-1}/1 - 0.5z^{-1})$, $H = 0.5z^{-1}$, $K = 1$, $L = z^{-1}$, $Q = I$. Notice that F is stable, G is nonminimum phase, $L_0 = LG^{-1}$ is unstable and does not obey condition (49c). The closed-loop is stable

$$W(z)=\begin{bmatrix}\dfrac{1-2z^{-1}+z^{-2}-0.5z^{-3}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}} & \dfrac{z^{-1}}{1-0.5z^{-2}}\\ \dfrac{1.5z^{-1}-1.5z^{-2}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}} & \dfrac{1}{1-0.5z^{-2}}\end{bmatrix}$$

The NMSF of $\phi_{yu}(z)=W(z)W^*(z)$ should be admissible because there is a delay in F, H and L

$$\bar{W}(z)=\begin{bmatrix}\dfrac{1-0.8z^{-1}+0.4z^{-2}-0.2z^{-3}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}} & \dfrac{0.4z^{-1}-0.2z^{-2}-0.15z^{-3}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}}\\ \dfrac{0.9z^{-1}-0.6z^{-2}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}} & \dfrac{1-0.2z^{-1}-0.45z^{-2}}{1-0.5z^{-1}-0.5z^{-2}+0.25z^{-3}}\end{bmatrix}$$

and

$$\bar{Q}=\begin{bmatrix}3.4 & -1.2\\ -1.2 & 1.6\end{bmatrix}.$$

We notice that $\bar{Q}$ is not block diagonal and $\bar{W}_{12}\bar{W}_{22}^{-1}\neq F$.

In the next theorem we drop the constraint that F is stable, but introduce a minimum phase constraint on G.

Theorem 5.3. Consider the closed-loop system (9) with assumptions A.1–A.3. The forward path $\{F, G\}$ is identifiable using I_2 if

(1) $\delta[H:K:L]_+=\delta[H_+]$. (55a)

(2) Q is block-diagonal.

(3) $G(z)$ is square, strictly minimum phase and $G(\infty)$ is nonsingular. (55c)

(4) There is a delay either in F or in H and L. (55d)

(5) $\delta[F:G]_+=\delta[G_+]$. (55e)

Note: Again condition (55e) is implied by A.2 and is included only for comparison purposes.

Proof: Let $A^{-1}[B:C]$ and $D^{-1}[M:N:R]$ be left coprime factorizations of $[F:G]$ and $[H:K:L]$, respectively. From (55c, e) it follows that $|C|$ has all its zeros in $|z|<1$.

The remainder of the proof follows a line completely parallel to the proof of Theorem 5.2. Therefore we only sketch it and stress the differences. With V_A and V_D as before we have

$$W(z)=\begin{bmatrix}V_AA & -V_AB\\ -V_DM & V_DD\end{bmatrix}^{-1}\begin{bmatrix}V_AC & 0\\ V_DR & V_DN\end{bmatrix}. \tag{56}$$

With $\bar{C}$ and $\bar{N}$ as in Theorem 5.2 we define a minimum phase

$$\hat{W}(z)=\begin{bmatrix}V_AA & -V_AB\\ -V_DM & V_DD\end{bmatrix}^{-1}\begin{bmatrix}\bar{C} & 0\\ V_DR(V_AC)^{-1}\bar{C} & \bar{N}\end{bmatrix} \tag{57}$$

$\hat{W}(z)$ is square. By the stationarity of the joint process the inverted matrix in (57) has all determinantal zeros in $|z|<1$, while $\det(V_AC)$ has all its zeros in $|z|<1$. Hence $\hat{W}(z)$ is stable; it is minimum phase because $\det\bar{C}\det\bar{N}\neq 0$ in $|z|\geq 1$

$$\hat{W}(\infty)=\begin{bmatrix}I & -F(\infty)\\ H(\infty) & I\end{bmatrix}^{-1}\times\begin{bmatrix}\lim\limits_{z\to\infty} z^{-n_A}\bar{C} & 0\\ L(\infty)G^{-1}(\infty)\lim z^{-n_A}\bar{C} & \lim z^{-n_D}\bar{N}\end{bmatrix} \tag{58}$$

with $G(\infty)$ finite and nonsingular by assumption. The remainder of the proof is identical to that of Theorem 5.2.

Comment: The assumptions of Theorem 5.3 are identical to those of Theorem 5.1 established for method I_1, except that a certain type of instability is allowed for $L(z)$ in Theorem 5.3: compare (45a) and (55a). Assumption (55c) and (55e) ensure that $G^{-1}F$ is stable (see the proof of Theorem 3.2).

We show now by a counterexample that, even with a minimum phase $G(z)$, the 'stability' condition (55e) on $F(z)$ is required to guarantee identifiability with I_2. We recall our comment made above that this condition is also necessary for I_1.

Example 5.3

Same F, G, H, K, L as in Example 5.1, and $Q=I$. Notice that F is unstable, while G is stable, so (55e) is violated

$$(I-HF)^{-1}=\frac{(z+0.5)(z-2)}{z^2-1.5z+0.8}.$$

This is stable, and since there is no unstable pole-zero cancellation in HF, the closed-loop is stable. The corresponding $W(z)$ is

$$W(z)=\frac{1}{d(z)}\begin{bmatrix}\dfrac{z^3-1.5z^2-0.1z-0.45}{z-0.5} & 0.9z\\ \dfrac{-(z-2)(z+0.5)}{z-0.5} & z(z-2)\end{bmatrix}$$

where $d(z) = z^2 - 1.5z + 0.8$. Notice that $W(z)$ is stable but not minimum phase. Also $W(\infty) = I$. The corresponding NMSF should be admissible because $F(\infty) = H(\infty) = L(\infty) = 0$. Now

$$\bar{W}(z) = \frac{1}{d(z)} \times \begin{bmatrix} \frac{z^3 - 0.3z^2 - 0.04z - 0.18}{z - 0.5} & \frac{0.3z^2 - 0.48z - 0.135}{z - 0.5} \\ \frac{-1.6z^2 + 0.6z + 0.4}{z - 0.5} & \frac{z^3 - 2.2z^2 + 1.45z + 0.3}{z - 0.5} \end{bmatrix}$$

and

$$\bar{Q} = \begin{bmatrix} 3.4 & -1.2 \\ -1.2 & 1.6 \end{bmatrix}.$$

Again we remark that $\bar{Q}$ is not diagonal and $\bar{W}_{12}\bar{W}_{22}^{-1} \neq F$.

We now give a last set of sufficient conditions in which we impose further constraints on F, G, H and K than earlier, but drop the requirement that $\{w_i\}$ and $\{v_i\}$ be uncorrelated.

Theorem 5.4. Consider the closed-loop system (9) obeying assumptions A.1 and A.2. Then $\{F, G\}$ is identifiable using I_2 if it is known *a priori* that

(1) $\delta[F : G]_+ = \delta[G_+]$, $\delta[H : K : L]_+ = \delta[H_+]$. (59a)

(2) G and K are minimum phase, while $G(\infty)$ and $K(\infty)$ are nonsingular. (59b)

(3) $F(\infty) = 0$. (59c)

Proof: By Theorem 3.2 $W(z)$ is minimum phase. By (7) and the assumptions, $W(z)$ is finite and nonsingular

$$W(\infty) = \begin{bmatrix} G(\infty) & 0 \\ H(\infty)G(\infty) + L(\infty) & K(\infty) \end{bmatrix}. \quad (60)$$

Therefore, $W(z) = \bar{W}(z)T$ for some constant T. Letting $z \to \infty$ shows that $T = W(\infty)$. The result then follows by Lemma 5.1.

6. CONTINUITY OF SPECTRAL FACTORS

In this section we first show that the NMSF $\{\bar{W}(z), \bar{Q}\}$ is a continuous function of the joint spectrum $\phi_{yu}(z)$, so that consistent estimates of $\phi_{yu}(z)$ will yield consistent estimates of $\bar{W}(z)$ and $\bar{Q}$. The continuity result will be established in the more general context of not necessarily rational, complex spectral density matrices. The real rational spectrum is a special case.

As a notational convenience, let us define ω such that $\exp(i\omega) = z$, so that if $|z| = 1$, ω is real. We then restrict attention to those spectra $\phi(\omega)$ for which

$$0 < c_1 I \leq \phi(\omega) \leq c_2 I < \infty \quad (61)$$

for some positive constants c_1, c_2 and $\omega \in [-\pi, \pi]$. We shall describe the spectral factorization result achievable for such spectra, based on the treatment in Rozanov (1967). Then we shall state and prove an intuitively reasonable continuity result.

Step 1: *normalization of* $\phi(\omega)$. Consider the spectrum $\hat{\phi}(\omega) = (2/c_1 + c_2)\phi(\omega)$ and set $q = (c_2 - c_1/c_2 + c_1)$. We shall solve the spectral factorization problem for $\hat{\phi}$ rather than ϕ, noting that $\phi(\omega) = I + M(\omega)$, where*

$$\|M(\omega)\| \leq q < 1 \quad \forall\omega \in [-\pi, \pi] \quad (62)$$

Step 2: *definition of certain operators*. Consider the space $L^2_{n\times n}$ of $n \times n$ matrix functions $\Phi(\omega)$, $\omega \in [-\pi, \pi]$, with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Phi(\omega)\|^2 \, d\omega < \infty. \quad (63)$$

With norm $\|\Phi(\cdot)\|^2_{\mathcal{H}}$ given by the integral on the left side of (6.3) and with

$$\langle \Phi, \psi \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Phi(\omega)\psi^*(\omega)\| \, d\omega \quad (64)$$

$L^2_{n\times n}$ becomes a Hilbert space. Every $\Phi(\cdot) \in L^2_{n\times n}$ can be written as

$$\Phi(\omega) = \sum_{-\infty}^{+\infty} a(n)e^{-j\omega n}$$

where

$$a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+j\omega n}\,\Phi(\omega)\, d\omega \quad (65)$$

and

$$\sum_{-\infty}^{+\infty} \|a(n)\|^2 = \|\Phi\|^2_{\mathcal{H}}. \quad (66)$$

*For a complex matrix A, we understand $\|A\| = [\lambda_{max}(A^*A)]^{1/2}$. For a hermitian A, $\|A\| = \lambda_{max}(A)$.

Let us define projections P_-, P_+ by

$$P_-\Phi(\omega) = \sum_{1}^{\infty} a(n)\,\mathrm{e}^{-j\omega n}, \qquad P_+\Phi(\omega) = \sum_{-\infty}^{-1} a(n)\,\mathrm{e}^{-j\omega n}. \tag{67}$$

Finally, define operators B_-^M, B_+^M on $L^2_{n\times n}$ by

$$B_-^M(\Phi) = P_-[\Phi(\omega)M(\omega)], \quad B_+^M(\Phi) = P_+[M(\omega)\Phi(\omega)]. \tag{68}$$

Step 3: *properties of* B_-^M, B_+^M. The spectral factorization formula depends on the following crucial properties of $B_\pm^M$. We use a subscript $\mathscr{H}$ to distinguish operator norms

$$\|B_\pm^M\|_{\mathscr{H}} \le q \tag{69}$$

from which it readily follows that $(I + B_-^M)^{-1}$ and $(I + B_+^M)^{-1}$ exist as operators on $L^2_{n\times n}$: $(I + B_\pm^M)^{-1} = I - B_\pm^M + (B_\pm^M)^2 - \cdots$.

Step 4: *construction of quantities appearing in spectral factorization.* Make the following definitions

$$\Psi_{-0}(\omega) = (I + B_-^M)^{-1}I, \qquad \psi_{0+}(\omega) = (I + B_+^M)^{-1}I \tag{70}$$

$$Q = \Psi_{-0}(I + M)\Psi_{0+}. \tag{71}$$

These definitions can be shown to ensure that

$$\Psi_{-0}(\omega) = \sum_{n=0}^{\infty} c(n)\,\mathrm{e}^{-j\omega n}, \qquad \psi_{0+}(\omega) = \sum_{\infty}^{0} d(n)\,\mathrm{e}^{-j\omega n} \tag{72}$$

for certain L_2 summable series $c(n)$, $d(n)$.

Step 5: *the spectral factorization theorem.*

(a) $\tilde{\Psi}_{-0}(z) = \sum_{n=0}^{\infty} c(n)z^{-n}$ is analytic, together with its inverse, outside and on the unit circle;

(b) $\tilde{\Psi}_{0+}(z) = \sum_{-\infty}^{0} d(n)z^{-n}$ is analytic, together with its inverse, inside and on the unit circle;

(c) Q is positive definite and constant;

(d) $I + M(\omega) = \Psi_{-0}^{-1}(\omega)Q\Psi_{0+}^{-1}(\omega)$; (73)

(e) $\Psi_{-0}(\omega) = \Psi_{0+}^{*}(-\omega)$; (74)

(f) $\tilde{\Psi}_{-0}(z)|_{z=\infty} = I = c(0)$. (75)

Evidently, $\tilde{W}(z) \triangleq \tilde{\Psi}_{-0}^{-1}(z)$ is the normalized minimum phase spectral factor, and Q is the covariation of the innovations. For the original spectrum ϕ, $\tilde{\Psi}_{-0}^{-1}(z)$ is still the NMSF, while the innovations covariance is

$$\bar{Q} = \frac{c_1 + c_2}{2}Q. \tag{76}$$

The calculations in the above procedure depend continuously on the given spectrum. The main result is as follows:

Theorem 6.1. Suppose there is given a hermitian $n \times n$ $\phi(\omega)$, $\omega \in [-\pi, \pi]$ satisfying (61) with $\phi(-\pi) = \phi(\pi)$, and define a norm for such matrix functions by

$$\|\phi\|_\infty = \sup_{\omega\in[-\pi,\pi]} \|\phi(\omega)\| \tag{77}$$

Then the quantities $\tilde{\Psi}_{-0}^{-1}(z)$ for $|z| \ge 1$ and $\bar{Q}$ defined by the above procedure depend continuously on ϕ. The theorem is proved in Appendix B.

The next result states that the canonical admissible spectral factor $\{\bar{W}(z), \bar{Q}\}$ depends also continuously on $\phi(\omega)$.

Corollary 6.1. Consider a closed-loop system of the form of Figs. 2 or 3, obeying assumptions A.1 and A.2. Let $\phi_{yu}(z)$ be the joint spectrum, and let $\{\bar{W}(z), Q\}$ be a corresponding canonical admissible spectral factorization of $\phi_{yu}(z)$, consistent with the structural assumptions considered in Sections 4 and 5. Then $\{\bar{W}(z), \bar{Q}\}$ depend continuously on $\phi_{yu}(z)$ in $|z| \ge 1$.

Proof: From the construction of $\bar{W}(z)$, $\bar{Q}$ in Sections 4 and 5 it follows that these quantities depend continuously on $\bar{W}(z)$, $\bar{Q}$, the NMSF. The result follows from Theorem 6.1.

We show now that, when the identifiability conditions of Section 4 are satisfied, F, $\bar{G}$, H, $\bar{K}$ can be approximately identified if an approximation $\hat{\phi}_{yu}(z)$ to the true spectrum $\phi_{yu}(z)$ is available. A similar result holds for F, $\bar{G}$ if the identifiability conditions of Section 5 are satisfied. Note that $\bar{G}$ and $\bar{K}$ are the minimum phase equivalents of the physical G and K.

Theorem 6.2. Consider a closed-loop system with assumptions A.1, A.2 and A.3. Suppose $\hat{\phi}_{yu}(z)$ is an approximation to the joint spectrum $\phi_{yu}(z)$, and let $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{K}$ be derived by (10) from the canonical admissible spectral factor $\hat{W}(z)$ of $\hat{\phi}_{yu}(z)$. If the closed-loop system has the form (28) and if any set of structural conditions of Theorem 4.1 is satisfied, $\{\hat{F}, \hat{H}\}$ approximate $\{F, H\}$, while $\{\hat{G}, \hat{K}\}$ approximate the minimum phase equivalents $\bar{G}$, $\bar{K}$ of G, K, the approximations being at all z in $|z| \ge 1$ away from poles of $\bar{W}_{11}^{-1}$ and $\bar{W}_{22}^{-1}$. If the closed-loop system has the form (9) and if the structural conditions of Theorem 5.2 or 5.3 are satisfied, then $\hat{F}$ approximates F, while $\hat{G}$ approximates $\bar{G}$ in $|z| \ge 1$.

Proof: Let $\bar{W}$ be the canonical admissible spectral factor of $\phi_{yu}(z)$, consistent with the structural knowledge about the system under consideration. Then, under the conditions of

Theorem 4.1, F, $\bar{G}$, H, $\bar{K}$ are identifiable from $\bar{W}(z)$ by the following formulas

$$F = \bar{W}_{12}\bar{W}_{22}^{-1}, \qquad \bar{G} = \bar{W}_{11} - \bar{W}_{12}\bar{W}_{22}^{-1}\bar{W}_{21} \tag{78a}$$

$$H = \bar{W}_{21}\bar{W}_{11}^{-1}, \qquad \bar{K} = \bar{W}_{22} - \bar{W}_{21}\bar{W}_{11}^{-1}\bar{W}_{12}. \tag{78b}$$

Similarly, under the conditions of Theorem 5.2 or 5.3, F, $\bar{G}$ are identifiable via (78a). By Corollary 6.1 $\hat{\bar{W}}$ approximates $\bar{W}$ in $|z| \geq 1$; and hence we conclude from (78) that, away from poles of $\hat{\bar{W}}_{22}^{-1}$ or $\hat{\bar{W}}_{11}^{-1}$ in $|z| \geq 1$, $\hat{F}$, $\hat{\bar{G}}$, $\hat{H}$, $\hat{\bar{K}}$ will approximate F, $\bar{G}$, H, $\bar{K}$ if the conditions of Theorem 4.1 are satisfied (and $\hat{F}$, $\hat{\bar{G}}$ will approximate F, $\bar{G}$ if the conditions of Theorems 5.2 or 5.3 are satisfied).

Following Theorem 4.2, it was argued that the orthogonality of the plant noise and the regulator in a generic system could be derived from the joint spectrum. We now examine what can be said when only an approximation $\hat{\phi}_{yu}$ of the joint spectrum is available. We shall show that if the NMSF $\{\hat{\bar{W}}, \hat{\bar{Q}}\}$ obtained from $\hat{\phi}_{yu}$ has an approximately block-diagonal $\hat{\bar{Q}}$, then the physical Q is approximately block-diagonal. We shall prove our main result with the help of a series of lemmas.

Lemma 6.1: Consider the closed-loop system (28) with assumptions A.1 and A.2, and the corresponding $W(z)$. Let $\phi(z) = W(z)QW^*(z)$. Assume that there exists a block-diagonal matrix D such that $D \leq Q \leq (1+\alpha)D$, for some $\alpha > 0$. Define $\phi_d(z) = WDW^*$. Let $\{\bar{W}(z), \bar{Q}\}$ be the NMSF of $\phi(z)$, and $\{\bar{W}_d(z), \bar{D}\}$ the NMSF of $\phi_d(z)$. Then $\alpha \to 0$ implies

$$\|\bar{W}_d - \bar{W}\|_{\mathscr{H}} \to 0 \text{ and } \|\bar{Q} - \bar{D}\| \to 0. \tag{79}$$

Proof: By the assumptions, $\phi_d \leq \phi \leq (1+\alpha)\phi_d$. The result then follows from Theorem 6.1.

Corollary 6.2. Consider the closed-loop system (28) with assumption A.2 and the following additional assumptions:

(1) the model is generic. (80)

(2) $F(\infty) = H(\infty) = 0$; $G(\infty)$ and $K(\infty)$ are nonsingular. (81)

(3) $D \leq Q \leq (1+\alpha)D$ for some block diagonal D and some $\alpha > 0$. (82)

Let $\phi(z) = WQW^*$, let $\{\bar{W}(z), \bar{Q}\}$ be the NMSF of $\phi(z)$, with $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{K}$ the corresponding NMR. Then $\bar{Q}$ is approximately block-diagonal, $\bar{F}$, $\bar{H}$ approximate F, H, while $\bar{G}$, $\bar{K}$ approximate the minimum phase equivalents of G, K.

Proof: By Theorem 3.3, $\bar{D}$ defined in Lemma 6.1 is block-diagonal and $\bar{W}_d$ yields, via (10), the original F and H and the minimum phase equivalents of G and K. The result then follows from (79).

Comment: Corollary 6.2 states that, under mild assumptions, if the original noise matrix is approximately block-diagonal, then the corresponding NMSF noise matrix $\bar{Q}$ is also approximately block-diagonal, and F, G, H, K are approximately identifiable from the NMSF $\bar{W}(z)$. The main result of this section, see Theorem 6.3 and 6.4 below, is a partial converse.

Theorem 6.3. Consider a 4-block closed-loop system (28) with F, G, H, K generic, $F(\infty) = H(\infty) = 0$ and $G(\infty)$, $K(\infty)$ nonsingular, and let $\phi_{yu}(z)$ be the joint spectrum. Suppose the NMSF $\{\bar{W}(z), \bar{Q}\}$ of $\phi_{yu}(z)$ is such that $\bar{Q}$ is approximately block-diagonal, and $\bar{W}(z)$ has a coprime MFD with denominator $\mathscr{A}$ and which can be approximated by a minimum phase $\bar{\bar{W}}(z)$ of the form

$$\bar{\bar{W}}(z) = \mathscr{A}^{-1}\begin{bmatrix} \bar{C} & 0 \\ 0 & \bar{N} \end{bmatrix} \text{ with } \bar{\bar{W}}(\infty) = I \tag{83}$$

where $\bar{C}$ and $\bar{N}$ obey the genericity conditions (25). Then the system noise covariance matrix Q is approximately block-diagonal.

Proof: See Appendix C.

Comment: We already knew from Lemma 6.1 that if the noise covariance matrix of the true system is approximately block-diagonal then the NMSF noise matrix is approximately block-diagonal; Theorem 6.3 states the converse result that if the NMSF noise matrix obtained from the true spectrum is approximately block-diagonal and the system is generic, then the noise matrix of the true system is approximately block-dagonal. In the next and final result we extend this to the case where only an approximate spectrum $\hat{\phi}_{yu}(z)$ is available.

Theorem 6.4. Consider a closed-loop system (28) with F, G, H, K generic, $F(\infty) = H(\infty) = 0$ and $G(\infty)$, $K(\infty)$ nonsingular. Let $\hat{\phi}_{yu}(z)$ be an approximation of the true spectrum $\phi_{yu}(z)$. Suppose the NMSF $\{\hat{\bar{W}}, \hat{\bar{Q}}\}$ of $\hat{\phi}_{yu}(z)$ has Q approximately block-diagonal, that $\hat{\bar{W}}$ has a coprime MFD that has approximately the form of $\hat{\bar{\bar{W}}}$ in (83), and that this form is generic. (The matrices $\hat{\bar{W}}$ and $\hat{\bar{\bar{W}}}$ have the same denominator matrix.) Then the true noise convariance is approximately block-diagonal and $\hat{\bar{W}}$ yields approximations of F, H and of the minimum phase equivalents of $\bar{G}$, $\bar{K}$ via (10), i.e. F, G, H, K are approximately identifiable from $\hat{\phi}_{yu}(z)$.

Proof: By Theorem 6.1 the NMSF depends continuously on $\phi_{yu}(z)$. Therefore the NMSF $\{\bar{W}, \bar{Q}\}$ of the true $\phi_{yu}(z)$ also has $\bar{Q}$ approximately block-diagonal and $\bar{W}(z)$ approximately of the form (83). Thus by Theorem 6.3 the true noise covariance is approximately block diagonal; the result then follows by Corollary 6.2.

We have shown in this section that under all conditions that guarantee identifiability from the exact spectrum $\phi_{yu}(z)$, the parameters (either F, G, or F, G, H, K) can also be approximately estimated from an approximate spectrum $\hat{\phi}_{yu}(z)$. This was the content of Theorem 6.2. We have also shown that the absence of correlation between plant noise and regulator noise, which can be detected for generic systems from the exact spectrum via the NMSF, can also be detected from the NMSF obtained from an approximate spectrum $\hat{\phi}_{yu}(z)$. This was the content of Theorem 6.4.

7. CONCLUSIONS

We have obtained a new set of identifiability results for the identifiability of linear feedback systems using the joint input-output identification method. These results have been obtained by extending previously known results in two directions.

First we have shown that the NMSF, which was the canonical factor previously used with the joint method, is not always compatible with the *a priori* knowledge about the structure of the feedback system. By introducing the concept of admissible canonical factors, new situations can be considered for which identifiability can be proved.

Next we have shown that there exists a whole class of spectral factors that lead to the same forward path model, even though the feedback models are different. This has allowed us to obtain new identifiability results for the forward path, in the presence of one-sided correlation between the regulator noise and the process noise, a situation that could not be handled by the joint input–output identification method I_2 before. The identifiability results obtained here for I_2 include all previously known results obtained with the direct prediction error method, at least for the linear case. They actually extend those results to a few new cases.

Finally, the continuity results of the last section show that the identifiability conditions obtained for I_2 are not merely of academic interest. In all cases where identifiability is guaranteed, consistent estimates of $\phi_{yu}(z)$ will yield consistent estimates of the system parameters. In addition, for all generic feedback systems, the absence of correlation between plant noise and regulator noise can be checked from approximate estimate of $\phi_{yu}(z)$.

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APPENDIX A: PROOF OF LEMMA 2.1

(a) First we consider $L = 0$. Observe that

$$W(z) = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}. \tag{A.1}$$

The row rank of $W(z)$ is therefore the sum of the row ranks of $G(z)$ and $K(z)$. If, say, W_{22} fails to have full row rank, then K fails to have full row rank because $W_{22} = (I - FH)^{-1}K$. Therefore $W(z)$ does not have full row rank, which contradicts assumption A.2. Same argument for W_{11}^{-1}.

(b) Now consider $L \neq 0$.

(1) Suppose first that $W(z)$ is square. The same argument as above shows that if W_{22} fails to have full rank, then so does K. But then, since K is square $[W_{21}/W_{22}]$ has less than full column rank by (7), and since $W(z)$ is square, $W(z)$ has less than full rank. This contradicts A.2.

(2) If $K(\infty)$ has full row rank, then $K(z)$ has full normal row rank, and so does $W_{22}(z) = (I - HF)^{-1}K$.

(3) If $L(z) = L_0(z)G(z)$, then

$$W(z) = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ L_0G & K \end{bmatrix}. \tag{A.2}$$

The row rank of $W(z)$ is therefore the row rank of

$$\begin{bmatrix} G & 0 \\ L_0G & K \end{bmatrix} = \begin{bmatrix} I & 0 \\ L_0 & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}.$$

This, in turn, is the sum of the row ranks of G and K. Therefore $K(z)$ and $W_{22}(z)$ have full normal row rank.

APPENDIX B: PROOF OF THEOREM 6.1

The main task is to prove that $\psi_{-0}^{-1}(\omega)$ and Q depend continuously on ϕ.

Lemma B.1. Let B_-^M, B_+^M be defined as earlier. Then these operators depend continuously on M, in fact

$$\|B_-^{M_1} - B_-^{M_2}\|_{\mathscr{X}} \le \max_\omega \|M_1(\omega) - M_2(\omega)\| \tag{A.3}$$

Proof: For arbitrary Φ, we have

$$[B_-^{M_1} - B_-^{M_2}](\Phi) = P_-\{\Phi(\omega)[M_1(\omega) - M_2(\omega)]\}$$
$$\|[B_-^{M_1} - B_-^{M_2}](\Phi)\|_{\mathscr{X}}^2 \le \|P_-\Phi(\omega)[M_1(\omega) - M_2(\omega)]\|_{\mathscr{X}}^2$$
$$\le \|\Phi(\omega)[M_1(\omega) - M_2(\omega)]\|_{\mathscr{X}}^2 \quad (P_- \text{ is a projection})$$
$$\le \max_\omega \|M_1(\omega) - M_2(\omega)\|^2 \int_{-\pi}^{\pi} \|\Phi(\omega)\|^2 \, d\omega$$

whence (A.3) is immediate. Same proof for B_+^M.

Lemma B.2. $(I + B_\pm^M)^{-1}$ depend continuously on M.

Proof: By Lemma 1 $I + B_\pm^M$ depend continuously on M. Since the inverse of both operators exist the result follows by a standard calculation, see Simmons (1963, p. 306, Theorem C).

Lemma B.3. $\Psi_{-0}(\omega)$, $\Psi_{0+}(\omega)$ depend continuously on M in $L_{n\times n}^2$, and also Q and $\bar{Q}$ depend continuously on M in the set of Hermitian complex conjugate matrices.

Proof: Follows by Lemma B.2 and the formulas (70), (71) and (76).

Lemma B.4. $\Psi_{-0}^{-1}(\omega)$, $\Psi_{0+}^{-1}(\omega)$ depend continuously on M in $L_{n\times n}^2$.

Proof: By (73), $\Psi_{-0}^{-1}(\omega) = [I + M(\omega)]\Psi_{0+}(\omega)Q^{-1}$. The result follows by Lemma B.3.

Lemma B.5. For arbitrary but fixed z with $|z| > 1$, $\bar{W}(z)$ depends continuously on M.

Proof: Let $\bar{W}(\omega) \triangleq \Psi_{-0}^{-1}(\omega)$, and let $W_1(\cdot)$ and $W_2(\cdot)$ be obtained from M_1 and M_2. Then by the previous lemmas

$$\|\bar{W}_1(\cdot) - \bar{W}_2(\cdot)\|_{\mathscr{X}} \le K\|M_1 - M_2\|_\infty$$

for some finite, positive K. Now, remembering (72), let

$$\bar{W}_1(\omega) = \sum_{n=0}^{\infty} k_1(n)\, e^{-j\omega n}$$

and

$$\bar{W}_2(\omega) = \sum_{n=0}^{\infty} k_2(n)\, e^{-j\omega n}.$$

Then by (66)

$$\|\bar{W}_1(\cdot) - \bar{W}_2(\cdot)\|_H^2 = \sum_0^\infty \|\tilde{k}(n)\|^2,$$

and hence

$$\sum_{n=0}^{\infty} \|k_1(n) - k_2(n)\|^2 \le K\|M_1 - M_2\|.$$

Therefore

$$\Psi_{-0}^{-1}(z) = \bar{W}(z) \triangleq \sum_0^\infty k(n) z^{-n}$$

whose existence in $|z| > 1$ is guaranteed by the spectral factorization theorem, depends continuously on M, and hence on $\phi(\omega)$.

APPENDIX C: PROOF OF THEOREM 6.3

Lemma C.1. Let Q be an $n \times n$ real symmetrical positive definite matrix. Let $B(z)$ be an $n \times n$ polynomial matrix of the form $B(z) = \mathrm{diag}\{z^{i_k}\}$ + polynomial terms of lower row degree, and such that $\det B(z)$ and $z^p \det B^*(z)$ are coprime, where $p = \sum_{k=1}^{n} i_k$. Let $M(z)$ be an $n \times n$ matrix of the form $M(z) = M_\nu \,\mathrm{diag}\{z^{i_k}\} + M_{\nu-1}\, \mathrm{diag}\,\{z^{i_k - 1}\} + \ldots + M_0 + \ldots + M_\nu^T\, \mathrm{diag}\,\{z^{-i_k}\}$ with $\nu = \max_{k=1,\ldots,n} \{i_k\}$ and $M(z) = M^T(z^{-1})$. Then there exist $\epsilon_0 > 0$ and, for any ϵ with $|\epsilon| < \epsilon_0$, $B_\epsilon(z) = \mathrm{diag}\,\{z^{i_k}\}$ + polynomial terms of lower row degree, and $Q_\epsilon > 0$ symmetric, such that

(i) $B_\epsilon(z)Q_\epsilon B_\epsilon^T(z^{-1}) = BQB^* + \epsilon M$; (A.4)
(ii) B_ϵ, Q_ϵ depend continuously on ϵ; (A.5)
(iii) $B_0 = B$, $Q_0 = Q$. (A.6)

Comment: This Lemma is a particular case of an implicit function theorem. (i) implicity defines B_ϵ, Q_ϵ as a function of ϵ. (iii) defines a solution of the equation at the value $\epsilon = 0$. (ii) is expected to be a conclusion of the implicit function theorem.

Proof: Consider the equation

$$(\Delta B)QB^* + B(\Delta Q)B^* + BQ(\Delta B^*) = \epsilon M \tag{A.7}$$

where ΔB is a polynomial matrix whose row degrees are lower than $i_1, \ldots, i_n$. This can be rewritten as

$$(B^{-1}\Delta B)Q + Q(\Delta B^*)(B^*)^{-1} + \Delta Q = \epsilon B^{-1} M (B^*)^{-1}. \tag{A8}$$

Now let $z = (1 - s/1 + s)$ and define

$$\hat{B}(s) \triangleq \mathrm{diag}\,\{(1+s)^{i_k}\} B\left(\frac{1-s}{1+s}\right)$$

$$\hat{M}(s) \triangleq \mathrm{diag}\,\{(1+s)^{i_k}\} M\left(\frac{1-s}{1+s}\right) \mathrm{diag}\,\{(1-s)^{i_k}\}.$$

Then $\hat{M}(s) = \hat{M}^T(-s)$, $\hat{B}(s)$ is polynomial, $\det \hat{B}(s)$ and $\det \hat{B}(-s)$ are coprime, and†

$$B^{-1}(z)M(z)B^{-*}(z) = B^{-1}\left(\frac{1-s}{1+s}\right)M\left(\frac{1-s}{1+s}\right)B^{-T}\left(\frac{1+s}{1-s}\right) = \hat{B}^{-1}(s)\hat{M}(s)\hat{B}^{-T}(-s).$$

Since $\det B(z)$ and $z^p \det B^*(z)$ have no common zeros, neither one has a zero at $z = -1$. By construction $M(z)$ has no poles at $z = -1$. Therefore $\lim_{z\to -1} B^{-1}(z)M(z)B^{-*}(z) = \lim_{s\to\infty} \hat{B}^{-1}(s)\hat{M}(s)\hat{B}^{-T}(-s)$ is finite. Hence $\hat{B}^{-1}(s)M(s)\hat{B}^{-*}(s)$ is proper, but not necessarily strictly proper. Therefore let

$$\hat{B}^{-1}(s)\hat{M}(s)\hat{B}^{-*}(s) = N + \hat{B}^{-1}(s)M_1(s)\hat{B}^{-*}(s) \quad (A.9)$$

where N is constant and symmetric, and $\hat{B}^{-1}(s)\hat{M}_1(s)\hat{B}^{-*}(s)$ is strictly proper. Now let $\hat{B}^{-1}M_1 = E + \hat{B}^{-1}F$, where E is polynomial and $\hat{B}^{-1}F$ is strictly proper. Then $E\hat{B}^{-*}$ is strictly proper because $\hat{B}^{-1}M_1\hat{B}^{-*}$ is strictly proper. Since $\det \hat{B}(s)$ and $\det \hat{B}(-s)$ are coprime, and $\hat{B}^{-1}F$ is strictly proper, there exist unique polynomial matrices X and Y [see proof of Appendix in Anderson and Bitmead (1977)] such that

$$\hat{B}(s)X(s) + Y(s)\hat{B}^T(-s) = F \quad (A.10)$$

and $\hat{B}^{-1}(s)Y(s)$ and $X(s)\hat{B}^{-*}(s)$ are strictly proper. Adding $\hat{B}(s)E(s)$ on both sides of (A.10) yields

$$\hat{B}(s)[X(s) + E(s)] + Y(s)\hat{B}^T(-s) = F + \hat{B}E = M_1(s) \quad (A.11)$$

Equivalently, with $X(s) + E(s) \triangleq Z(s)$

$$Z(s)\hat{B}^{-*}(s) + \hat{B}^{-1}(s)Y(s) = \hat{B}^{-1}(s)M_1(s)\hat{B}^{-*}(s) \quad (A.12)$$

with $Z\hat{B}^{-*}$ and $\hat{B}^{-1}Y$ strictly proper, and Z and Y uniquely defined by M_1 and $\hat{B}$. Since $M_1(s) = M_1^T(-s)$, we have $Y(s) = Z^*(s) = Z^T(-s)$. Substituting (A.9) in (A.12) yields

$$Z(s)\hat{B}^{-*}(s) + \hat{B}^{-1}(s)Y(s) + N = \hat{B}^{-1}(s)\hat{M}(s)\hat{B}^{-*}(s). \quad (A.13)$$

†B^{-T} is used to denote $(B^T)^{-1}$, and $B^{-*} = (B^*)^{-1}$. In the s-plane $B^*(s)$ will denote $B^T(-s)$, while $B^{-*}(s) = B^{-T}(-s)$.

Now if we define $\Delta\hat{B}(s) \triangleq \operatorname{diag}\{(1+s)^{i_k}\}\Delta B(1 - s/1 + s)$, then (A.8) can be rewritten

$$\hat{B}^{-1}(s)\Delta\hat{B}(s)Q + Q\Delta\hat{B}^*(s)\hat{B}^{-*}(s) + \Delta Q = \epsilon\hat{B}^{-1}(s)\hat{M}(s)\hat{B}^{-*}(s). \quad (A.14)$$

Comparing with (A.13) shows that (A.14) has a unique solution given by

$$\Delta B(s) = \epsilon Y(s)Q^{-1}, \qquad \Delta Q = \epsilon N. \quad (A.15)$$

Converting back to the z-plane, and noting that $s = (1 - z/1 + z)$, and $1 + s = (2/1 + z)$, this defines $\Delta B(z)$ through

$$\Delta B(z) = \operatorname{diag}\left\{\left(\frac{z+1}{2}\right)^{i_k}\right\}\Delta\hat{B}\left(\frac{1-z}{1+z}\right)\Big\}.$$

$\Delta B(z)$ is polynomial because $\hat{B}^{-1}Y$ is strictly proper. $\hat{B}$ is row proper and $i_1, \ldots, i_n$ are the row degrees of $\hat{B}$. Recognizing that the left-hand side of (A.7) is the first variation of BQB^* (with variables B and Q), we have shown that (A.14) defines a unique solution for B_ϵ and Q_ϵ; (A.5) and (A.6) follow from (A.15).

Proof of Theorem: Let $\bar{\bar{Q}}$ be the block diagonal approximation of $\bar{Q}$, and let the true $W(z)$, corresponding to F, G, H, K, be written as $W = \mathcal{A}^{-1}\mathcal{B}$, while $\bar{W} = \mathcal{A}^{-1}\bar{\mathcal{B}}$. Since W has minimal degree by the genericity assumption, there is no loss in generality in assuming that W, $\bar{W}$ and $\bar{\bar{W}}$ have the same denominator matrix (see Lemma 3.1). Then $\mathcal{A}^{-1}\mathcal{B}\mathcal{Q}\mathcal{B}^*\mathcal{A}^{-*} = \mathcal{A}^{-1}[\bar{B}\bar{\mathcal{Q}}\bar{B}^*]\mathcal{A}^{-*}$.

By the assumptions, there exists an $\epsilon > 0$ and a $M(z)$, satisfying the conditions of Lemma C.1, such that

$$\mathcal{B}\mathcal{Q}\mathcal{B}^* = \bar{\mathcal{B}}\bar{\mathcal{Q}}\bar{\mathcal{B}} = \begin{bmatrix}\bar{C} & 0\\ 0 & \bar{N}\end{bmatrix}\bar{\bar{Q}}\begin{bmatrix}\bar{C}^* & 0\\ 0 & \bar{N}^*\end{bmatrix} + \epsilon M. \quad (A.16)$$

Now let

$$\begin{bmatrix}\bar{C} & 0\\ 0 & \bar{N}\end{bmatrix} = \bar{B}.$$

Since $\bar{B}$ is minimum phase, and $\bar{C}$ and $\bar{N}$ satisfy the genericity assumption, it follows that $\det\bar{\mathcal{B}}$ and $z^p\det\bar{\mathcal{B}}^*$ are coprime, where $p = \deg\det\bar{\mathcal{B}}$. Therefore, by Lemma C.1, $\mathcal{B}$ and Q depend continuously on ϵ, and so, if ϵ is small, Q is close to $\bar{Q}$ and, therefore, approximately block-diagonal.