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Representations of jointly stationary stochastic feedback processes†

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Stable constant linear closed-loop systems relating an input vector u to an output vector y and vice versa produce a jointly stationary (y, u)-process. On the other hand it is often natural to split up a stationary vector random process z into component vectors y and u, and to examine the closed-loop relations between y and u. This paper presents a number of new results on the spectral factorization and the closed-loop representation of a jointly stationary vector (y, u)-process. Conditions are derived on the closed-loop models for the joint process model to be of minimal degree, stable and minimum-phase. Relations between different joint process models producing the same spectrum $\phi_{yu}(z)$ are established.

1. Introduction

We present a number of new results on the spectral factorization and on the representation of discrete-time stationary vector random processes, in which the vector process z can be thought of as made up of two subvectors y and u, i.e. $z=[y^T, u^T]^T$. Such situations can arise in many applications in which a vector process is composed of two different subprocesses that play different roles. For example, in engineering applications u and y can be the input and the output of a linear system. In econometric time series one may consider some of the variables as exogenous and others as endogenous. Assuming that the joint spectral-density matrix function $\phi_{yu}(z)$ is available, it is then reasonable to investigate what can be inferred about the dynamical relationship between the u and y processes from $\phi_{yu}(z)$. More specifically one might want to answer questions such as the following. If a spectral factor is obtained from $\phi_{yu}(z)$, can we derive a dynamical model relating u and y from it ? Assuming an underlying ' true ' dynamical model exists relating u to y, under what conditions can it be uniquely obtained from $\phi_{yu}(z)$? Is it possible to detect the presence of feedback between two time series (viz. u and y) from their joint spectrum ?

As can be seen from these questions, the study of jointly stationary stochastic processes is an important problem in the field of multivariate time-series analysis. In addition it has direct implications for the solution of two problems that have received a fair degree of attention : the detection of feedback between stationary time series (see, e.g. Granger 1963, Caines and Chan 1975, and the references therein), and the identifiability of linear systems operating under feedback (see, e.g. Vorchik *et al.* 1973, Gustavsson *et al.* 1977, Wellstead and Edmunds 1975, Ng *et al.* 1977, Anderson and Gevers 1979, and

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the references therein). In this paper we shall establish some basic results on spectral factorization of jointly stationary processes, and on the relations between the spectral density function of a joint process and the matrix transfer-function realizations of the individual processes. These results, which we believe to be a novel contribution to the fields of spectral factorization and matrix fraction representations of stationary stochastic processes, should lead to the solution of a number of problems that involve jointly stationary stochastic processes. In particular, they have enabled us to establish more precise conditions for the identifiability of stationary feedback processes and for the detection of feedback between stationary time series. These results are presented elsewhere (Anderson and Gevers 1981 a, Gevers and Anderson 1981).

The paper is organized as follows. In § 2, we present a number of results on the factorization of spectral-density matrices. We establish some new results relating the spectral-density matrix and the co-prime matrix fraction description (MFD) of the spectral factors. Section 3 presents different representations for joint processes. Two points of view will be considered. If the (y, u)-process originates from a physical linear closed-loop system, a 6-block model containing six transfer-function matrices may be necessary to represent the closed-loop system :

$$\begin{aligned} y_i &= Fu_i + m_i \\ u_i &= Hy_i + n_i \end{aligned} \quad \text{with} \quad \begin{bmatrix} m_i \\ n_i \end{bmatrix} = \begin{bmatrix} G & J \\ L & K \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix}$$

$F(z)$, $H(z)$, $G(z)$, $J(z)$, $L(z)$ and $K(z)$ are transfer-function matrices, m_i and n_i are stationary noise processes, and (w, v) is a white-noise process. But if no underlying physical system is assumed, then without loss of generality a 4-block model will be sufficient to represent the (y, u)-process :

$$\begin{aligned} y_i &= Fu_i + Gw_i \\ u_i &= Hy_i + Kv_i \end{aligned}$$

where (w, v) is a white-noise process. Relations will be established between the (4 or 6) transfer-function matrices of a closed-loop representation for the (y, u)-process, and the transfer-function matrix of a white-noise driven model for the joint process

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} = W(z) \begin{bmatrix} w_i \\ v_i \end{bmatrix}$$

where $W(z)$ is a transfer-function matrix and (w, v) is a white-noise process. In § 4 the 4-block closed-loop representations of the (y, u)-process are studied. Necessary and sufficient conditions on the 4-block transfer-function matrices and their MFDs are given for the joint process model $W(z)$ to be either stable or minimum-phase. When a model $W(z)$ for the joint process (y, u) arises from a 4-block closed-loop representation with orthogonal noise sources (i.e. w_i orthogonal to v_j, for all i, j), several questions should be raised about the transfer matrix $W(z)$, such as : Is $W(z)$ of minimal degree ? What is the class of equivalent transfer matrices (i.e. giving rise to the same joint spectrum $\phi_{yu}(z)$) ? Do all equivalent models have orthogonal noise sources w_i, v_i ?

These questions will be answered in § 5 ; they have important implications in terms of the identifiability properties of the closed-loop system. Throughout the paper we shall rely heavily on matrix fraction descriptions of transfer-function matrices, thereby establishing some new results on MFDs for joint processes.

2. Factorization of discrete process spectral-density matrices

For completeness we first recall some basic results about spectral factorization theory. These results are discrete-time extensions of continuous-time results of Youla (1961). The proofs of Youla can easily be extended and will therefore be omitted.

The spectral-density matrix of a stationary real n-vector valued random process $\{y_i\}$ is given by

$$\phi(z) \triangleq \sum_{k=-\infty}^{\infty} R_k z^{-k}$$

where

$$R_k \triangleq E\{y_i y_{i-k}^{\mathrm{T}}\}$$

The $n \times n$ matrix $\phi(z)$ of real functions of z has the following properties :

(1) $\phi(z) = \phi^{\mathrm{T}}(z^{-1})$

(2) $\phi(z)$ is non-negative-definite for $|z| = 1$

(3) $\phi(z)$ is analytic for $|z| = 1$

In addition, we shall assume

(4) $\phi(z)$ is of full normal rank, i.e. $\phi(z)$ is of rank n for almost all z

Except when stated otherwise, we shall also make the following assumption :

(5) $\phi(z)$ is rational, i.e. the elements of $\phi(z)$ are real rational functions of z

A spectral factorization of $\phi(z)$ is a decomposition of $\phi(z)$ as

$$\phi(z) = W(z) Q W^*(z) \tag{2.1}$$

where $W(z)$ is a causal matrix, $W(z) = \sum_{k=0}^{\infty} W_k z^{-k}$ (with $W(z)$ rational if $\phi(z)$ is rational), $W^*(z)$ denotes $W^{\mathrm{T}}(z^{-1})$, and Q is a symmetric real non-negative-definite matrix, possibly equal to the identity matrix. We shall use the standard definitions for causal, strictly causal, stable, minimum-phase and strictly minimum-phase transfer-function matrices: $W(z)$ is causal if $W(\infty) < \infty$, strictly causal if $W(\infty) = 0$, stable if all entries have poles inside $|z| < 1$, minimum-phase if $W(z)$ is square and $W^{-1}(z)$ is causal and analytic in $|z| > 1$, strictly minimum-phase if in addition $W^{-1}(z)$ is analytic in $|z| \geqslant 1$. We can now state the spectral factorization theorem for discrete processes (see Youla (1961) for the continuous-time results).

Spectral factorization theorem

Let $\phi(z)$ be an $n \times n$ real rational full-rank spectral-density matrix.

(*a*) There exists a real $n \times n$ rational matrix $H(z)$ such that $\phi(z) = H(z)H^*(z)$, where $H(\infty)$ is finite and non-singular, $H(z)$ is stable and minimum-phase. If $\phi(z)$ is positive-definite on $|z| = 1$, $H(z)$ is strictly minimum-phase.

(*b*) Any other factorization $\phi(z) = K(z)K^*(z)$, in which $K(z)$ is real and rational, is such that $K(z) = H(z)V(z)$, where $V(z)$ is a real rational para-unitary matrix (i.e. $V(z)V^*(z) = I$). Moreover, $V(z)$ is stable if and only if $K(z)$ is stable.

(*c*) Any other factorization $\phi(z) = K(z)K^*(z)$, in which $K(\infty)$ is finite and non-singular, $K(z)$ is real and rational, stable and minimum-phase, is such that $K(z) = H(z)V$, where V is a real orthogonal matrix.

(*d*) There exists a unique factorization of the form $\phi(z) = \bar{W}(z)\bar{Q}\bar{W}^*(z)$, in which $\bar{W}(z)$ is real and rational, stable, minimum-phase and such that $\bar{W}(\infty) = I$, with $\bar{Q}$ positive-definite symmetric.

(*e*) Any other factorization of the form $\phi(z) = W(z)QW^*(z)$, in which $W(z)$ is real and rational, and Q is non-negative-definite symmetric, is such that $W(z) = \bar{W}(z)V(z)$, where $V(z)$ is a real and rational scaled para-unitary matrix, i.e. $V(z)QV^*(z) = \bar{Q}$. Moreover, $V(z)$ is stable if and only if $W(z)$ is stable.

(*f*) Any other factorization of the form $\phi(z) = W(z)QW^*(z)$, in which $W(\infty)$ is finite and non-singular, $W(z)$ is $n \times n$ real and rational, stable and minimum-phase, and Q is positive-definite symmetric, is such that $W(z) = \bar{W}(z)V$, where V is a real non-singular constant matrix, with $VQV^T = \bar{Q}$.

Notice that $K(z)$ in (*b*) and $W(z)$ in (*e*) can have more columns than rows; however, they must have full row rank, because $\phi(z)$ is assumed to have full rank.

Definition 1

We shall call $\bar{W}(z)$, defined in part (*d*) of the spectral factorization theorem, the normalized minimum-phase spectral factor (NMSF) of $\phi(z)$.

The NMSF of $\phi(z)$ can be called canonical because it is uniquely defined by $\phi(z)$. However, there are other ways of defining canonical spectral factors. For example, two other canonical factors can be defined as follows.

(1) $\phi(z) = H_1(z)H_1^*(z)$, where $H_1(z)$ is $n \times n$ real and rational, stable, minimum-phase and $H_1(\infty)$ is lower triangular with positive diagonal elements.

$H_1(z)$ is obtained from the NMSF $\bar{W}(z)$ as $H_1(z) = \bar{W}(z)L$, where L is the uniquely defined real lower triangular factor of Q that has positive diagonal elements: $Q = LL^T$.

(2) Similarly there is a unique factorization $Q = UU^T$, where U is real upper triangular with positive diagonal elements. This defines another canonical spectral factor $H_2(z) = \bar{W}(z)U$. For a further discussion on canonical factors of discrete-process spectral-density matrices, see, e.g. Phadke (1973). It should be clear from the last two examples that in general the factorization $\phi(z) = H(z)H^*(z)$, in which $H(z)$ is only required to be stable, minimum-phase, with $H(\infty)$ finite and non-singular, is not canonical.

We notice that if a pair $\{W(z), Q\}$ is given with $W(z)$ a stable, real and rational transfer-function matrix and Q a non-negative-definite (white-noise covariance) matrix, this pair completely specifies a spectrum. We shall say that $\{W(z), Q\}$ is a spectral factor of $\phi(z)$ if $\phi(z) = W(z)QW^*(z)$, although we shall occasionally omit Q and refer to $W(z)$ as a spectral factor of $\phi(z)$, when no confusion is possible. In the light of these conventions we introduce the following definitions.

Definition 2

Two spectral factors $\{W_1(z), Q_1\}$ and $\{W_2(z), Q_2\}$ will be called equivalent if

$$W_1(z)Q_1W_1^*(z) = W_2(z)Q_2W_2^*(z)$$

Definition 2′

Given a spectral factor $\{W_1(z), Q_1\}$ we shall say that $W_2(z)$ is an equivalent spectral factor if there exists a symmetric real non-negative-definite matrix Q_2 such that

$$W_1(z)Q_1W_1^*(z) = W_2(z)Q_2W_2^*(z)$$

Throughout this paper it will be useful to work with polynomial matrix fraction descriptions of rational transfer-function matrices. We recall that $A^{-1}(z)B(z)$ is a left *polynomial matrix fraction description* (in short MFD) of the rational transfer-function matrix $W(z)$ if the elements of $A(z)$ and $B(z)$ are polynomials in z, and if $W(z) = A^{-1}(z)B(z)$. This MFD is termed *left co-prime* if and only if every greatest common left divisor of $A(z)$ and $B(z)$ is a unimodular matrix. A square polynomial matrix $P(z)$ is *unimodular* if and only if its determinant if a non-zero constant. Right MFDs are similarly defined with the corresponding properties.

We shall also need the notions of row-degree and row-proper polynomial matrices. We recall that the *row degree* of the kth row of a polynomial matrix is the degree of the highest-degree polynomial appearing as an element of this row. If a real $n \times n$ polynomial matrix $A(z)$ has row degrees $a_1, \ldots, a_m$, then $A(z)$ can be uniquely written as

$$A(z) = D(z)A_{\mathrm{hr}} + L(z) \tag{2.2}$$

where $D(z) = \mathrm{diag}\,(z^{a_1}, \ldots, z^{a_n})$, A_{hr} is a matrix of real numbers and $L(z)$ is a polynomial matrix whose row degrees are strictly less than the corresponding row degrees of $A(z)$. The polynomial matrix is called *row-proper* (or row-reduced) if the matrix A_{hr} is non-singular. Finally, any polynomial matrix $A(z)$ can be transformed to a row-proper matrix by left multiplication by a unimodular matrix. Therefore any rational matrix $H(z)$ can always be written as $H(z) = A^{-1}(z)B(z)$ with A row-proper, and A and B co-prime. For further details on MFDs, see Rosenbrock (1970), Wolovich (1974) and Kailath (1980).

The *degree* (or McMillan degree) of a causal rational transfer-function matrix $W(z)$, denoted $\delta[W]$, is defined as the dimension of a minimal realization of $W(z)$ (see, e.g. McMillan 1952, Kalman 1965, Anderson and Vongpanitlerd 1973). It can be shown (Wolovich 1974) that, if $W(z) = A^{-1}(z)B(z)$, with $A(z)$ and $B(z)$ left co-prime, then $\delta[W] = \partial[|A|]$, where $|A|$ is the determinant of $A(z)$ and $\partial[|A|]$ denotes the *polynomial degree* of $|A|$.

Equivalent spectral factors of a same spectrum $\phi(z)$ may not have the same degree. It makes sense, therefore, to pay particular attention to the class of spectral factors of a particular $\phi(z)$ that have minimal degree among all possible spectral factors. As shown in Anderson (1967), the degree of such spectral factors is one-half the degree of $\phi(z)$.

We now establish relations between MFDs of equivalent spectral factors of a given spectrum. Given a spectral factor $W(z)=A^{-1}(z)B(z)$, we shall also derive necessary and sufficient conditions for $W(z)$ to be a minimal-degree factor of $\phi(z)=W(z)W^*(z)$. We start with two technical lemmas that will be needed in various proofs.

Lemma 1

Let $A(z)$ be an $n \times n$ non-singular polynomial matrix and $B(z)$ an $n \times m$ polynomial matrix such that $A^{-1}(z)B(z)$ is proper. Let $i_1, \ldots, i_n$ be the row degrees of A and let $D(z)=\text{diag}\,(z^{d_1}, \ldots, z^{d_n})$. Then the following three statements are equivalent :

(1) $d_k \geqslant i_k$, $k=1, \ldots, n$

(2) $D^{-1}A$ is proper

(3) A^*D is a polynomial matrix

In addition, any of these conditions implies that B^*D is a polynomial matrix.

Proof

Let the (k, j)th element of A be $[A]_{k,j}=a_{kj}(z)$. Then $[D^{-1}A]_{k,j}=a_{kj}(z)/z^{d_k}$ and $[A^*D]_{j,k}=z^{d_k}a_{kj}(z^{-1})$. Clearly the elements of the kth row of $D^{-1}A$ are proper, and the elements of the kth column of A^*D are polynomials if and only if $d_k \geqslant i_k$. Therefore (2) and (3) hold if and only if $d_k \geqslant i_k$ for all k. Now suppose (2) holds. Since $D^{-1}B=D^{-1}A\,.\,A^{-1}B$, it follows that $D^{-1}B$ is proper. Therefore, by (3), B^*D is polynomial. □

Lemma 2

Let $A(z)$ be an $n \times n$ row-proper polynomial matrix with row degrees $d_1, \ldots, d_n$, and let $D(z)=\text{diag}\,(z^{d_1}, \ldots, z^{d_n})$. Then $\det\,(A^*D)=z^p \det A(z^{-1})$, where $p=\deg\,|A(z)|=\sum_1^n d_i$.

Proof

Since $A(z)$ is row-proper, we can write by (2.2)

$$A(z)=D(z)A_{\text{hr}}+L(z)$$

Since the row degrees of $L(z)$ are strictly less than the corresponding row degrees of $A(z)$, and since A_{hr} is non-singular, $\deg\,|A(z)|=\deg\,|D(z)|=\sum_{i=1}^n d_i \triangleq p$. Also $A^*(z)=A^{\text{T}}(z^{-1})$. Therefore $\det\,(A^*D)=z^p \det A(z^{-1})$. □

The next theorem establishes relations between MFDs of equivalent spectral factors of a given spectrum.

Theorem 1

Let $\phi(z)$ be an $n \times n$ real and rational full-rank power-spectral-density matrix, and let $\{\overline{W}(z), \overline{Q}\}$ be the NMSF of $\phi(z)$. In addition, let $\overline{A}^{-1}(z)\overline{B}(z) = \overline{W}(z)$ be a left co-prime MFD of $\overline{W}(z)$. Then

(i) Any other stable spectral factor $W(z)$ of $\phi(z)$ has a co-prime MFD of the form

$$W(z) = (P\overline{A})^{-1}B \tag{2.3}$$

where P and B are polynomial matrices and P is stable†.

(ii) The NMSF is a minimal-degree spectral factor.

(iii) If $W(z) = A^{-1}B$ is any other minimal-degree spectral factor of $\phi(z)$, with A and B co-prime, then $A = P\overline{A}$ for some unimodular matrix $P(z)$, and $W(z) = \overline{A}^{-1}C$ for some polynomial matrix $C(z)$.

Proof

(i) Let $W(z)$ be an equivalent spectral factor of $\phi(z)$, and let $W(z) = A^{-1}(z)B(z)$ with A and B co-prime. Without loss of generality we can assume A and $\overline{A}$ to be row-proper. Hence for some non-negative-definite Q, we have

$$\phi(z) = (\overline{A})^{-1}\overline{B}\overline{Q}\overline{B}^*(\overline{A}^*)^{-1} = A^{-1}BQB^*(A^*)^{-1}$$

Therefore

$$A\overline{A}^{-1}\overline{B}\overline{Q} = BQB^*(A^*)^{-1}A^*(\overline{B}^*)^{-1} \tag{2.4}$$

Let $a_1, \ldots, a_n$ be the row degrees of the rows of $A(z)$, and let $D(z) = \text{diag}\,(z^{a_1}, \ldots, z^{a_n})$. Then by Lemma 1, B^*D and A^*D are polynomial, and by Lemma 2, $\det\,(A^*D) = z^p \det A(z^{-1})$, where $p = \partial|A(z)|$. Similarly let $\bar{a}_1, \ldots, \bar{a}_n$ be the row degrees of the rows of $\overline{A}(z)$, and let $\overline{D}(z) = \text{diag}\,(z^{\bar{a}_1}, \ldots, z^{\bar{a}_n})$. Then $\overline{A}^*\overline{D}$ and $\overline{B}^*\overline{D}$ are polynomial. Also, since $\lim_{z\to\infty} \overline{A}^{-1}(\infty)\overline{B}(\infty) = I$, it follows that the row degrees of $\overline{B}$ are equal to the corresponding row degrees of $\overline{A}$. Therefore $\det\,(\overline{B}^*\overline{D}) = z^q \det \overline{B}(z^{-1})$, where $q = \deg|\overline{B}(z)|$. We can now rewrite (2.4) as follows :

$$A\overline{A}^{-1}\overline{B}\overline{Q} = BQB^*D(A^*D)^{-1}\overline{A}^*\overline{D}(\overline{B}^*\overline{D})^{-1} \tag{2.5}$$

The left-hand side (LHS) of (2.5) is analytic in $|z| \geqslant 1$ because $\overline{W}(z)$ is stable, and hence $\overline{A}(z)$ is stable. The right-hand side (RHS) is analytic in $|z| < 1$ because

(*a*) $W(z)$ is stable, hence A^{-1} is analytic in $|z| \geqslant 1$, and therefore $(A^*D)^{-1}$ is analytic in $|z| \leqslant 1$;

(*b*) $\overline{W}^{-1}(z)$ is analytic in $|z| > 1$, hence $\overline{B}^{-1}$ is analytic in $|z| > 1$, and therefore $(\overline{B}^*\overline{D})^{-1}$ is analytic in $|z| < 1$;

(*c*) B^*D and $\overline{A}^*\overline{D}$ are polynomial.

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† For the sake of brevity we shall often omit the argument z of polynomial matrices or rational matrices. We shall also say that a polynomial matrix $P(z)$ is stable if its determinant $|P(z)|$ has all its roots strictly inside the unit circle, and that a rational matrix $H(z)$ is stable if the denominator matrix $A(z)$ of any co-prime MFD $H(z) = A^{-1}(z)B(z)$ is stable.

Therefore both sides of (2.5) are polynomial matrices. Now $\bar{A}$ and $\bar{B}$ are left co-prime by assumption, and since $\bar{Q}$ is constant and non-singular, $\bar{A}$ and $\bar{B}\bar{Q}$ must also be left co-prime. Since the LHS of (2.5) is polynomial, it follows that $\bar{A}$ divides A, i.e. $A = P\bar{A}$, for some polynomial matrix P. Since A and $\bar{A}$ are stable, it follows that P is stable. This concludes the proof of part (i).

(ii) By part (i) it follows that the degree of any spectral factor $W(z)$ is equal to $\partial|P\bar{A}|$, whereas the degree of the NMSF is $\partial|\bar{A}|$. It follows that the NMSF is of minimum degree.

(iii) By part (i), $A = P\bar{A}$; and since $A^{-1}B$ has minimal degree with A and B co-prime, it follows that $|P|$ is constant. Therefore

$$W(z) = A^{-1}B = (P\bar{A})^{-1}B = \bar{A}^{-1}C$$

where $C \triangleq P^{-1}B$ is polynomial because P is unimodular. □

Corollary 1

Let $\phi(z)$ be an $n \times n$ real and rational full-rank power-spectral-density matrix, and let $\{W(z), Q\}$ be an arbitrary stable minimal-degree spectral factor of $\phi(z)$, with $W(z) = A^{-1}(z)B(z)$, and A and B co-prime. Then any other stable minimal-degree spectral factor $\hat{W}(z)$ of $\phi(z)$ can be written $\hat{W}(z) = A^{-1}(z)C(z)$ for some polynomial $C(z)$, with A and C co-prime.

Proof

Let $\bar{W}(z)$ be the NMSF of $\phi(z)$, and let $\bar{W}(z) = \bar{A}^{-1}(z)\bar{B}(z)$, with $\bar{A}$, $\bar{B}$ left co-prime. Then by Theorem 1, there exists a unimodular $P(z)$ and a polynomial $C_1(z)$ such that $A(z) = P(z)\bar{A}(z)$, and $\hat{W}(z) = \bar{A}^{-1}(z)C_1(z)$. Therefore, $\hat{W}(z) = A^{-1}(z)P(z)C_1(z) = A^{-1}(z)C(z)$, with $C = PC_1$. A and C are co-prime, because $W(z)$ is minimal degree, so $\delta[\hat{W}] = \partial|\bar{A}| = \partial|A|$. □

In the sequel we shall often obtain a global model $\{W(z), Q\}$ for a joint (y, u)-process via the combination of a forward model describing y as a function of u, and a feedback model describing u as a function of y. An important question then is whether the global transfer-function matrix $W(z)$ is of minimal degree, i.e. whether the spectrum $\phi(z) = W(z)QW^*(z)$ produced by the (y, u)-process cannot be realized by a spectral factor $\hat{W}(z)$ of smaller degree than $W(z)$. Considering results (*b*) and (*e*) of the spectral factorization theorem, this would mean that $W(z)$ contains a para-unitary factor that unnecessarily augments its degree. In the next two results we give precise conditions that allow us to test whether a matrix transfer function is of minimal degree. Since the covariance matrix Q can always be factored into, say, $Q = LL^T$ and L can be absorbed in $W(z)$, we shall only consider the case where $Q = I$.

Lemma 3

Let $U(z)$ be an $n \times m$ $(m \geqslant n)$ stable, real and rational proper transfer-function matrix of full normal row rank. Let $A^{-1}(z)B(z)$ be a left co-prime MFD of $U(z)$, with $A(z)$ row-proper. Finally, let $D(z) = \text{diag}\,(z^{i_1}, \ldots, z^{i_n})$, where i_k is the row degree of the kth row of $A(z)$. Then $U(z)U^*(z) = Q$, with Q real-symmetric and positive-definite (i.e. $U(z)$ is scaled para-unitary) if and only if $A(z)$ divides $B(z)B^*(z)D(z)$, in the sense that $A^{-1}BB^*D$ is polynomial.

Proof

(i) *Necessity*. Suppose $U(z)$ is scaled para-unitary, i.e. $U(z)U^*(z)=Q$. Then $A^{-1}BB^*(A^*)^{-1}=Q$, and hence

$$A^{-1}(z)B(z)B^*(z)D(z)=QA^*(z)D(z)$$

Now by Lemma 1, B^*D and A^*D are polynomial ; therefore A divides BB^*D.

(ii) *Sufficiency*. Suppose

$$A^{-1}BB^*D=L(z) \tag{2.6}$$

for some polynomial $L(z)$. Then $L^*=D^*BB^*(A^*)^{-1}$, and hence $BB^*(A^*)^{-1}=(D^*)^{-1}L^*=DL^*$, because $(D^*)^{-1}=D$. Now $LD^{-1}=A^{-1}BB^*=UB^*$. Since U and B^* are proper, so is LD^{-1} ; therefore by Lemma 1 DL^* is polynomial. Now

$$\begin{aligned}\phi(z)\triangleq UU^*=A^{-1}BB^*(A^*)^{-1}&=A^{-1}BB^*D(A^*D)^{-1}\\ =L(A^*D)^{-1}&=A^{-1}(DL^*)\end{aligned} \tag{2.7}$$

Now (2.7) gives two alternative MFDs for $\phi(z)$. But $|A|$ has all its zeros in $|z|<1$ by stability of $U(z)$, while $|A^*D|$ has all its zeros in $|z|>1$, because by Lemma 2 $\det(A^*D)=z^p\det A(z^{-1})$, where $p=\deg|A(z)|$. Therefore $\phi(z)$ must be polynomial. But since U and U^* are proper and since U has full normal row rank, it follows that ϕ must be real and positive-definite. Therefore U is scaled para-unitary. □

Theorem 2

Let $W(z)$ be an $n\times m$ $(m\geqslant n)$ stable, real and rational proper transfer function matrix of full normal row rank and let $\phi(z)=W(z)W^*(z)$. Let $W(z)=A^{-1}(z)B(z)$ be a left co-prime MFD of $W(z)$, with $A(z)$ row-proper. Let $D(z)=\operatorname{diag}(z^{i_1},\ldots,z^{i_n})$, where i_k is the row degree of the kth row of $A(z)$. Then $W(z)$ is not a minimal-degree spectral factor of $\phi(z)$ if and only if $A(z)$ and $B(z)B^*(z)D(z)$ have a non-unimodular common left divisor $P(z)$. In addition, if $P(z)$ is a greatest common left divisor of A and BB^*D, and if $A(z)=P(z)\bar{A}(z)$, then any stable minimal-degree spectral factor $\hat{W}(z)$ of $\phi(z)$ can be written $\hat{W}(z)=\bar{A}(z)^{-1}C(z)$, for some polynomial $C(z)$.

Proof

(i) *Necessity*. Suppose $W(z)$ is not of minimal degree, and let $\phi(z)=\hat{W}(z)\hat{W}^*(z)$, where $\hat{W}(z)$ is a minimal-degree proper spectral factor. Then, by Theorem 1, $\hat{W}(z)=\bar{A}^{-1}C$ for some C, where $\delta[\hat{W}]=\partial|\bar{A}|=\frac{1}{2}\delta[\phi(z)]$, and $A=P\bar{A}$ for some P. Since $W(z)$ is not of minimal degree, P is not unimodular. Now

$$\begin{aligned}\phi(z)=(\bar{A})^{-1}CC^*(\bar{A}^*)^{-1}&=A^{-1}BB^*(A^*)^{-1}\\ &=(\bar{A})^{-1}P^{-1}BB^*(P^*)^{-1}(\bar{A}^*)^{-1}\end{aligned} \tag{2.8}$$

Therefore

$$P^{-1}BB^*D=CC^*P^*D \tag{2.9}$$

Now $D^{-1}A$ is proper by construction. Therefore $D^{-1}PC$ is proper, because $D^{-1}PC=D^{-1}P\bar{A}(\bar{A})^{-1}C=D^{-1}A(\bar{A})^{-1}C=D^{-1}A\hat{W}(z)$. Therefore, by Lemma 1, C^*P^*D is polynomial. Hence, by (2.9), P divides BB^*D, which implies that A and BB^*D have a common left divisor.

(ii) *Sufficiency.* Suppose A and BB^*D are not left co-prime, and let P be a non-unimodular greatest common left divisor, i.e.

$$A = P\bar{A}, \quad BB^*D = PL \tag{2.10}$$

for some polynomial matrices $\bar{A}$ and L. Since $W(z)$ is stable, P is analytic in $|z| \geqslant 1$. Then

$$\phi(z) = A^{-1}BB^*(A^*)^{-1} = (\bar{A})^{-1}P^{-1}BB^*(P^*)^{-1}(\bar{A}^*)^{-1} \tag{2.11}$$

We must show that $\psi(z) \triangleq P^{-1}BB^*(P^*)^{-1} = CC^*$ for some polynomial $C(z)$. Now

$$\psi(z) = P^{-1}BB^*(P^*)^{-1} = P^{-1}BB^*D(P^*D)^{-1} = L(P^*D)^{-1} \tag{2.12}$$

Also, from (2.10), $L^* = D^*BB^*(P^*)^{-1}$, and therefore $BB^*(P^*)^{-1} = (D^*)^{-1}L^* = DL^*$. Hence $\psi(z)$ can also be written

$$\psi(z) = P^{-1}BB^*(P^*)^{-1} = P^{-1}(DL^*) \tag{2.13}$$

Let a_i, $i = 1, \ldots, n$ be the rows of A, $\bar{a}_i$, $i = 1, \ldots, n$ the rows of $\bar{A}$, and p_{ij}, $i, j = 1, \ldots, n$ the elements of P. Then from (2.10)

$$a_i = \sum_{i=1}^{n} p_{ik}\bar{a}_k, \quad i = 1, \ldots, n \tag{2.14}$$

Since $\bar{A}$ is non-singular, no $\bar{a}_k$ is identically zero ; therefore (2.14) implies that the row degrees of P are smaller than or equal to the corresponding row degrees of A. Let $p_1, \ldots, p_n$ be the row degrees of P. Then D can be factored as $D = RS$, with $R = \operatorname{diag}(z^{p_1}, \ldots, z^{p_n})$ and $S = \operatorname{diag}(z^{s_1}, \ldots, z^{s_n})$, with $s_k = i_k - p_k$, $k = 1, \ldots, n$. (Recall that i_k is the row degree of the kth row of A.) By the above argument $s_k \geqslant 0$, $k = 1, \ldots, n$. Therefore in (2.12) $P^*D = (P^*R)T$, and $\det(P^*D) = z^p \det P(z^{-1}) \det T$, where $p \geqslant \deg |P|$, with equality holding if P is row-proper. Hence the poles of $L(P^*D)^{-1}$ in (2.12) are the inverses of the zeros of $|P|$ plus some poles at $z = 0$. Now we examine (2.13) which is identical to (2.12). Since L is polynomial, L^* has all its poles at $z = 0$, and so has DL^*. Hence the poles of $P^{-1}(DL^*)$ are the zeros of $|P|$ plus some poles at $z = 0$. Because P is stable, it follows that $\psi(z)$ has all its poles at $z = 0$. Therefore we can write $\psi(z) = H(z)H^*(z)$, where $H(z)$ is a real and rational matrix, not necessarily proper, that has all its poles at zero. Hence, without loss of generality, we can write $H(z) = C(z)/z^k$, where $C(z)$ is polynomial and z^k is the least common denominator of all the elements of $H(z)$. Therefore $\psi(z) = C(z)C^*(z)$, and, finally, $\phi(z) = \bar{A}^{-1}CC^*(\bar{A}^*)^{-1}$. This concludes the proof. □

3. Representations for joint processes

In this section we describe various transfer-function representations for joint (y, u)-processes. We restrict ourselves, as in the whole paper, to linear stationary processes.

A. *Representations of physical closed-loop processes*

First we consider the case where the process (y, u) is generated by a physical closed-loop system (see Fig. 1). It can then be represented, without loss of generality, by the following representation :

$$y_i = F(z)u_i + m_i \tag{3.1 a}$$

$$u_i = H(z)y_i + n_i \tag{3.1 b}$$

where $F(z)$ and $H(z)$ are causal, real, rational transfer-function matrices, and m_i and n_i are stationary noise processes, y_i and $m_i \in \mathbb{R}^p$, u_i and $n_i \in \mathbb{R}^q$. The process $\{u_i\}$ can be thought of as the input, and the process $\{y_i\}$ as the output, of a closed-loop system ; $\{m_i\}$ and $\{n_i\}$ are then noises in the feedforward and in the feedback path. The representation (3.1) will be called a *closed-loop representation* of the joint (y, u) process because the closed-loop relations between y and u are explicitly expressed. In addition to the assumptions made above, the following two standing assumptions will be made.

A 1 : There exists a delay somewhere in the closed loop, i.e. $F(\infty)H(\infty) = 0$, where $F(\infty) = \lim_{z \to \infty} F(z)$.

A 2 : The joint process (y, u) is a stationary full-rank bounded stochastic process.

By Assumption A 2, (m, n) is a full-rank process. Without loss of generality the joint noise process (m, n) can be represented as

$$\begin{bmatrix} m_i \\ n_i \end{bmatrix} = \begin{bmatrix} G(z) & J(z) \\ L(z) & K(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} \tag{3.2 a, b}$$

where G, J, L and K are causal, real, rational transfer-function matrices, $\dim w_i \geqslant \dim m_i = p$, $\dim v_i \geqslant \dim n_i = q$, and (w, v) is a white-noise process with a non-negative-definite covariance matrix Q :

$$E\left\{ \begin{bmatrix} w_i \\ v_i \end{bmatrix} [w_i^{\mathrm{T}} \;\; v_j^{\mathrm{T}}] \right\} = Q\delta_{ij}, \quad Q \geqslant 0 \tag{3.3}$$

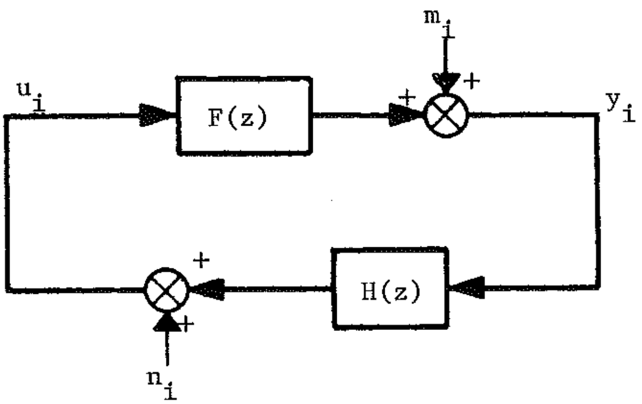

Figure 1.

We shall often, but not always, consider the case where $\dim w_i = p$ and $\dim v_i = q$, and where therefore Q is positive-definite by the full-rank assumption on (m, n).

From the 6-block closed-loop model (3.1)–(3.2), we can derive a matrix transfer-function model for the joint process (y, u) :

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & J \\ L & K \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} = \begin{bmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} = W(z) \begin{bmatrix} w_i \\ v_i \end{bmatrix} \quad (3.4)$$

This model will be called a joint matrix transfer-function model, or, in short, *joint model* for the (y, u) process. It is clear from (3.4) that, given the 6-block transfer-function matrices F, G, J, H, K and L of a closed-loop model, a joint model $W(z)$ can be uniquely computed :

$$\left. \begin{aligned} W_{11} &= (I - FH)^{-1}(G + FL), \quad & W_{12} &= (I - FH)^{-1}(J + FK) \\ W_{21} &= (I - HF)^{-1}(HG + L), \quad & W_{22} &= (I - HF)^{-1}(HJ + K) \end{aligned} \right\} \quad (3.5)$$

(The inverses exist by Assumption A 1.) The converse is, of course, not true : F, G, J, H, K and L cannot be recovered from the W_{ij}'s.

An important simplification is when $J(z) \equiv L(z) \equiv 0$ in (3.2). This corresponds to the fact that the forward-path noise m_i and the feedback-path noise n_i are generated by two separate white-noise sources that have at most instantaneous correlation. In such a case we have

$$\left. \begin{aligned} m_i &= G(z) w_i \\ n_i &= K(z) v_i, \quad w_i \perp v_j \quad \text{for } i \neq j \end{aligned} \right\} \quad (3.6)$$

The joint model now becomes

$$\begin{bmatrix} y_i \\ n_i \end{bmatrix} = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} \quad (3.7)$$

i.e.

$$W(z) \triangleq \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} (I - FH)^{-1}G & (I - FH)^{-1}FK \\ (I - HF)^{-1}HG & (I - HF)^{-1}K \end{bmatrix} \quad (3.8)$$

We call (3.7) a 4-block closed-loop model (see Fig. 2). An important property of this 4-block model is that the relation between F, G, H, K and $W(z)$ is now one-to-one and onto by (3.8) and the inverse relations†

$$\left. \begin{aligned} F &= W_{12} W_{22}^{-1}, \quad & H &= W_{21} W_{11}^{-1} \\ G &= W_{11} - W_{12} W_{22}^{-1} W_{21}, \quad & K &= W_{22} - W_{21} W_{11}^{-1} W_{12} \end{aligned} \right\} \quad (3.9)$$

† When G and K are not square (i.e. when $\dim w_i > p$, and/or $\dim v_i > q$), the inverses in (3.9) are understood to be right inverses. Their existence is guaranteed by the full-rank assumption on (y, u).

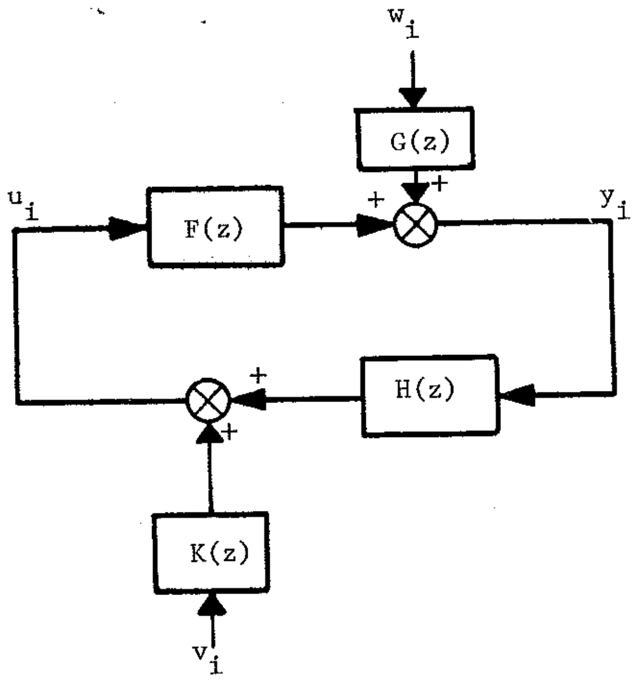

Figure 2.

A further simplification occurs when the noises m_i and n_i in the forward and feedback paths are completely uncorrelated. In such case m_i and n_i can be modelled by

$$\left.\begin{aligned} m_i &= G(z)w_i \\ n_i &= K(z)v_i, \quad w_i \perp v_j \quad \text{for all } i, j \end{aligned}\right\} \tag{3.10}$$

Many properties hold for 4-block models that do not hold for 6-block models; other properties will in addition require that $\{w_i\}$ and $\{v_j\}$ be uncorrelated for all i, j. This distinction is particularly important in the problem of identifiability of feedback processes (see Anderson and Gevers (1981 a) and Sin and Goodwin (1980)). In the remainder of this paper we shall restrict ourselves to processes that can be represented by 4-block closed-loop models, i.e. we shall assume $J(z) = L(z) = 0$; models with one-sided correlation between $\{m\}$ and $\{n\}$, namely with $J(z) = 0$ but $L(z) \neq 0$, are considered in Anderson and Gevers (1981 a).

In the 4-block model of Fig. 2, the noise process $\{m\}$ represents the effect on the output y of all the noises acting on the forward path. However, it might very well be that the actual physical noise enters the system at some internal part, or parts, of the plant $F(z)$. The same can be true for the feedback path. Therefore we shall also consider a generalization of the 4-block model represented by Fig. 3:

$$\left.\begin{aligned} y_i &= F_2(F_1 u_i + G_1 w_i) \\ u_i &= H_2(H_1 y_i + K_1 v_i) \end{aligned}\right\} \tag{3.11}$$

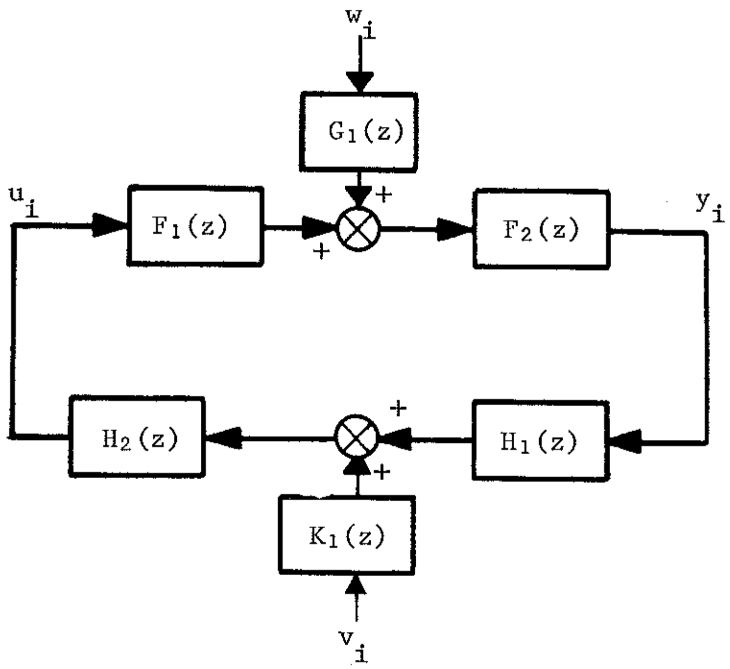

Figure 3.

If the noises are referred to the outputs, the system (3.11) is equivalent to the 4-block model (3.7) provided

$$\left.\begin{aligned} F&=F_2F_1, \quad G=F_2G_1 \\ H&=H_2H_1, \quad K=H_2K_1 \end{aligned}\right\} \tag{3.12}$$

However, as we shall see later, if the actual physical system has the structure of Fig. 3, the stability constraints on the matrix transfer functions F, G, H and K of the equivalent (by (3.12)) 4-block model are less stringent than if the noises really act on the input and output as in Fig. 2.

B. *Representation of a joint stationary process*

If no assumption is made about the existence of a physical interconnection structure for the joint (y, u) process, then any real stationary full-rank bounded stochastic process (y, u) with a rational power spectral density $\phi_{yu}(z)$ can be represented as

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} \tag{3.13}$$

where $W(z)$ is a stable rational transfer-function matrix, and (w, v) is a white-noise process satisfying (3.3). Here $\{W(z), Q\}$ is any spectral factor of $\phi_{yu}(z)$, for example the NMSF defined in § 2. Since a quadruple $\{F, G, H, K\}$ can be uniquely derived from $W(z)$ via (3.9), any joint process (y, u) can also be represented by a closed-loop 4-block model, provided it is stable :

$$y_i = F(z)u_i + G(z)w_i \tag{3.14 a}$$

$$u_i = H(z)y_i + K(z)v_i \tag{3.14 b}$$

with (w, v) satisfying (3.3). Conditions for stability of (3.14), obtained from a spectrum $\phi_{yu}(z)$, will be examined in § 4.

We stress again that the 4-block model (3.14) is valid in so far as it correctly describes the behaviour of the joint process (y, u). If the joint process does, in fact, result from the physical interconnections of a forward system $(u \rightarrow y)$ and a feedback system $(y \rightarrow u)$, then the individual models (3.14 *a*) and (3.14 *b*), obtained from a spectral factor $\{W(z), Q\}$ of $\phi_{yu}(z)$, will in general not be correct models of the true forward and feedback systems. The reason for this is that there exists an infinity of equivalent spectral factors that will lead to different quadruples $\{F, G, H, K\}$. The conditions under which the input–output and the feedback system can be separately identified from joint process data is the problem of the identifiability of a feedback process, which will not be pursued here. We refer the reader to Anderson and Gevers (1981 a).

We conclude this section by noting that the 4-block model (3.14) is equivalent to

$$A(z)y_i = B(z)u_i + C(z)w_i \tag{3.15 a}$$

$$D(z)u_i = M(z)y_i + N(z)v_i \tag{3.15 b}$$

A, B, C, D, M and N are polynomial matrices obtained via left co-prime MFDs of $[F \vdots G]$ and $[H \vdots K]$:

$$[F \vdots G] = A^{-1}[B \vdots C], \quad [H \vdots K] = D^{-1}[M \vdots N] \tag{3.16}$$

For given F, G, H and K, the triples $\{A, B, C\}$ and $\{D, M, N\}$ are unique up to left multiplication by unimodular matrices. Although we shall basically use the 4-block transfer function model (3.14) in the sequel, we shall often resort to the MFDs (3.16) in our proofs. In the process we shall have proved a number of results for the closed-loop difference-equation model (3.15).

4. Stability of 4-block closed-loop representations of joint processes

We have shown in § 3 that a joint process model for a joint process (y, u) can always be transformed into a 4-block closed-loop model. Also, in many practical cases, physical closed-loop processes can be represented by 4-block models. In this section we shall derive conditions for the 4-block models to be stable and minimum-phase.

We shall make frequent usage of the Smith canonical forms, and of the Smith–McMillan form (see, e.g. Rosenbrock (1970) and Kailath (1980)). Without going into any detail here we shall need to introduce some definitions. Let $A(z)$ be an $n \times n$ polynomial matrix. Then there exist unimodular matrices $M(z)$ and $N(z)$, and a diagonal matrix $\Lambda(z)$ such that

$$A(z) = M(z)\Lambda(z)N(z) \tag{4.1}$$

$\Lambda(z) = \text{diag}\,(\lambda_1(z), \ldots, \lambda_r(z), 0, \ldots, 0)$, where r is the normal rank of $A(z)$, and the $\lambda_i(z)$ are the invariant polynomials of $A(z)$. The decomposition (4.1) is termed the Smith form of $A(z)$. We shall often factor $\Lambda(z)$ into

$$\Lambda(z) = \Lambda_-(z)\Lambda_+(z) = \Lambda_+(z)\Lambda_-(z)$$

where $|\Lambda_-(z)|$ has all its roots in $|z| < 1$, and $|\Lambda_+(z)|$ has all its roots in $|z| \geqslant 1$, and $\Lambda_\pm$ are diagonal. Substituting in (4.1) gives a useful factorization of $A(z)$:

$$A(z) = (M\Lambda_-)(\Lambda_+ N) = A_-(z)A_+(z) \tag{4.2}$$

Then

$$|A(z)| = |A_-(z)| \cdot |A_+(z)| = a_-(z) \cdot a_+(z) = a(z) \tag{4.3}$$

where $a_-(z)$ has all its roots in $|z| < 1$ and $a_+(z)$ has all its roots in $|z| \geqslant 1$. In the sequel we shall often factor polynomial matrices $A(z)$ as $A_+(z)A_-(z)$, or $A_-(z)A_+(z)$ following the procedure just described ; accordingly the determinant $a(z)$ will be factored into $a_-(z)a_+(z)$ as in (4.3). It will be implicitly understood that $|A_-(z)|$ $(= a_-(z))$ has all its roots in $|z| < 1$ and $|A_+(z)|$ $(= a_+(z))$ has all its roots in $|z| \geqslant 1$. We shall sometimes refer to such factorizations as SPUP factorizations (SPUP for stable part, unstable part).

Let now $H(z)$ be a proper rational transfer-function matrix.

Definition 3

The *unstable part* of $H(z)$, written $H_+(z)$, is the sum of those terms in a partial-fraction expansion of $H(z)$ that have poles in $|z| \geqslant 1$.

The *stable part* is then defined as

$$H_-(z) \triangleq H(z) - H_+(z) \tag{4.4}$$

Evidently, since H_+ and H_- have no poles in common,

$$\delta[H] = \delta[H_+] + \delta[H_-] \tag{4.5}$$

Lemma 4

Let $H(z)$ be a proper matrix transfer function with $H(z) = A^{-1}(z)B(z)$, a co-prime MFD. Let $|A(z)| = a_+(z)a_-(z)$, where $a_+(z)$ has all its zeros in $|z| \geqslant 1$ and $a_-(z)$ in $|z| < 1$. Then

$$\delta[H_+] = \partial[a_+], \quad \delta[H_-] = \partial[a_-] \tag{4.6}$$

where $\partial[a_+]$ is the polynomial degree of $a_+(z)$.

Proof

Let $H_+(z) = E^{-1}F$, $H_- = JG^{-1}$, co-prime MFDs. Then $H = E^{-1}[FG + EJ]G^{-1}$. Now let $[FG + EJ]G^{-1} = L^{-1}K$, a co-prime MFD. Then $\delta[H] = \delta[H_+] + \delta[H_-] = \partial[|E|] + \partial[|G|] \geqslant \partial[|E|] + \partial[|L|]$. But $H = (LE)^{-1}K$, so $\delta[H] \leqslant \partial[|E|] + \partial[|L|]$. Hence $\delta[H] = \partial[|E|] + \partial[|L|]$, $\partial[|G|] = \partial[|L|]$, and $(LE)^{-1}K$ is co-prime. Then $|LE| = |A| = a_+(z)a_-(z)$. Evidently, owing to the location of the zeros of $|E|$ and $|G|$, $|E| = ka_+(z)$, $|G| = k^{-1}a_-(z)$, with k constant. □

We shall also have to consider pole-zero cancellations between transfer-function matrices.

Definition 4

Let F and G be two proper transfer-function matrices. We shall say that there is no pole-zero cancellation in the product FG if $\delta[FG]=\delta[F]+\delta[G]$. We shall say that there is no unstable pole-zero cancellation in the product FG if $\delta[(FG)_+]=\delta[F_+]+\delta[G_+]$.

This definition is, of course, consistent with the definition of pole-zero cancellations for scalar transfer functions. We have shown in Anderson and Gevers (1981 b) that this definition is equivalent to another definition based on the co-primeness of certain polynomial matrices obtained from MFDs of F and G.

After these preliminaries, we now consider the stability of the closed-loop representations of the (y, u) processes. As in § 3 we shall first consider the case where the joint (y, u) process is generated by a physical closed-loop system.

A. *Stability of closed-loop systems*

First we consider closed-loop models of the form of Fig. 1.

Lemma 5

Consider the closed-loop model of Fig. 1 obeying Assumption A 1. Let $F(z)=A^{-1}(z)B(z)$, a co-prime MFD, and $H(z)=C^{-1}(z)D(z)$, also co-prime. Then the closed-loop system is stable if and only if

$$\det\begin{bmatrix} A & -B \\ -D & C \end{bmatrix} \neq 0 \quad \text{for } |z| \geqslant 1 \tag{4.7}$$

The joint process (y, u) is stationary if and only if (4.7) holds and (m, n) is stationary.

Proof

From Fig. 1, we have

$$y_i = Fu_i + m_i$$

$$u_i = Hy_i + n_i$$

This is equivalent to

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} A & -B \\ -D & C \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} m_i \\ n_i \end{bmatrix} = W(z) \begin{bmatrix} m_i \\ n_i \end{bmatrix} \tag{4.8}$$

The inverses exist by Assumption A 1. Now (4.8) is a left co-prime MFD of $W(z)$. Indeed

$$\text{rank}\begin{bmatrix} A & -B & A & 0 \\ -D & C & 0 & C \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & -B & A & 0 \\ -D & 0 & 0 & C \end{bmatrix}\begin{matrix} \}p \\ \}q \end{matrix} \tag{4.9}$$

The rank of (4.9) is $p+q$ because both A, B and C, D are co-prime (Rosenbrock 1970). Therefore $W(z)$ is stable if and only if (4.7) holds, and (y, u) is stationary if and only if $W(z)$ is stable and (m, n) is stationary. □

The result of Lemma 5 was established in Callier and Desoer (1976) for a more general class of feedback systems using a different proof. In the next lemma we establish closed-loop stability using F and H directly.

Lemma 6

The closed-loop system of Fig. 1 obeying Assumption A 1 is stable if and only if either

(i) $$\left.\begin{array}{c}(I-FH)^{-1},\quad (I-FH)^{-1}F,\quad (I-HF)^{-1}H \text{ and} \\ (I-HF)^{-1} \text{ have all their poles in } |z|<1\end{array}\right\} \tag{4.10 a}$$

or

(ii) $(I-FH)^{-1}$ is stable and there is no unstable pole-zero cancellation in FH. (4.10 *b*)

Proof

Part (i) follows immediately from

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} (I-FH)^{-1} & (I-FH)^{-1}F \\ (I-HF)^{-1}H & (I-HF)^{-1} \end{bmatrix} \begin{bmatrix} m_i \\ n_i \end{bmatrix} \tag{4.11}$$

See also Desoer and Chan (1976) where it is shown that stability of one of the four blocks in (4.11) is not sufficient to guarantee (4.10 *a*). Part (ii) is established in Anderson and Gevers (1981 b). □

From the preceding discussion it follows that the joint (y, u) process is stationary if and only if the closed loop is stable and the entering noises are stationary. Conditions for the closed loop to be stable are either (4.7) or (4.10).

We now consider the case where the noises m_i and n_i are modelled by the shaping filters $m_i = G(z)w_i$ and $n_i = K(z)v_i$, and where these noises actually enter the closed loop as is shown on Fig. 2. The following result follows immediately from Lemmas 5 and 6.

Corollary 2

Consider the closed-loop system (3.14) and assume that the noises m_i and n_i actually enter the system at the outputs of the forward and feedback path, respectively (see Fig. 2). Then the process (y, u) is stationary if and only if

(1) the closed loop is stable, i.e. (4.7) or (4.10) holds ; (4.12 *a*)

(2) $G(z)$ and $K(z)$ are stable. (4.12 *b*)

If the noises enter at some internal part of the plant, as in Fig. 3, we have the following result.

Corollary 3

Consider the closed-loop system (3.11) represented by Fig. 3; then the joint process (y, u) is stationary if and only if

(1) the closed loop is stable (i.e. (4.7) or (4.10) holds, with F, G, H and K defined in (3.12)); (4.13 *a*)

(2) $G_1(z)$ and $K_1(z)$ are stable. (4.13 *b*)

Now in many cases a 4-block model $\{F, G, H, K\}$ will be used to represent a physical closed-loop process that actually has the structure of Fig. 3. Suppose thus that the model of Fig. 2 has been obtained by referring to the outputs noise sources that actually enter the plant and the regulator at some internal points. Then we can derive stability conditions on $\{F, G, H, K\}$ from (4.13), which will be weaker than the conditions (4.12). The main result is Theorem 3 below, and it depends on two lemmas which follow.

Lemma 7

Let the closed-loop model (3.14) (Fig. 2) be obtained from a closed-loop model (3.11) (Fig. 3). Suppose that G_1 and K_1 are stable, and that there are no unstable pole-zero cancellations in F_2F_1, nor in H_2H_1. Then

$$\delta[F \vdots G]_+ = \delta[F]_+ \quad \text{and} \quad \delta[H \vdots K]_+ = \delta[H]_+ \tag{4.14}$$

where $\delta[F \vdots G]_+$ is the McMillan degree of the unstable part of the $p \times (q+p)$ transfer-function matrix $[F(z) \vdots G(z)]$†, etc.

Proof

By assumption, $F = F_2F_1$, $G = F_2G_1$ and G_1 is stable. Therefore $G_{1+}(z) = 0$, and hence $\delta[F_1 \vdots G_1]_+ = \delta[F_1]_+$. Let $[F_1 \vdots G_1] = AB^{-1}$, co-prime, and let $|B| = b_+(z)b_-(z)$, a SPUP factorization. Then, by Lemma 4, $\delta[F_1 \vdots G_1]_+ = \delta[F_1]_+ = \partial[b_+(z)]$. Now let $F_2 = C^{-1}D$, co-prime, and let $|C| = c_+(z)c_-(z)$, defined similarly. Then $\delta[F_2]_+ = \partial[c_+(z)]$. Since there are no unstable pole-zero cancellations in $F = F_2F_1$, $\delta[F]_+ = \delta[F_1]_+ + \delta[F_2]_+ = \partial[b_+(z)] + \partial[c_+(z)]$. Also $[F \vdots G] = F_2[F_1 \vdots G_1] = C^{-1}DAB^{-1}$. Let $DAB^{-1} = T^{-1}U$, co-prime, and let $|T| = t_+(z)t_-(z)$ with $t_+(z)$ and $t_-(z)$ defined as usual. Then

$$\delta[DAB^{-1}]_+ = \delta[T^{-1}U]_+ = \partial[t_+(z)] \leqslant \partial[b_+(z)]$$

Also $[F \vdots G] = (TC)^{-1}U$ and $|TC| = t_+(z)c_+(z)t_-(z)c_-(z)$. Therefore

$$\delta[F \vdots G]_+ \leqslant \partial[t_+] + \partial[c_+] \leqslant \partial[b_+] + \partial[c_+] = \delta[F]_+$$

But, obviously, $\delta[F \vdots G]_+ \geqslant \delta[F]_+$. Therefore $\delta[F \vdots G]_+ = \delta[F]_+$. By the same argument $\delta[H \vdots K]_+ = \delta[H]_+$. □

Conversely we now show that, if a 4-block model satisfies assumptions (4.14), then it can be represented by a closed-loop model (3.11) (Fig. 3) with a stable G_1 and a stable K_1, and no unstable pole-zero cancellations in F_2F_1 nor in H_2H_1. We present the result for the forward-path model; an identical argument holds for the feedback model.

† For simplicity of notation we shall often use $\delta[F \vdots G]_+$ as a shorthand for $\delta[(F \vdots G)_+]$. By extension $\delta[F]_+$ will often be used for $\delta[F_+]$.

Lemma 8

Let F and G be proper rational matrices with the same number of rows, and assume that

$$\delta[F \vdots G]_+ = \delta[F]_+ \tag{4.15}$$

Then there exist proper rational matrices F_1, G_1 and F_2 with

$$\delta[F_2]_+ = \delta[F_2] = \delta[G]_+, \quad \delta[G_1]_+ = 0 \tag{4.16 a}$$

$$F = F_2 F_1, \quad G = F_2 G_1 \tag{4.16 b}$$

and there are no unstable pole zero cancellations in $F_2 F_1$.

Proof

Let $G = E^{-1}C_1$, co-prime. Let E_-E_+ be a SPUP factorization of E with E_+ row-proper. Then $\delta[G]_+ = \partial[|E_+|]$. Let $[F \vdots G] = A^{-1}[B \vdots C]$, a co-prime MFD. Then $A = PE$, $C = PC_1$ for some polynomial P, and hence

$$[F \vdots G] = E_+^{-1} E_-^{-1} P^{-1}[B \vdots C] \tag{4.17}$$

Let $D = \text{diag}(z^{i_1}, \ldots, z^{i_p})$, where p = number of rows of F, and i_k = row-degree of the kth row of E_+. Then $[F \vdots G] = E_+^{-1} D(PE_-D)^{-1}[B \vdots D]$ where $E_+^{-1} D$ and $(PE_-D)^{-1}[B \vdots C]$ are proper by construction of D. We now define

$$F_2 \triangleq E_+^{-1} D, \quad F_1 \triangleq (PE_-D)^{-1}B, \quad G_1 \triangleq (E_-D)^{-1}C_1 \tag{4.18}$$

Then $F = F_2 F_1$, $G = F_2 G_1$, $\delta[G_1]_+ = 0$ because E_- and D have all their roots in $|z| < 1$; also $\delta[F_2]_+ = \delta[F_2] = \partial[|E_+|] = \delta[G]_+$, because E_+ and D are co-prime, and so are E and C_1. It remains to be shown that there are no unstable pole-zero cancellations in $F_2 F_1$. Let $PE_- = M_-M_+$, a SPUP factorization. Then by (4.17) $[F \vdots G] = (M_+E_+)^{-1}M_-^{-1}[B \vdots C]$. Since this factorization is co-prime, we have

$$\delta[F \vdots G]_+ = \partial[|M_+|] + \partial[|E_+|] = \delta[F]_+ \tag{4.19}$$

the last equality by assumption (4.15). But by (4.18), the form of D and the above factorization of PE_-, we have $\delta[F_1]_+ = \partial[|M_+|]$ and $\delta[F_2]_+ = \partial[|E_+|]$. Therefore $\delta[F]_+ = \delta[F_1]_+ + \delta[F_2]_+$, and hence there are no unstable pole-zero cancellations in $F_2 F_1$. □

Combining Lemmas 7 and 8, we have now proved the following theorem.

Theorem 3

Consider a joint process (y, u) described by a 4-block closed-loop model (3.14) (see Fig. 2). Then the joint process (y, u) is stationary if and only if

(i) the closed loop is stable (i.e. (4.10) or (4.7) holds); (4.20 *a*)

(ii) $\delta[F \vdots G]_+ = \delta[F_+]$, $\delta[H \vdots K]_+ = \delta[H_+]$. (4.20 *b*)

Note : It is understood here that the 4-block model of Fig. 2 may or may not result from a model such as in Fig. 3.

Proof

(*a*) Suppose (4.20) holds. Then by Lemma 8, there exists a representation of the form (3.11) for (y, u), where G_1 and K_1 are stable and with no unstable pole-zero cancellations in the loop $F_2F_1H_2H_1$. The result follows by Corollary 3.

(*b*) If the joint process is stationary, then the closed loop must be stable by (4.12) or (4.13). If the 4-block model represents the interconnection structure of the physical system, then G and K must also be stable by (4.12): this obviously implies (4.20). If, on the other hand, the 4-block model is obtained by reduction of a model of the form (3.11) (or Fig. 3), then by (4.13 *b*) G_1 and K_1 must be stable, and by (4.13 *a*) the closed loop must be stable. Therefore (Anderson and Gevers 1981 b) there can be no unstable pole-zero cancellations in the closed loop. This, in turn, implies (4.20) by Lemma 7. □

Next we give necessary and sufficient conditions for stationarity of the joint process as a function of the MFDs of the 4-block transfer functions F, G, H and K.

Theorem 4

Consider the joint process (y, u) represented by the 4-block closed-loop model (3.14). Let $A^{-1}[B \vdots C]$ be a left co-prime MFD of $[F \vdots G]$, and $D^{-1}[M \vdots N]$ be a left co-prime MFD of $[H \vdots K]$. Then the joint process is stationary if and only if

$$\deg \begin{bmatrix} A & -B \\ -M & D \end{bmatrix} \neq 0 \quad \text{for } |z| \geqslant 1 \tag{4.21}$$

Proof

The model (3.14) is equivalent to

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} A & -B \\ -M & D \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} = W(z) \begin{bmatrix} w_i \\ v_i \end{bmatrix} \tag{4.22}$$

(*a*) *Necessity*. (Note: The result would be immediate if the factorization in (4.22) were left co-prime, but this is not always so.) Let $F = A_1^{-1} B_1$, co-prime, and $H = D_1^{-1} M_1$, also co-prime. By Theorem 3 and Lemma 5

$$\det \begin{bmatrix} A_1 & -B_1 \\ -M_1 & D_1 \end{bmatrix} \neq 0 \quad \text{for } |z| \geqslant 1 \tag{4.23}$$

Now suppose $A = PA_1$, $B = PB_1$, and similarly $D = QD_1$, $M = QM_1$ for some polynomial matrices P and Q. Then $[F \vdots G] = (PA_1)^{-1}[PB_1 \vdots C]$, co-prime. By Theorem 3, $\delta[F \vdots G]_+ = \delta[F]_+ = \partial[|A_1|_+]$. Therefore $\partial|P_+| = 0$, i.e. $|P| \neq 0$ for $|z| \geqslant 1$. By the same argument $|Q| \neq 0$ for $|z| \geqslant 1$. Hence (4.21) follows from (4.23) and the fact that we can write

$$\begin{bmatrix} A & -B \\ -M & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A_1 & -B_1 \\ -M_1 & D_1 \end{bmatrix} \tag{4.24}$$

(*b*) *Sufficiency.* Assumption (4.21) and the form (4.24) implies that (4.23) holds and that $|P|$ and $|Q|$ have all their zeros in $|z|<1$. Therefore by Lemma 5 the closed loop is stable, and from $F=A_1^{-1}B_1$, co-prime and $[F \vdots G]=(PA_1)^{-1}[PB_1 \vdots C]$, co-prime, it follows that

$$\delta[F_+]=\delta[F \vdots G]_+=\partial[|A_1|_+] \tag{4.25}$$

Similarly for $\delta[H_+]=\delta[H \vdots K]_+$. Therefore, by Theorem 3, the joint process is stationary. □

The next two results give necessary and sufficient conditions on F, G, H and K and their MFDs for $W(z)$ to be minimum-phase.

Lemma 9

Consider a stationary joint process (y, u) represented by the closed-loop model (3.14). Let $A^{-1}[B \vdots C]$ be a co-prime MFD of $[F \vdots G]$, and $D^{-1}[M \vdots N]$ be a co-prime MFD of $[H \vdots K]$. Let $W(z)$ be the matrix transfer function of the joint process computed from (4.22). Then $W(z)$ is minimum-phase if and only if

$$\det C(z) \det N(z) \neq 0 \quad \text{for } |z|>1$$

Proof

$$W(z)=\begin{bmatrix} A & -B \\ -M & D \end{bmatrix}^{-1}\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$$

By Theorem 4, (4.21) holds. Therefore any common factor in the above MFD of $W(z)$ must be stable. Therefore $W(z)$ is minimum-phase if and only if $|C|$ and $|N|$ have all their zeros in $|z| \leqslant 1$. □

Theorem 5

Consider a stationary joint process (y, u) represented by the closed-loop model (3.14). Let $W(z)$ be the joint process matrix transfer function computed from (3.14) via (3.8). Then $W(z)$ is minimum-phase if and only if

(i) $$\delta[F \vdots G]_+=\delta[G]_+, \quad \delta[H \vdots K]_+=\delta[K]_+ \tag{4.26 a}$$

(ii) G and K are minimum-phase (4.26 *b*)

Proof

(*a*) *Sufficiency.* Suppose (4.26) holds. Then, by the stationarity of (y, u), we have, using Theorem 3,

$$\delta[F \vdots G]_+=\delta[G]_+=\delta[F]_+, \quad \delta[H \vdots K]_+=\delta[K]_+=\delta[H]_+ \tag{4.27}$$

Let $[F \vdots G]=A^{-1}[B \vdots C]$, co-prime, and let $A=A_-A_+$ be a SPUP factorization of A, with A_+ row-proper. Let $D=\operatorname{diag}(z^{i_1}, \ldots, z^{i_p})$, where $i_1, \ldots, i_p$ are the row degrees of A_+. Then

$$[F \vdots G]=A_+^{-1}D(A_-D)^{-1}[B \vdots C] \tag{4.28}$$

where $A_+^{-1} D$ and $(A\_D)^{-1}[B \vdots C]$ are proper by construction of D. We now define $F_2 \triangleq A_+^{-1} D$, $F_1 \triangleq (A\_D)^{-1}B$, $G_1 \triangleq (A\_D)^{-1}C$. Then

$$F = F_2F_1, \quad G = F_2G_1, \quad \delta[F_1]_+ = \delta[G_1]_+ = 0 \tag{4.29 a}$$

$$\delta[F_2]_+ = \delta[F_2] = \partial[|A_+|] = \delta[F \vdots G]_+ \tag{4.29 b}$$

Now, by assumption (4.27),

$$\delta[F_+] = \delta[G_+] = \delta[F \vdots G]_+ = \delta[F_2]_+ = \delta[F_1]_+ + \delta[F_2]_+ = \delta[G_1]_+ + \delta[F_2]_+$$

by (4.29). Therefore there are no unstable pole-zero cancellations in F_2F_1 nor in F_2G_1. A similar argument shows that one can write

$$H = H_2H_1, \quad K = H_2K_1, \quad \delta[H_1]_+ = \delta[K_1]_+ = 0 \tag{4.30 a}$$

$$\delta[H_2]_+ = \delta[H_2] = \delta[H \vdots K]_+ \tag{4.30 b}$$

where H_2H_1 and H_2K_1 have no unstable pole-zero cancellations. Therefore the 4-block model is equivalent to a model of the form of Fig. 3, where F_1, F_2 and G_1, and H_1, H_2 and K_1 have the stability properties indicated by (4.29) and (4.30). The matrix transfer function $W(z)$ can be written

$$W(z) = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}$$

It is stable by the stationarity assumption on (y, u). Evidently $W^{-1}(z)$ exists precisely when $G^{-1}(z)$ and $K^{-1}(z)$ exist. Then we have

$$W^{-1}(z) = \begin{bmatrix} G^{-1} & -G^{-1}F \\ -K^{-1}H & K^{-1} \end{bmatrix}$$

By (4.29) and (4.30)

$$G^{-1}F = G_1^{-1} F_1, \quad K^{-1}H = K_1^{-1} H_1, \quad \delta[F_1]_+ = \delta[H_1]_+ = 0$$

Also by (4.26 *b*), G^{-1} and K^{-1} are analytic in $|z| > 1$, and therefore so are G_1 and K_1. Hence $W(z)$ is minimum-phase.

(*b*) *Necessity*. Let $[F \vdots G] = A^{-1}[B \vdots C]$, co-prime, and $[H \vdots K] = D^{-1}[M \vdots N]$, co-prime. Then by Lemma 9, $W(z)$ minimum-phase implies $\det C \det N \neq 0$ for $|z| > 1$. Since $G = A^{-1}C$ and $K = D^{-1}N$, it follows that G and K are minimum-phase. Now let $G = E^{-1}C_1$, co-prime, and suppose $A = PE$, $C = PC_1$ for some polynomial P. Then $|P| \neq 0$ in $|z| > 1$, because $|C| \neq 0$ for $|z| > 1$. Let $E = E\_E_+$, a SPUP factorization. Then $\delta[G]_+ = \partial[|E_+|]$. Also $[F \vdots G] = A^{-1}[B \vdots C] = E_+^{-1}(PE\_)^{-1}[B \vdots C]$. Since this factorization is co-prime, it follows that $\delta[F \vdots G]_+ = \partial[|E_+|] = \delta[G]_+$. A similar argument shows that $\delta[H \vdots K]_+ = \delta[K]_+$. □

Corollary 4

Consider a stationary joint process (y, u) represented by the closed-loop model (3.14). Let $W(z)$ be the associated joint process matrix transfer function and let $F(z)$ and $H(z)$ be stable. Then $W(z)$ is minimum-phase if and only if $G(z)$ and $K(z)$ are minimum-phase.

Proof

If F and H are stable, it follows from Theorem 3 that $\delta[F \vdots G]_+ = \delta[H \vdots K]_+ = 0$ (by (4.20 *b*)). Hence condition (4.26 *a*) of Theorem 5 is obviously satisfied.$\square$

B. *Stability of closed-loop models of joint processes*

So far in this section we have given conditions under which a joint process (y, u), generated by a closed-loop system $\{F, G, H, K\}$ is stationary, which in turn implies that $W(z)$ is stable. In the last part of this section we consider the case where a joint stationary process (y, u) is given, described by either a white-noise driven transfer-function model $W(z)$ or a spectrum $\phi_{yu}(z)$. The question then arises as to whether there always exists an equivalent stable closed-loop representation of (y, u). Our first example shows that this is not always so.

Example 1

$$W(z) = \begin{bmatrix} \dfrac{z-0\cdot7}{z+0\cdot8} & \dfrac{1}{z+0\cdot8} \\ \dfrac{1\cdot95}{z+0\cdot8} & \dfrac{z-0\cdot5}{z+0\cdot8} \end{bmatrix}$$

From (3.9), $F = 1/(z-0\cdot5)$, $G = (z-2)/(z-0\cdot5)$, $H = 1\cdot95/(z-0\cdot7)$ and $K = (z-2)/(z-0\cdot7)$. $W(z)$ is stable, and so are G and K; but $(1-HF)^{-1} = \{(z-0\cdot5)(z-0\cdot7)\}/\{(z-2)(z+0\cdot8)\}$ is unstable. Hence $W(z)$ does not have an equivalent stable 4-block closed-loop model.

We now show that for almost every joint spectrum $\phi_{yu}(z)$ one can construct a stable 4-block closed-loop model.

Theorem 6

Let $\phi_{yu}(z)$ be the real rational spectral-density matrix of a full-rank stationary (y, u)-process and assume that $\phi_{yu}(z)$ is positive-definite on $|z| = 1$. Let $W(z)$ be a stable minimum-phase spectral factor of $\phi_{yu}(z)$. Then the 4-block closed-loop model (3.14), obtained from $W(z)$ via (3.9), is stable.

Proof

Let $W(z)$ be partitioned as usual, and let

$$\begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} A_1^{-1}, \quad \begin{bmatrix} W_{12} \\ W_{22} \end{bmatrix} = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} A_2^{-1}$$

be right co-prime factorizations. Then

$$W(z) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^{-1} \tag{4.31}$$

By stability of $W(z)$, $|A_1|$ and $|A_2|$ have all their zeros in $|z| < 1$. Therefore, it follows from the minimum-phase assumption on $W(z)$ and the fact that $\phi_{yu}(z)$ has no zero on $|z|=1$, that

$$\det \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \neq 0 \quad \text{in } |z| \geqslant 1 \tag{4.32}$$

From (3.9), $F=W_{12}W_{22}^{-1}=B_{12}B_{22}^{-1}$ and $H=W_{21}W_{11}^{-1}=B_{21}B_{11}^{-1}$. Then, using a result of Callier and Desoer (1976), (4.32) implies that the closed loop in Fig. 2 is stable, i.e. (4.10) holds. It remains to show that (4.20 *b*) holds. From (3.9), $G=(B_{11}-B_{12}B_{22}^{-1}\,B_{21})A_1^{-1}$. Therefore

$$[F \vdots G]=B_{12}B_{22}^{-1}[I \vdots -B_{21}A_1^{-1}]+[0 \vdots B_{11}A_1^{-1}]$$

Since $|A_1|$ has all its zeros in $|z|<1$, we have

$$\begin{aligned}\delta[F \vdots G]_+ &\leqslant \delta[B_{12}B_{22}^{-1}]_+ + \delta[I \vdots -B_{21}A_1^{-1}]_+ + \delta[0 \vdots B_{11}A_1^{-1}]_+ \\ &= \delta[B_{12}B_{22}^{-1}]_+ = \delta[F_+]\end{aligned}$$

Obviously, also, $\delta[F \vdots G]_+ \geqslant \delta[F_+]$. Therefore $\delta[F \vdots G]_+ = \delta[F_+]$. By the same argument $\delta[H \vdots K]_+ = \delta[H_+]$, and so, by Theorem 3, the closed-loop system is stable. □

Theorem 6 tells us that any stable strictly minimum-phase $W(z)$ produces a stable closed-loop model F, G, H, K. However the strictly minimum-phase condition on $W(z)$ is not necessary as the following example indicates. One reason for this is that the factorizations $F=B_{12}B_{22}^{-1}$ and $H=B_{21}B_{11}^{-1}$ may not be co-prime ; therefore (4.32) is not a necessary and sufficient condition for stability.

Example 2

$$W(z)=\begin{bmatrix} \dfrac{z-1}{z+0\cdot5} & \dfrac{1}{z+0\cdot5} \\[2ex] \dfrac{-1\cdot5(z-1)}{z+0\cdot5} & \dfrac{z-1}{z+0\cdot5} \end{bmatrix}, \quad \text{stable, non-minimum phase}$$

$$F=\frac{1}{z-1}, \quad G=1, \quad H=-1\cdot5, \quad K=1$$

Note : $(1-HF)^{-1}F=1/(z+0\cdot5)$ and the other three blocks of (4.11) are also stable. The factorization of $W(z)$ in the form (4.31) is

$$W(z)=\begin{bmatrix} z-1 & 1 \\ -1\cdot5(z-1) & z-1 \end{bmatrix}\begin{bmatrix} z+0\cdot5 & 0 \\ 0 & z+0\cdot5 \end{bmatrix}^{-1}$$

Two things can be observed : (1) this factorization is not co-prime ; (2) (B_{21}, B_{11}) is not co-prime either.

In a last example we show that if $\phi_{yu}(z)$ has a zero on the unit circle, then a stable (necessarily non-strictly) minimum-phase spectral factor $W(z)$ can yield an unstable closed-loop model.

Example 3

$$\phi_{yu}(z)=\frac{1}{d(z)}\begin{bmatrix} 1\cdot 25 & -z-0\cdot 75+0\cdot 5z^{-1} \\ 0\cdot 5z-0\cdot 75-z^{-1} & 4\cdot 25-1\cdot 5(z+z^{-1}) \end{bmatrix}$$

with $d(z)=(z-0\cdot 4)(z-0\cdot 6)(z^{-1}-0\cdot 4)(z^{-1}-0\cdot 6)$. Note that $\phi_{yu}(1)$ is singular. The following is a stable minimum-phase spectral factor :

$$W(z)=\begin{bmatrix} \dfrac{z}{z-0\cdot 4} & \dfrac{0\cdot 5}{z-0\cdot 6} \\ \dfrac{-1}{z-0\cdot 4} & \dfrac{z-1\cdot 5}{z-0\cdot 6} \end{bmatrix}$$

This leads to $(1-HF)^{-1}=\{z(z-1\cdot 5)\}/\{(z-1)(z-0\cdot 5)\}$, which has an unstable pole at $z=1$.

5. Relations between closed-loop representations of equivalent spectral factors

In this section we investigate some connections between the 4-block representations of equivalent spectral factors, and we present a set of conditions on the closed-loop models that will guarantee that the corresponding $W(z)$ is of minimal degree. The results presented here have importance in the solution of the identifiability question for closed-loop processes. The first lemma will be needed in several subsequent proofs. Assumptions A 1 and A 2 stated in the beginning of § 3 will stand throughout this section and will not be explicitly restated.

Lemma 10

Consider the 4-block closed-loop process (3.14), and let $A^{-1}[B \vdots C]$ be a left co-prime MFD of $[F \vdots G]$, and $D^{-1}[M \vdots N]$ be a left co-prime MFD of $[H \vdots K]$. If for all z, one at least of $C(z)$ and $N(z)$ has full row rank, then

$$\begin{bmatrix} A & -B \\ -M & D \end{bmatrix}^{-1}\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}=W(z) \tag{5.1}$$

is a left co-prime MFD of $W(z)$.

Proof

Suppose the two factors are not left co-prime. Then there must exist z_0 such that the following matrix is not full-rank :

$$\begin{bmatrix} A & -B & C & 0 \\ -M & D & 0 & N \end{bmatrix} \tag{5.2}$$

Hence one at least of $C(z_0)$, $N(z_0)$ looses full rank. Suppose it is $C(z_0)$. Since A, B and C are left co-prime, the matrix $[A \;\; -B \;\; C \;\; 0]_{z=z_0}$ has full rank. Since the matrix (5.2) is not full-rank, it follows that $N(z_0)$ does not have full rank, a contradiction. □

The condition that $C(z)$ and $N(z)$ do not drop in rank at the same point is not a necessary condition for co-primeness, as can be checked for the case $F=1/z$, $G=1$, $H=0{\cdot}25/z$, $K=1$. This leads to $A=z$, $B=1$, $C=z$, $D=z$, $M=0{\cdot}25$, $N=z$, and the co-prime factorization

$$W(z)=\begin{bmatrix} z & -1 \\ -0{\cdot}25 & z \end{bmatrix}^{-1}\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

On the other hand, (5.1) can be not-co-prime. Consider

$$F=\frac{1}{z+0{\cdot}5}, \quad G=\frac{z+0{\cdot}6}{z+0{\cdot}4}, \quad H=0, \quad K=\frac{z+0{\cdot}5}{z}$$

With $A=(z+0{\cdot}5)(z+0{\cdot}4)$, $B=z+0{\cdot}4$, $C=(z+0{\cdot}6)(z+0{\cdot}5)$, $D=z$, $M=0$, $N=z+0{\cdot}5$, it is easy to check that the factorization (5.1) is not co-prime.

The next theorem gives conditions for a closed-loop model to be of minimal degree.

Theorem 7

Consider the stationary closed-loop process (3.14). Let $[F \,\vdots\, G]=A^{-1}[B \,\vdots\, C]$, co-prime, and $[H \,\vdots\, K]=D^{-1}[M \,\vdots\, N]$, co-prime, with A and D row-proper. Suppose that for all z at least one of $C(z)$ and $N(z)$ has full rank. Let $D_1(z)=\operatorname{diag}(z^{a_1}, \ldots, z^{a_p})$, where a_k is the row degree of the kth row of A, and let $D_2(z)=\operatorname{diag}(z^{d_1}, \ldots, z^{d_q})$, where d_k is the row degree of the kth row of D. Let $W(z)$ be the corresponding matrix transfer function obtained by (3.8) or (5.1), and let $\phi(z)=W(z)W^*(z)$. Then $W(z)$ is of minimal degree if and only if the polynomial matrix

$$\begin{bmatrix} A & -B & CC^*D_1 & 0 \\ -M & D & 0 & NN^*D_2 \end{bmatrix} \tag{5.3}$$

has full rank for all z.

Proof

Since $A^{-1}B=F$ is proper, a_k is the row degree of the kth row of $[A \,\vdots\, -B]$, $k=1, \ldots, p$; similarly d_k is the row degree of the kth row of $[-M \,\vdots\, D]$, $k=1, \ldots, q$. The result is then a consequence of Theorem 2. □

Lemma 10, Theorem 7 and Corollary 5 offer important tools to check whether the MFD for $W(z)$, obtained from a 4-block closed-loop model by (4.22), is both co-prime and of minimal degree. We now label such models.

Definition 5

Let F, G, H and K be the transfer-function matrices of a 4-block closed-loop system (3.14), and let $[F \,\vdots\, G]=A^{-1}[B \,\vdots\, C]$ and $[H \,\vdots\, K]=D^{-1}[M \,\vdots\, N]$ be left co-prime factorizations. The model $\{F, G, H, K\}$ is called *generic* if for every z

(i) $\det CC^*D_1$ and $\det NN^*D_2$ are co-prime (5.4)

(ii) the matrix (4.3) has full rank (5.5)

Condition (5.4) implies that C and N do not drop in rank at the same z_0; hence the factorization (5.1) is co-prime by Lemma 10. Condition (5.5) implies that $W(z)$ is a minimal-degree spectral factor of $\phi_{yu}(z)$. In the next theorem we give sufficient conditions for genericity in terms of F, G, H and K. With $F=A^{-1}B$, co-prime and $D=\text{diag}\,(z^{i_1}, \ldots, z^{i_k})$, where $i_1, \ldots, i_k$ are the row degrees of the rows of B, we introduce the following notation : $p(F)=\{$poles of $F\}$, $p^{-1}(F)=\{$finite inverses of the poles of $F\}$, $z(FF^*)=\{z_0$ such that BB^*D is singular at $z_0\}$, $z^{-1}(FF^*)=\{$finite inverses of the elements of $z(FF^*)\}$. Note that $z(FF^*)$ is *not* obtained by using a left co-prime realization of FF^*, say $M^{-1}N$, and defining $z(FF^*)$ as the set of z_0 for which $N(z_0)$ is singular. For square F, we denote $z(F)=\{$finite zeros of $F\}$, $z^{-1}(F)=\{$finite inverses of the zeros of $F\}$.

Theorem 8

Consider the stationary closed-loop system (3.14) and its corresponding $W(z)$ defined by (3.8). Assume that $G(\infty)$ and $K(\infty)$ have full rank. The system is generic if the following conditions all hold :

(1) $$p(G)\cap\{p(F)\cup p^{-1}(F)\cup z(GG^*)\} \text{ is empty} \tag{5.6 a}$$

(2) $$p(K)\cap\{p(H)\cup p^{-1}(H)\cup z(KK^*)\} \text{ is empty} \tag{5.6 b}$$

(3) $$\{p(F)\cup p^{-1}(F)\cup z(GG^*)\}\cap\{p(H)\cup p^{-1}(H)\cup z(KK^*)\} \text{ is empty} \tag{5.7}$$

Proof

Let $[F\,\vdots\,G]=A^{-1}[B\,\vdots\,C]$, co-prime, and $[H\,\vdots\,K]=D^{-1}[M\ N]$, co-prime, with A and D row-proper. Then $\lim_{z\to\infty} A^{-1}C=G(\infty)$ has full rank. Therefore the row degrees of A equal the corresponding row degrees of C. Let D_1 and D_2 be constructed as in Theorem 7. Now let $G=A_1^{-1}C_1$, co-prime ; then $A=PA_1$, $C=PC_1$ for some polynomial P, and the zeros of $|P|$ belong to $p(F)$. Then, using Lemma 1, it is easy to see that the zeros of $|CC^*D_1|$ belong to $\{p(F)\cup p^{-1}(F)\cup z(GG^*)\}$. Similarly the zeros of $|NN^*D_2|$ belong to $\{p(H)\cup p^{-1}(H)\cup z(KK^*)\}$.

Now consider (5.3). By (5.7), CC^*D_1 and NN^*D_2 cannot be singular at the same z_0. Suppose $|CC^*D_1|_{z_0}=0$. Then (5.3) has full rank if $[A\,\vdots\,-B]_{z_0}$ has full rank ; since $[A\,\vdots\,-B]$ can only drop in rank for $z_0\in p(G)$, this is guaranteed by (5.6 a). If $|NN^*D_2|_{z_0}=0$, (5.3) has full rank by (5.6 b). □

For square $W(z)$, Theorem 8 can easily be simplified.

Corollary 5

Under the same conditions as in Theorem 8, but with $G(z)$ and $K(z)$ square, the system is generic if the following conditions all hold :

(1) $$p(G)\cap\{z^{-1}(G)\cup p^{-1}(F)\} \text{ is empty} \tag{5.8 a}$$

(2) $$p(K)\cap\{z^{-1}(K)\cup p^{-1}(H)\} \text{ is empty} \tag{5.8 b}$$

(3) $$\{p(F)\cup p^{-1}(F)\cup z(G)\cup z^{-1}(G)\} \cap\{p(H)\cup p^{-1}(H)\cup z(K)\cup z^{-1}(K)\} \text{ is empty} \tag{5.9}$$

We notice that, save in exceptional cases, a closed-loop system represented by (3.14) will always obey the conditions (5.6)–(5.7). Hence the word ' generic ' for systems that obey those conditions. We recall that there is a bijective relation (one-to-one and onto) between $\{F, G, H, K\}$ and $W(z)$. This motivates the idea of genericity for $W(z)$.

Definition 6

A joint process matrix transfer function $W(z)$ is generic if it has a stable equivalent 4-block model and if $\{F, G, H, K\}$ obeys the genericity conditions (5.6)–(5.7).

In the remainder of this section, we present some results concerning the relations between generic matrix transfer-function models $W(z)$ for processes (y, u) generating the same spectrum. We first need the following technical result.

Lemma 11

Let C, N, L_i and E_i be polynomial matrices of dimensions indicated satisfying the following equations :

$$\begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} r_1 & r_2 \\ \multicolumn{2}{c}{\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}} \end{array} = \begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} s_1 & s_2 \\ \multicolumn{2}{c}{\begin{bmatrix} L_1 & L_3 \\ L_4 & L_2 \end{bmatrix}} \end{array}\begin{array}{cc} r_1 & r_2 \\ \multicolumn{2}{c}{\begin{bmatrix} E_1 & E_3 \\ E_4 & E_2 \end{bmatrix}} \end{array} \tag{5.10}$$

with $s_1 \leqslant r_1$ and $s_2 \leqslant r_2$. Suppose that for all z_0 rank $C(z_0)=s_1$ or rank $N(z_0)=s_2$, and that $C(z)$ and $N(z)$ possess normal ranks s_1 and s_2. Then there exists a unimodular matrix U such that

$$U\begin{bmatrix} E_1 & E_3 \\ E_4 & E_2 \end{bmatrix} = \begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} r_1 & r_2 \\ \multicolumn{2}{c}{\begin{bmatrix} \bar{E}_1 & 0 \\ 0 & \bar{E}_2 \end{bmatrix}} \end{array}, \quad \begin{bmatrix} L_1 & L_3 \\ L_4 & L_2 \end{bmatrix} U^{-1} = \begin{bmatrix} \bar{L}_1 & 0 \\ 0 & \bar{L}_2 \end{bmatrix}$$

Proof

Using unimodular operations we can assume that $\begin{bmatrix} E_1 & E_3 \\ E_4 & E_2 \end{bmatrix}$ is in upper triangular Hermite form :

$$\begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} l_1 & l_2 \\ \multicolumn{2}{c}{\begin{bmatrix} \bar{E}_1 & \bar{E}_3 \\ 0 & \bar{E}_2 \end{bmatrix}} \end{array} \quad \text{with } l_1 + l_2 = r_1 + r_2$$

Let l_1 be maximal, i.e. the first column of E_2 is non-zero. Then, by the Hermite character $l_1 \geqslant s$, and rank $E_1 = s_1$. If $l_1 \leqslant r_1$, $L_4 E_1 = 0$ by (5.10) and so $L_4 = 0$. If $l_1 > r_1$, then $L_1 E_1 = [C \,\vdots\, 0]$ by (5.10). Since C has normal rank s_1, so must L_1 and so the columns of E_1 beyond the r_1th are zero. Therefore, once again, $L_4 E_1 = 0$ and hence $L_4 = 0$. Now

$$\begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} r_1 & r_2 \\ \multicolumn{2}{c}{\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}} \end{array} = \begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{cc} s_1 & s_2 \\ \multicolumn{2}{c}{\begin{bmatrix} L_1 & L_3 \\ 0 & L_2 \end{bmatrix}} \end{array}\begin{array}{cc} l_1 & l_2 \\ \multicolumn{2}{c}{\begin{bmatrix} E_1 & E_3 \\ 0 & E_2 \end{bmatrix}} \end{array} \tag{5.11}$$

By (5.11) normal rank $L_2 = s_2$, because normal rank $N = s_2$. Therefore the first column of $L_2 E_2$ is non-zero, and hence $l_1 \geq r_1$. Therefore we can always redefine E_1, E_2 and E_3 in such a way that

$$\begin{array}{c} \\ s_1 \\ \\ s_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \begin{bmatrix} C & 0 \\ & \\ 0 & N \end{bmatrix} \end{array} = \begin{array}{c} \\ s_1 \\ \\ s_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} L_1 & L_3 \\ & \\ 0 & L_2 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \begin{bmatrix} \bar{E}_1 & \bar{E}_3 \\ & \\ 0 & \bar{E}_2 \end{bmatrix} \end{array} \tag{5.12}$$

with $\bar{E}_2$ in (5.11) having zeros in its first $(l_1 - r_1)$ columns, in case $l_1 > r_1$. From (5.12)

$$C = L_1 \bar{E}_1, \quad N = L_2 \bar{E}_2, \quad L_1 \bar{E}_3 + L_3 \bar{E}_2 = 0$$

Let $L_1^{-1} L_3 = -M_1 M_2^{-1}$, right co-prime. Then $\bar{E}_3 = -L_1^{-1} L_3 \bar{E}_2 = M_1 M_2^{-1} \bar{E}_2$, and since $\bar{E}_3$ is polynomial, $\bar{E}_2 = M_2 K$ for some polynomial K. Then $N = L_2 M_2 K$ and $C = L_1 \bar{E}_1$ with $|M_2|$ a divisor of $|L_1|$. Since N and C have no rank reduction at the same point, $|M_2|$ is constant, and so M_2 is unimodular. Let $M_2^{-1} = P$. Then $L_3 = -L_1 M_1 P$ and $\bar{E}_3 = M_1 P \bar{E}_2$. Therefore

$$\begin{bmatrix} C & 0 \\ & \\ 0 & N \end{bmatrix} = \begin{bmatrix} L_1 & -L_1 M_1 P \\ & \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \bar{E}_1 & M_1 P \bar{E}_2 \\ & \\ 0 & \bar{E}_2 \end{bmatrix}$$

$$= \begin{bmatrix} L_1 & 0 \\ & \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \bar{E}_1 & 0 \\ & \\ 0 & \bar{E}_2 \end{bmatrix} \qquad \square$$

Theorem 9

Consider the model (3.14) of Fig. 2 with the following additional assumptions :

(i) the model is generic ;

(ii) $F(\infty) = H(\infty) = 0$, and $G(\infty)$ and $K(\infty)$ have full rank ;

(iii) Q is block-diagonal, i.e. $w_i \perp v_i$ for all i.

Let $W(z)$ be the corresponding joint process model, and $\phi_{yu}(z) = W(z) Q W^*(z)$. If $\{\hat{W}(z), \hat{Q}\}$ is any other minimal-degree spectral factor of $\phi_{yu}(z)$ with $\hat{W}(\infty)$ block-diagonal and non-singular, then

(*a*) $\hat{Q}$ is block-diagonal ;

(*b*) the scaled para-unitary matrix $V(z) = \hat{W}(z)^{-1} W(z)$ is block-diagonal : $V(z) = \text{diag}\,(V_1(z), V_2(z))$;

(*c*) the model $\hat{F}, \hat{G}, \hat{H}, \hat{K}$ corresponding to $\hat{W}(z)$ is generic and $F = \hat{F}$, $G = \hat{G} V_1$, $H = \hat{H}$, $K = \hat{K} V_2$.

Proof

Without loss of generality we can assume $Q = I$, because otherwise the block-diagonal elements of Q can be factored and absorbed in $W(z)$. With the usual co-prime MFDs of F, G, H and K, and the genericity assumption, the factorization (5.1) of $W(z)$ is co-prime and $W(z)$ has minimal degree. Let the dimensions of C and N be $(p \times r_1)$ and $(q \times r_2)$, respectively. By Corollary 1, $\hat{W}(z)$ can be written

$$\hat{W}(z) = \begin{bmatrix} A & -B \\ -M & D \end{bmatrix}^{-1} P \tag{5.13}$$

for some square polynomial $P(z)$. Now let $V(z) = J^{-1}E$, co-prime. Then $VV^* = \hat{Q}$ and

$$PV = PJ^{-1}E = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} \tag{5.14}$$

Hence $PJ^{-1} = L$ for some polynomial L, and by the genericity assumption and Lemma 11, we can assume

$$E = \begin{matrix} & \begin{matrix} r_1 & r_2 \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \end{matrix}, \quad L = \begin{matrix} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \end{matrix} \tag{5.15}$$

Now let $\hat{Q} = SS^{\mathrm{T}}$, with S lower-triangular, and let $U(z) = S^{-1}V(z)$. Then $UU^* = I$, while

$$U(z) = K^{-1} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \tag{5.16}$$

where $K = JS$. In addition $U(\infty)$ is lower-block triangular and has full rank. Let $D_1 = \operatorname{diag}(z^{i_1}, \ldots, z^{i_p})$, where i_j are the row degrees of E_1, and define D_2 similarly. Then from (5.16) and $UU^* = I$, it follows that

$$\begin{bmatrix} E_1E_1{}^*D_1 & 0 \\ 0 & E_2E_2{}^*D_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} K_{11}{}^*D_1 & K_{21}{}^*D_2 \\ K_{12}{}^*D_1 & K_{22}{}^*D_2 \end{bmatrix} \tag{5.17}$$

and so

$$\begin{bmatrix} K_{11}{}^*D_1 & K_{21}{}^*D_2 \\ K_{12}{}^*D_1 & K_{22}{}^*D_2 \end{bmatrix} \begin{bmatrix} (E_1E_1{}^*D_1)^{-1}K_{11} & (E_1E_1{}^*D_1)^{-1}K_{12} \\ (E_2E_2{}^*D_2)^{-1}K_{21} & (E_2E_2{}^*D_2)^{-1}K_{22} \end{bmatrix} = I \tag{5.18}$$

The $E_i{}^*D_i$ and $K_{ij}{}^*D_i$ are polynomial by Lemma 1 and the fact that $U(z)$ is proper. The (1–2) term in (5.18) gives

$$K_{11}{}^*D_1(E_1E_1{}^*D_1)^{-1}K_{12} + K_{21}{}^*D_2(E_2E_2{}^*D_2)^{-1}K_{22} = 0 \tag{5.19}$$

Since $C = L_1E_1$ and $N = L_2E_2$, it follows from the genericity assumption that $|E_1E_1{}^*D_1|$ and $|E_2E_2{}^*D_2|$ are co-prime. Also by (5.16)

$$\begin{bmatrix} I & K_{11}{}^{-1}K_{12} \\ K_{22}{}^{-1}K_{21} & I \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} K_{11}{}^{-1}E_1 & 0 \\ 0 & K_{22}{}^{-1}E_2 \end{bmatrix} \tag{5.20}$$

From the lower-block triangular structure and the full-rank property of $U(\infty)$, it follows that $\lim_{z\to\infty} K_{11}^{-1} K_{12}=0$, while $\lim_{z\to\infty} K_{22}^{-1} K_{21}$ and $\lim_{z\to\infty} K_{11}^{-1} E_1$ are finite. Using (5.20) and $UU^*=I$, established at $z=\infty$, gives

$$\lim_{z\to\infty} K_{11}^{-1} E_1E_1^*(K_{11}^*)^{-1}=I$$

Since the first term of (5.19) is $[K_{11}^{-1}(E_1E_1^*)(K_{11}^*)^{-1}]^{-1}K_{11}^{-1} K_{12}$, it tends to zero as $z\to\infty$. Since the two terms of (5.19) have no common poles, and the first is strictly proper, they are both zero. Hence $K_{12}=K_{21}^*D_2=K_{21}=0$, and $U(z)$ is block-diagonal. Since $V(\infty)$ is block-diagonal, so is S. Therefore $\hat{Q}$ and $V(z)$ are block-diagonal.

We have shown that

$$V(z)=\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}=\begin{bmatrix} J_1^{-1} E_1 & 0 \\ 0 & J_2^{-1} E_2 \end{bmatrix} \tag{5.21}$$

Therefore by (5.14), $P=\text{diag}\,(\hat{C}, \hat{N})$, with $\hat{C}J_1^{-1} E_1=C$, $\hat{N}J_2^{-1} E_2=N$. Let $\hat{Q}=\text{diag}\,(\hat{Q}_1, \hat{Q}_2)$, and let $D_3=\text{diag}\,(z^{a_1}, \ldots, z^{a_p})$, $D_4=\text{diag}\,(z^{d_1}, \ldots, z^{d_q})$, where a_k and d_k are the row degrees of the kth row of A and D, respectively. Then $J_1^{-1} E_1E_1^*(J_1^*)^{-1}=\hat{Q}_1$ and $\hat{C}\hat{Q}_1\hat{C}^*D_3=CC^*D_3$. Since $\hat{Q}_1$ is real and $\hat{C}$ is square, it follows that det $\hat{C}\hat{C}^*D_3$ and det CC^*D_3 have the same zeros. Hence det $\hat{C}\hat{C}^*D_3$ and det $\hat{N}\hat{N}^*D_4$ are co-prime and, since $\hat{W}(z)$ has minimal degree, the model $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{K}$ corresponding to $\hat{W}(z)$ is generic. Finally, from (5.13) and the form of P, we have $\hat{F}=F$, $\hat{G}=A^{-1}\hat{C}=GV_1^{-1}$, $\hat{H}=H$, $\hat{K}=D^{-1}\hat{N}=KV_2^{-1}$.

Theorem 9 shows that if a closed-loop model is generic, with a delay in the forward and the feedback paths, and if the noises acting on the system are uncorrelated, then any square minimal-degree equivalent model with a delay in both paths will also have uncorrelated noise sources. This result was first obtained by Sin and Goodwin (1980) under a stronger set of assumptions, using state variable representations. However, they did not consider that $W(z)$ could have non-minimal degree and did not investigate the genericity conditions which guarantee not only that $W(z)$ is of minimal degree but that the realization obtained by combining minimal realizations of the forward and feedback systems is also minimal. The importance of Theorem 9 for the identifiability of closed-loop stationary processes is that, if it is known that the forward and feedback path have a delay and that G and K are square, and if the NMSF $\{\bar{W}(z), \bar{Q}\}$ of the joint process leads to a block-diagonal $\bar{Q}$, then we know that the underlying physical closed-loop process has uncorrelated noise sources. This question is further pursued in Anderson and Gevers (1981 a).

6. Concluding remarks

We have established a number of new results on the spectral factorization and the closed-loop representation of jointly stationary vector processes having a joint spectrum $\phi_{yu}(z)$. Conditions have been derived on the forward and the feedback paths of the closed-loop model to produce a minimal-degree, stable, minimum-phase white-noise driven model for the joint (y, u) process. New closed-loop stability conditions have been obtained for the case where the noises enter the system at some internal part of the plant and/or regulator, and where the models are obtained by referring those noises to the output. Finally we have established some relations between joint process models producing the same spectrum for the case where the processes y and u are generated by uncorrelated noise sources. In Anderson and Gevers (1981 a) we use the results established here to study further the relations between equivalent joint process models and closed-loop models of joint processes. This has enabled us to produce conditions under which a closed-loop model can be uniquely identified from closed-loop spectral data. In that paper we also consider more general types of correlations between the process noise and the regulator noise; namely we allow one-sided correlation between the regulator noise v_i and the past process noise w_j, i.e. we consider the case where $L \neq 0$ in the 6-block model (3.4). The question of feedback-free processes is investigated in Gevers and Anderson (1981).

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