Informative data: how to get just sufficiently rich?

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Abstract— **Prediction error identification requires that data be informative with respect to the chosen model structure. Whereas sufficient conditions for informative experiments have been available for a long time, there were surprisingly no results of necessary and sufficient nature. With the recent surge of interest in optimal experiment design, it is of interest to know the minimal richness required of the externally applied signal to make the experiment informative. We provide necessary and sufficient conditions on the degree of richness of the applied signal to generate an informative experiment, both in open loop and in closed loop. In a closed-loop setup, where identification can be achieved with no external excitation if the controller is of sufficient degree, our results provide a precisely quantifiable trade-off between controller degree and required degree of external excitation.**

I. INTRODUCTION

This paper takes a new look at informative experiments for linear time-invariant systems, both in open-loop and in closed-loop identification. The generation of data that are informative with respect to a chosen model structure, together with the identifiability of that model structure, are the two essential ingredients for a well-defined Prediction Error Identification (PEI) problem [6]. These two concepts will be recalled in Section III.

Some readers might think that everything has been written about these concepts, which were much studied in the 1970's. We shared the same view ... until recently. The motivation for our renewed interest into these fundamental questions is the recent surge of interest in experiment design, itself triggered by the new concept of *least costly identification experiment for robust control* [2], [3], [5], [4]. In this context, questions like the following become relevant:

1) what is the smallest amount of external excitation that is required to generate informative data?
2) assuming that the system operates in closed-loop, when can the noise by itself generate informative data?
3) if noise excitation is not sufficient, then how much additional reference excitation is required?
4) assuming that excitation can be applied at different entry points of a multi-input system operating in closed loop, is it necessary to excite each input?

Sufficient conditions for informativity using noise excitation only (question 2) have been given, under different sets of assumptions, in [7], [8], [3]. The key condition is in terms of the complexity of the controller. Question 4 has been addressed in [1] where it is shown that, when informative data cannot be generated using noise excitation only, this does not imply that all reference inputs must be excited.

In attempting to address questions 1 and 3 above, we discovered that these questions do not seem to have been solved before. As is well-known, besides the choice of an identifiable model structure, the key ingredient for a unique global minimum of the identification criterion is the informativity of the experiment. In open-loop identification, and in all closed-loop identification experiments where the noise excitation by itself does not make the experiment informative, informativity is achieved by applying a sufficiently rich external signal: see Section III for definitions. Whereas the literature abunds with sufficient conditions on input signal richness, there appear to be no result on the smallest possible degree of richness that delivers informative data in a given identification setup. In other words, necessary conditions on input richness that will guarantee an informative experiment are strangely lacking. The recent resurgence of interest in optimal experiment design makes this question all the more relevant, because optimal solutions are most often expressed as multisines. It is then important to know how many different frequencies are required to ensure that these optimal inputs produce informative data.

The purpose of this contribution is to find the smallest possible degree of richness of the excitation signal that makes an experiment informative with respect to a chosen model structure, both in open-loop and in closed-loop identification. We address the following two questions:

- assuming open-loop identification, what is the smallest degree of input signal richness that is necessary to achieve an informative experiment with respect to a chosen model structure?
- assuming closed-loop identification with a controller that is not sufficiently complex to yield informative data using noise excitation only, what is then the smallest degree of reference signal excitation that is necessary to achieve an informative experiment with respect to a chosen model structure?

The results of this paper provide necessary and sufficient conditions for informativity of the experiment, in open loop

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and in closed loop, for all commonly used single-input single-output model structures (ARMAX, ARX, BJ and OE). While the results for open-loop experiments are to be expected, our results for closed-loop experiments provide a remarkable and quantifiable trade-off between controller complexity and required degree of richness of the external excitation.

The paper is organized as follows. In Section II we set up the notations and the key tools of the prediction error identification framework. In Section III, we recall the basic concepts of identifiability, informative experiments, and the degree of richness of a signal. The body of our results is in Section IV where we derive the minimal degree of richness required of the external signal to provide informative data, in both an open-loop and a closed-loop setup. In line with common practice, we conclude with conclusions.

II. PREDICTION ERROR IDENTIFICATION SETUP

Consider the identification of a linear time-invariant discrete-time single-input single-output process

$$\mathcal{S} : y(t) = G_0(z)u(t) + H_0(z)e(t), \tag{1}$$

where z is the forward-shift operator, $G_0(z)$ and $H_0(z)$ are the process transfer functions, $u(t)$ is the control input and $e(t)$ is white noise with variance σ_e^2. Both $G_0(z)$ and $H_0(z)$ are rational and proper; furthermore, $H_0(\infty) = 1$, i.e. the impulse response $h(t)$ of the filter $H_0(z)$ satisfies $h(0) = 1$. This true system may be under feedback control with a proper rational stabilizing controller $K(z)$:

$$u(t) = K(z)[r(t) - y(t)]. \tag{2}$$

The system (1) is identified using a model structure parametrized by a vector $\theta \in \mathcal{R}^d$:

$$M(\theta) : y(t) = G(z,\theta)u(t) + H(z,\theta)e(t). \tag{3}$$

It is assumed that the loop transfer function $G(z)K(z)$ has a non-zero delay, both for $G_0(z)$ and for all $G(z,\theta)$. The set of models $M(\theta)$, for all θ in some set $D_\theta \in \mathcal{R}^d$, defines the model set $\mathcal{M} \triangleq \{M(\theta) \mid \theta \in D_\theta\}$. The true system belongs to this model set, $\mathcal{S} \in \mathcal{M}$, if there is a θ_0 such that $M(\theta_0) = \mathcal{S}$. In a PEI framework, a model $[G(z,\theta) \;\; H(z,\theta)]$ uniquely defines the one-step-ahead predictor of $y(t)$ given all input/output data up to time t:

$$\hat{y}(t|t-1,\theta) = W_u(z,\theta)u(t) + W_y(z,\theta)y(t), \tag{4}$$

where $W_u(z,\theta)$ and $W_y(z,\theta)$ are stable filters obtained from the model $[G(z,\theta) \;\; H(z,\theta)]$ as follows:

$$W_u(z,\theta) = H^{-1}(z,\theta)G(z,\theta),\; W_y(z,\theta) = [1 - H^{-1}(z,\theta)].$$

Since there is a $1-1$ mapping between $[G(z,\theta),\; H(z,\theta)]$ and $[W_u(z,\theta),\; W_y(z,\theta)]$, the model $M(\theta)$ will in future refer indistinctly to either one of these equivalent descriptions. For later use, we introduce the following vector notations:

$$W(z,\theta) \triangleq [W_u(z,\theta) \;\; W_y(z,\theta)], \qquad z(t) \triangleq \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \tag{5}$$

We consider throughout this paper that the process $z(t)$ is quasistationary [6], so that the spectral density matrix $\Phi_z(\omega)$ is well defined. The one-step-ahead prediction error is defined as:

$$\begin{aligned} \varepsilon(t,\theta) &\triangleq y(t) - \hat{y}(t|t-1,\theta) = y(t) - W(z,\theta)z(t) \\ &= H^{-1}(z,\theta)\left[y(t) - G(z,\theta)u(t)\right]. \end{aligned} \tag{6}$$

Using a set of N input-output data and a least squares prediction error criterion yields the estimate $\hat{\theta}_N$ [6]:

$$\hat{\theta}_N = \arg\min_{\theta \in D_\theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta). \tag{7}$$

Under reasonable conditions [6], $\hat{\theta}_N \stackrel{N\to\infty}{\longrightarrow} \theta^* \triangleq \arg\min_{\theta \in D_\theta} \bar{V}(\theta)$, with $\bar{V}(\theta) \triangleq \bar{E}[\varepsilon^2(t,\theta)]$ where

$$\bar{E}[f(t)] \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E[f(t)]. \tag{8}$$

If $\mathcal{S} \in \mathcal{M}$ and if $\hat{\theta}_N \stackrel{N\to\infty}{\longrightarrow} \theta_0$, then $\sqrt{N}(\hat{\theta}_N - \theta_0) \stackrel{N\to\infty}{\longrightarrow} N(0, P_\theta)$, with

$$\begin{aligned} P_\theta &= [I(\theta)]^{-1}\,|_{\theta=\theta_0}, && (9) \\ I(\theta) &= \frac{1}{\sigma_e^2}\bar{E}\left[\psi(t,\theta)\psi(t,\theta)^T\right], && (10) \\ \psi(t,\theta) &= -\frac{\partial\varepsilon(t,\theta)}{\partial\theta} = \frac{\partial\hat{y}(t|t-1,\theta)}{\partial\theta} \\ &= \nabla_\theta W(z,\theta)z(t), && (11) \end{aligned}$$

where $\nabla_\theta W(z,\theta) \triangleq \frac{\partial W(z,\theta)}{\partial\theta}$. The matrix $I(\theta_0)$ is called the information matrix.

III. IDENTIFIABILITY, INFORMATIVITY AND RICHNESS

Several concepts of identifiability have been proposed in the scientific literature. Here we adopt a *uniqueness-oriented definition* proposed in [6], which deals with the injectivity of the mapping from parameter vector to model.

Definition 3.1: **(Identifiability)** A parametric model structure $M(\theta)$ is locally identifiable at a value θ_1 if $\exists\delta > 0$ such that, for all θ in $\|\theta - \theta_1\| \leq \delta$:

$$W(z,\theta) = W(z,\theta_1) \;\; \forall z \;\Rightarrow\; \theta = \theta_1.$$

The model structure is globally identifiable at θ_1 if the same holds for $\delta \to \infty$. Finally, a model structure is globally identifiable if it is globally identifiable at almost all θ_1. ■

Identifiability is a property of the parametrization $[G(z,\theta),\; H(z,\theta)]$ or, equivalently, $[W_u(z,\theta),\; W_y(z,\theta)]$. If the model structure is globally identifiable at some θ_1, then there is no other parameter value $\theta \neq \theta_1$ that yields the same predictor as $M(\theta_1)$. However, it does not guarantee that the minimum, say θ_1, of the asymptotic criterion $\bar{V}(\theta)$ is unique. This requires, additionally, that the data set is informative enough to distinguish between different predictors, which leads us to the definition of informative data with respect to a model structure.

Definition 3.2: **(Informative data)** [6] A quasistationary data set $z(t)$ is informative with respect to a parametric model set $\{M(\theta), \theta \in \mathcal{D}_\theta\}$ if, for any two models $W(z,\theta_1)$ and $W(z,\theta_2)$ in that set,

$$\bar{E}\{[W(z,\theta_1) - W(z,\theta_2)]z(t)\}^2 = 0 \tag{12}$$

implies

$$W(e^{j\omega},\theta_1) = W(e^{j\omega},\theta_2) \quad \text{for almost all } \omega. \tag{13}$$

■

The definition of informative data is with respect to a model set, not with respect to the true system. In an identification experiment, one first selects a globally identifiable model structure; this is a user's choice. Experimental conditions must then be selected that make the data informative with respect to that structure; this is again a user's choice. But the data are generated by the true system, in open or in closed loop. Thus, the conditions that make a data set $z(t)$ informative with respect to a model structure depend on the true system and on the possible feedback configuration.

Convergence of an identification algorithm to the exact θ_0 when $\mathcal{S} \in \mathcal{M}$ rests on the simultaneous satisfaction of two different conditions: (i) the use of a model structure that is identifiable, at least at the global minimum θ_0 of the asymptotic criterion $\bar{V}(\theta)$; (ii) the application of experiments that are informative with respect to the model structure used. This paper focuses on the latter condition, and is not restricted to the case where the system is in the model set.

In order to examine the requirements on the excitation signals that generate informative data, we recall the definitions of a persistently exciting regression vector and of richness of a signal.

Definition 3.3: **(Persistently Exciting regressor)** A quasistationary vector signal $\psi(t)$ is called persistently exciting (**PE**) if $\bar{E}[\psi(t)\psi^T(t)] > 0$. ■

Definition 3.4: **(Richness of a signal)** A quasistationary scalar signal $u(t)$ is sufficiently rich of order n (denoted **SRn**) if the following regressor is PE:

$$\phi_{1,n}(t) \triangleq \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ z^{-2} \\ \vdots \\ z^{-n} \end{bmatrix} u(t) \tag{14}$$

■

For future use, we introduce the notation:

$$\mathcal{B}_{k,n}(z) \triangleq \begin{bmatrix} z^{-k} & z^{-k-1} & \dots & z^{-n} \end{bmatrix}^T, \quad \text{for} \quad k \le n. \tag{15}$$

Observe that, by our assumption of quasistationarity, $u(t)$ is SRn if $\mathcal{B}_{k+1,k+n}(z)u(t)$ is PE for any k. **We denote by $\mathcal{U}_n$ the set of all SRn signals.** Equivalent definitions of sufficient richness are given by the following proposition.

Proposition 3.1: [6], [8] A scalar quasistationary signal $u(t)$ is SRn if

- its spectral density is nonzero in at least n frequency points in the interval $(-\pi, \pi]$.
- it cannot be filtered to zero by a FIR filter $\alpha_1 z^{-1} + \ldots + \alpha_n z^{-n}$ of degree $n-1$. ■

The equivalence comes by observing that

$$\begin{aligned} &\alpha^T \bar{E}[\phi_{1,n}(t)\phi_{1,n}^T(t)]\alpha \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\alpha_1 e^{-j\omega} + \ldots + \alpha_n e^{-jn\omega}|^2 \Phi_u(\omega) d\omega. \end{aligned}$$

The main contribution of this paper is to describe the weakest possible richness conditions on the external signal $u(t)$ (in open-loop identification) or $r(t)$ (in closed-loop identification) that make the data informative with respect to a given model structure.

IV. INFORMATIVITY OF THE DATA SET FOR ARMAX AND BJ MODEL STRUCTURES

In this section we derive necessary and sufficient conditions for the informativity of the data set for ARMAX and BJ model structures, as well as for the special cases of ARX and OE model structures.

A. Open-loop identification

Recall Definition 3.2 for informative data. We introduce the following shorthand notation (see (5)):

$$\begin{aligned} \triangle W_u &\triangleq W_u(z,\theta_1) - W_u(z,\theta_2) \\ \triangle W_y &\triangleq W_y(z,\theta_1) - W_y(z,\theta_2). \end{aligned} \tag{16}$$

For open loop data, it follows from (5) and (1) that:

$$\begin{aligned} &[W(z,\theta_1) - W(z,\theta_2)]z(t) \\ &= [\triangle W_u + \triangle W_y G_0]u(t) + \triangle W_y H_0 e(t). \end{aligned} \tag{17}$$

Given the independence between u and e, condition (12) is therefore equivalent with the following set of conditions:

$$\bar{E}\{[\triangle W_u + \triangle W_y G_0]u(t)\}^2 = 0 \tag{18}$$

$$\bar{E}\{\triangle W_y H_0 e(t)\}^2 = 0 \tag{19}$$

We now seek the smallest degree of richness of u for which the conditions (18)-(19) imply $\triangle W_u \equiv 0$ and $\triangle W_y \equiv 0$. Since (19) implies $\triangle W_y \equiv 0$, this is equivalent with finding necessary and sufficient conditions on the richness of u such that

$$E\{[\triangle W_u]u(t)\}^2 = 0 \implies \triangle W_u \equiv 0 \tag{20}$$

ARMAX model structure

Consider first the ARMAX model structure

$$A(z^{-1})y(t) = B(z^{-1})u(t) + C(z^{-1})e(t) \tag{21}$$

where $A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a}$, $B(z^{-1}) = b_1 z^{-1} + \ldots + b_{n_b} z^{-n_b}$, and $C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_{n_c} z^{-n_c}$. For ARMAX model structures, one must consider as generic the possible existence of common roots between the polynomials A and B, as well as between A and C. However, the three polynomials A, B and C must be coprime at any identifiable θ. We then have the folllowing result.

Theorem 4.1: For the ARMAX model structure (21), the data set is informative if and only if $u(t)$ is SRk, where $k = n_b + \min\{n_a, n_c\}$.

Proof: Since $\triangle W_y \equiv 0$ it follows that

$$\frac{C(z,\theta_1)}{A(z,\theta_1)} = \frac{C(z,\theta_2)}{A(z,\theta_2)}$$

for any pair of values θ_1 and θ_2. Let $U(z)$ be the greatest common factor of $A(z,\theta_1)$ and $C(z,\theta_1)$:

$$A(z,\theta_1) = A_1(z,\theta_1)U(z), \quad C(z,\theta_1) = C_1(z,\theta_1)U(z),$$

with $U(z) = 1 + u_1 z^{-1} + \ldots + u_{n_u} z^{-n_u}$. Then $\triangle W_y \equiv 0$ is equivalent with

$$A(z,\theta_2)C_1(z,\theta_1) - C(z,\theta_2)A_1(z,\theta_1) = 0 \tag{22}$$

with $A_1(z,\theta_1)$ and $C_1(z,\theta_1)$ coprime. The general solution of (22) is

$$A(z,\theta_2) = A_1(z,\theta_1)T(z), \quad C(z,\theta_2) = C_1(z,\theta_1)T(z),$$

where $T(z)$ is an arbitrary monic polynomial of the same degree as U. For this ARMAX model structure, the left hand side of (20), expressed at θ_1 and θ_2, is equivalent with:

$$\bar{E}\{[A(z,\theta_2)B(z,\theta_1) - A(z,\theta_1)B(z,\theta_2)]u(t)\}^2 = 0, \tag{23}$$

which is equivalent with

$$\bar{E}\{[T(z)B(z,\theta_1) - U(z)B(z,\theta_2)]u(t)\}^2 = 0. \tag{24}$$

This implies

$$T(z)B(z,\theta_1) - U(z)B(z,\theta_2) \equiv 0 \tag{25}$$

if and only if $u(t) \in \mathcal{U}_{n_b+n_u}$. Since $U(z)$ cannot have a common factor with $B(z,\theta_1)$, the general solution of (25) is

$$B(z,\theta_2) = M(z)B(z,\theta_1), \quad T(z) = M(z)U(z).$$

However, since $T(z)$ and $U(z)$ have the same degree, n_u, and since both are monic, the only solution is $M(z) = 1$, which implies that $\theta_1 = \theta_2$. We conclude that the predictor at θ_1 is identical to the predictor at any other value θ_2 if and only if $u(t) \in \mathcal{U}_{n_b+n_u}$ where n_u is the number of common roots between $A(z,\theta)$ and $C(z,\theta)$ at θ_1. For the data to be informative with respect to the ARMAX model structure, this must hold *at all values of* θ_1. The stated result then follows, since the maximum number of common roots between $A(z,\theta)$ and $C(z,\theta)$ is $\min\{n_a, n_c\}$. ∎

The result for an ARX model structure follows immediately.

Corollary 4.1: For the ARX model structure $A(z^{-1})y(t) = B(z^{-1})u(t) + e(t)$ with A and B as above, the data set is informative if and only if $u(t) \in \mathcal{U}_{n_b}$. ∎

BJ model structure

Consider now the BJ model structure:

$$y(t) = \frac{B(z^{-1})}{F(z^{-1})}u(t) + \frac{C(z^{-1})}{D(z^{-1})}e(t) \tag{26}$$

where $B(z^{-1})$ and $C(z^{-1})$ are as above, with $F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_{n_f} z^{-n_f}$ and $D(z^{-1}) = 1 + d_1 z^{-1} + \ldots + d_{n_d} z^{-n_d}$. We have the following result.

Theorem 4.2: For the BJ model structure (26) operating in open loop, the data are informative if and only if $u(t)$ is SRk, where $k = n_b + n_f$.

Proof: It follows from (19) that $\frac{C(z,\theta_1)}{D(z,\theta_1)} = \frac{C(z,\theta_2)}{D(z,\theta_2)}$. Therefore, (20) is equivalent with $\bar{E}\{\frac{D(z,\theta_1)}{C(z,\theta_1)}[\frac{B(z,\theta_1)}{F(z,\theta_1)} - \frac{B(z,\theta_2)}{F(z,\theta_2)}]u(t)\}^2 = 0$, or equivalently,

$$\bar{E}\{[F(z,\theta_2)B(z,\theta_1) - F(z,\theta_1)B(z,\theta_2)]u(t)\}^2 = 0. \tag{27}$$

Since the degree of the polynomial that filters $u(t)$ in (27) is $n_b + n_f$, the result then follows immediately. ∎

Corollary 4.2: For the OE model structure $y(t) = \frac{B(z^{-1})}{F(z^{-1})}u(t) + e(t)$, the richness condition on $u(t)$ is identical to that for the BJ model structure. ∎

B. Closed-loop identification

For closed-loop data, it follows from (5), (1) and (2) that

$$\begin{aligned}&[W(z,\theta_1) - W(z,\theta_2)]z(t) = \\ &KS[\triangle W_u + \triangle W_y G_0]r(t) + H_0 S[\triangle W_y - K\triangle W_u]e(t)\end{aligned} \tag{28}$$

Given the independence between r and e, condition (12) is therefore equivalent with the following set of conditions:

$$\bar{E}\{KS[\triangle W_u + \triangle W_y G_0]r(t)\}^2 = 0 \tag{29}$$

$$\bar{E}\{H_0 S[\triangle W_y - K\triangle W_u]e(t)\}^2 = 0 \tag{30}$$

These conditions, in turn, are equivalent with the following:

$$\triangle W_y \equiv K\triangle W_u, \tag{31}$$

$$\bar{E}\{K\triangle W_u r(t)\}^2 = 0, \tag{32}$$

where the second expression follows by substituting the first in (29) and using $S = (1 + KG_0)^{-1}$. Note that, given the first condition, the second is equivalent with

$$\bar{E}\{\triangle W_y r(t)\}^2 = 0. \tag{33}$$

For the controller $K(z)$ of (2) we shall consider a coprime factorization $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$, with $X(z^{-1}) = x_0 + x_1 z^{-1} + \ldots + x_{n_x} z^{-n_x}$ and $Y(z^{-1}) = 1 + y_1 z^{-1} + \ldots + y_{n_y} z^{-n_y}$.

ARMAX model structure

We first consider the ARMAX model structure (21) under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$. For simplicity, we consider only parameter values θ at which the following assumption holds.

Assumption 1: The polynomials $A(z^{-1})Y(z^{-1}) + B(z^{-1})X(z^{-1})$ and $C(z^{-1})$ are coprime.

Notice that the subset of θ values at which these polynomials have a common root has measure zero in the parameter space. They correspond to parameter values that cause a pole-zero cancellation between the closed-loop poles of the model and the zeros of the noise model. We then have the following result.

Theorem 4.3: Consider the ARMAX model structure (21) under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$, with Assumption 1 holding.
(i) Let $r(t) \equiv 0$. Then the data are informative if and only if

$$max(n_x - n_a, n_y - n_b) \geq 0. \quad (34)$$

(ii) Let $max(n_x - n_a, n_y - n_b) < 0$. Then the data are informative for almost all $r(t) \in \mathcal{U}_k$ if and only if

$$k \geq min(n_a - n_x, n_b - n_y). \quad (35)$$

Proof: For the ARMAX model structure (21), the identity (31) is equivalent with[1]

$$(A_2Y + B_2X)C_1 = (A_1Y + B_1X)C_2, \quad (36)$$

where the pairs of polynomials C_1 and $A_1Y + B_1X$, as well as C_2 and $A_2Y + B_2X$ are coprime by Assumption 1. Therefore, the general solution of (36) is

$$A_2Y + B_2X = M(A_1Y + B_1X), \quad C_2 = MC_1, \quad (37)$$

where $M(z)$ is an arbitrary polynomial. But since $C_1(z)$ and $C_2(z)$ are monic with the same degree, the only solution is $M(z) = 1$, i.e.

$$C_1 = C_2 \;\text{ and }\; (A_1 - A_2)Y + (B_1 - B_2)X = 0. \quad (38)$$

Since X and Y are coprime, the general solution of this last equality is given by

$$\triangle A = TX, \quad \triangle B = -TY, \quad (39)$$

where $T(z) = t_1z^{-1} + \ldots + t_{n_t}z^{-n_t}$ is again an arbitrary polynomial with $\deg T(z) = n_t \triangleq \min\{n_a - n_x, n_b - n_y\}$. $T(z) = 0$ is the only solution of (39), and hence the data are informative without any external excitation if and only if condition (34) holds.

Now consider the case where $\max(n_x - n_a, n_y - n_b) < 0$, i.e. $\min\{n_a - n_x, n_b - n_y\} \geq 1$. It then follows from $C_1 = C_2$ and $\triangle B = -TY$ that condition (32) is equivalent with $\bar{E}\{XTr(t)\}^2 = 0$. This implies $T(z) \equiv 0$, and hence the informativity of the data, if and only if $r(t) \in \mathcal{U}_k$ with $k \geq min(n_a - n_x, n_b - n_y)$ provided the points of support of $r(t)$ do not coincide with possible zeroes of X on the unit circle. This proves part (ii) of the Theorem. ■

Comment. An ARMAX model identified in closed loop is identifiable from noise information only if the controller is sufficiently complex with respect to the model structure, in a way that is made precise by condition (34); this condition is known and can be found in [8]. What is novel and, we believe, remarkable in Theorem 4.3 is that, when that complexity condition is not satisfied by the controller, then the degree of richness required of the reference signal is precisely determined by how much that condition is violated. In other words, the degree of richness required of $r(t)$ is precisely equal to the difference between the complexity required by expression (34) and the actual complexity of the controller.

[1]To keep notations simple, we drop the z argument here.

Corollary 4.3: For the ARX model structure $A(z^{-1})y(t) = B(z^{-1})u(t) + e(t)$ under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$, the richness conditions are identical to those given in Theorem 4.3 for the ARMAX model structure.

Proof: The proof follows immediately by setting $C(z^{-1}) = 1$ everywhere in the proof of Theorem 4.3. ■

BJ model structure

We now consider the BJ model structure (26) under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$. For simplicity, we shall again exclude parameter values θ that cause a pole-zero cancellation between the closed-loop poles of the model and the zeros of the noise model. This corresponds to the following assumption.

Assumption 2: The polynomials $F(z^{-1})Y(z^{-1}) + B(z^{-1})X(z^{-1})$ and $C(z^{-1})$ are coprime.

We then have the following result.

Theorem 4.4: Consider the BJ model structure (26) under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$, with Assumption 2 holding.
(i) With $r(t) \equiv 0$, the data are informative if and only if

$$max(n_x - n_f, n_y - n_b) \geq n_d + min\{n_x, n_f\}. \quad (40)$$

(ii) Let $max(n_x - n_f, n_y - n_b) < n_d + min\{n_x, n_f\}$. Then the data are informative for almost all $r(t) \in \mathcal{U}_k$ if and only if

$$k \geq n_d + min\{n_x, n_f\} + min(n_f - n_x, n_b - n_y). \quad (41)$$

Proof: For the BJ model structure, the identity (31) is equivalent with

$$\frac{(F_1Y + B_1X)D_1}{C_1F_1} = \frac{(F_2Y + B_2X)D_2}{C_2F_2}. \quad (42)$$

Suppose first that D_1 and F_1 have a common polynomial factor $H(z) = 1 + h_1z^{-1} + \ldots + h_{n_h}z^{-n_h}$, so that $D_1 = \bar{D}_1H$ and $F_1 = \bar{F}_1H$, with $\bar{D}_1$ and $\bar{F}_1$ coprime. Consider, additionally, that there are possible pole-zero cancellations at θ_1 between the zeroes of the controller and the poles of $G(\theta_1)$, and let $M(z) = 1 + m_1z^{-1} + \ldots + m_{n_m}z^{-n_m}$ be the greatest common factor between X and $\bar{F}_1$, so that $X = X_1M$ and $\bar{F}_1 = F_1^*M$ with X_1 and F_1^* coprime. Note that $F_1 = F_1^*MH$. Then (42) is equivalent with

$$\frac{(F_1^*HY + B_1X_1)\bar{D}_1}{C_1F_1^*} = \frac{(F_2Y + B_2X)D_2}{C_2F_2}, \quad (43)$$

where $C_1F_1^*$ and $(F_1^*HY + B_1X_1)\bar{D}_1$ are now coprime because C_1 is coprime with D_1 (and hence with $\bar{D}_1$) and also with $F_1Y + B_1X$ (and hence with $F_1^*HY + B_1X_1$) by Assumption 1. In addition, F_1^* is coprime with B_1, $\bar{D}_1$ and X_1, and hence with the whole numerator. The general solution is therefore

$$C_2F_2 = C_1F_1^*T \quad (44)$$
$$(F_2Y + B_2X)D_2 = (F_1^*HY + B_1X_1)\bar{D}_1T, \quad (45)$$

where $T = 1+t_1 z^{-1}+\ldots+t_{n_t} z^{-n_t}$ of degree $n_t = n_m+n_h$. Equation (45) can be rewritten as

$$(F_2 D_2 - F_1^* H \bar{D}_1 T)Y + (B_2 D_2 M - B_1 \bar{D}_1 T)X_1 = 0 \quad (46)$$

with X_1 and Y coprime. Note that the leading term of the polynomials multiplying Y and X_1 is zero. The general solution of (46) is given by

$$F_2 D_2 - F_1^* H \bar{D}_1 T = X_1 U \quad (47)$$
$$B_2 D_2 M - B_1 \bar{D}_1 T = -YU \quad (48)$$

where U is an arbitrary polynomial of the form $U(z) = u_1 z^{-1} + \ldots + u_{n_u} z^{-n_u}$, with $n_u = n_d + n_m + min\{n_f - n_x, n_b - n_y\}$. $U(z) = 0$ is the only solution of (47)-(48) if and only if $\max\{n_x - n_f, n_y - n_b\} \geq n_d + n_m$. Suppose this is the case; it then follows from (47)-(48) that

$$F_2 D_2 = F_1^* H \bar{D}_1 T \quad \text{and} \quad (49)$$
$$B_2 D_2 M = B_1 \bar{D}_1 T \quad (50)$$

Combining (44) and (49) yields

$$\frac{C_2}{D_2} = \frac{C_1}{H\bar{D}_1} = \frac{C_1}{D_1},$$

while combining (49) and (50) yields

$$\frac{B_2}{F_2} = \frac{B_1}{F_1^* HM} = \frac{B_1}{F_1}.$$

Together, these last two identities imply that the data are sufficiently informative to distinguish between the model at θ_1 and at any other θ. Part (i) of the Theorem then follows from the fact that n_m is the number of common roots between the zeroes of the controller and the poles of the plant model and that the largest possible number of such common roots, for any value of θ_1, is $min\{n_x, n_f\}$.

Suppose now that the controller is not sufficiently complex to produce informative data with noise excitation only, i.e. condition (40) is violated. We then seek necessary and sufficient richness conditions on $r(t)$ under which conditions (31) and (32) or, equivalently (33), imply $\triangle W_u \equiv 0$ and $\triangle W_y \equiv 0$. It follows from the previous derivations that

$$\begin{aligned}\triangle W_y &= \frac{C_1 D_2 - D_1 C_2}{C_1 C_2} \\ &= \frac{1}{C_1 C_2}[C_1 D_2 - D_1 \frac{C_1 F_1^* T}{F_2}] \\ &= \frac{1}{F_2 C_2}[D_2 F_2 - D_1 F_1^* T] = \frac{X_1 U}{F_2 C_2} \quad (51)\end{aligned}$$

Thus, (33) implies $\triangle W_y \equiv 0$ if and only if $E\{\frac{X_1 U}{F_2 C_2} r(t)\}^2 = 0$ implies $U(z) \equiv 0$. Remember that $deg(U) \triangleq n_u = n_d + n_m + min\{n_f - n_x, n_b - n_y\}$ where n_m is the number of common roots between the controller zeroes and the poles of $G(z, \theta)$ at the considered θ_1. Since the largest number of such pole-zero cancellations at any θ_1 is $min\{n_x, n_f\}$, it then follows that $U(z) \equiv 0$ if and only if $r(t) \in \mathcal{U}_k$ for all $k \geq n_d + min\{n_x, n_f\} + min(n_f - n_x, n_b - n_y)$ provided the points of support of $u(t)$ do not coincide with possible zeroes of X on the unit circle. This proves part (ii) of the Theorem. ■

Comment. Just like in the case of an ARMAX model structure identified in closed loop, the degree of richness required of the external excitation signal $r(t)$ is precisely equal to the difference between the complexity required by expression (40) and the actual complexity of the controller.

Corollary 4.4: For the OE model structure $y(t) = \frac{B(z^{-1})}{F(z^{-1})}u(t)+e(t)$, under feedback control with the stabilizing controller $K(z) = \frac{X(z^{-1})}{Y(z^{-1})}$, the data set is informative if and only if $K(z) \neq 0$.

Proof: Since $W_y(z) = 0$ for an OE model, condition (30) is equivalent with $K \triangle W_u \equiv 0$. Since $W_u = \frac{B_1}{F_1} - \frac{B_2}{F_2}$ this implies $\frac{B_1}{F_1} = \frac{B_2}{F_2}$ if and only if $K(z) \neq 0$. ■

This confirms a result obtained in [3] where it was shown that identification with an OE model structure in a closed loop setup yields a unique global minimum without external excitation if the controller is not identically zero.

V. CONCLUSIONS

We have provided necessary and sufficient conditions on the external signals to achieve informative data for all commonly used input-output models, under both open-loop and closed-loop experimental conditions. Our objective has been to find the smallest possible degree of richness of the external signal (u in open loop, r in closed loop) that delivers an informative data set. While the open-loop results were either known or to be expected, the novel contribution of this paper lies with the closed-loop results. They show a remarkable and quantifiable trade-off between the controller complexity and the required degree of external excitation. Our conditions on the required controller complexity in the absence of external excitation are identical to conditions derived in [3] for the existence of a unique minimum of the identification criterion in the absence of excitation. However, the results in [3] have been obtained under the assumption that the system is in the model set, while the results of this paper do not require such assumption.

REFERENCES

[1] A.S. Bazanella, M. Gevers, and L. Mišković. Closed-loop identification of MIMO systems: a new look at identifiability and experiment design. In *CD-ROM Proc. of European Control Conference*, pages 5694–5699, Kos, Greece, July 2007.

[2] X. Bombois, G. Scorletti, M. Gevers, R. Hildebrand, and P.M.J. Van den Hof. Cheapest open-loop identification for control. In *CD-ROM Proc. 33rd IEEE Conf on Decision and Control*, pages 382–387, The Bahamas, December 2004.

[3] X. Bombois, G. Scorletti, M. Gevers, P.M.J. Van den Hof, and R. Hildebrand. Least costly identification experiment for control. *Automatica*, 42(10):1651–1662, October 2006.

[4] M. Gevers, L. Mišković, D. Bonvin, and A. Karimi. Identification of multi-input systems: variance analysis and input design issues. *Automatica*, 42(4):559–572, April 2006.

[5] H. Jansson and H. Hjalmarsson. Optimal experiment design in closed loop. In *16th IFAC World Congress on Automatic Control, paper 04528*, July 2005.

[6] L. Ljung. *System Identification: Theory for the User, 2nd Edition*. Prentice-Hall, Englewood Cliffs, NJ, 1999.

[7] T. S. Ng, G. C. Goodwin, and B. D. O. Anderson. Identifiability of MIMO linear dynamic systems operating in closed loop. *Automatica*, 13:477–485, 1977.

[8] T. Söderström and P. Stoica. *System Identification*. Prentice-Hall International, Hemel Hempstead, Hertfordshire, 1989.