V. Conclusion

The similarity between the decentralized control of finite state Markov chains and the centralized cases was shown in Sections III and IV; centralized methods are used to derive algorithms for one step delay sharing decentralized problems, essentially by defining the "state" to be the one step delayed "shared information." In Section III each controller's action depends on the one step delay sharing information pattern only via his present observation, the past state information, and the past control information for noiseless recoverable problems. For the infinite horizon problem with average expected cost, sufficient conditions for the existence of a stationary optimal policy were obtained. The "policy iteration algorithm" was modified slightly to be applicable to this problem. The development of the general case (noisy observations) displayed a structure similar to that in the centralized case [20]. Sondik's algorithm was readily applied with some modifications. Also, some properties concerning the existence of a stationary optimal policy and the structure of the system were revealed.

As noted previously, one of the important points that makes the extension of the classical results to the decentralized case possible is due to the separation principle [6]. In [6] this principle is proved to hold with the one step delay sharing information pattern. In addition, the separation principle is proved to break down when the delay is more than one step. One interesting application of the decentralized control of Markov chains is in multiaccess broadcast communication systems; the reader is referred to [28], where detailed modeling, analytical, and simulation results are presented.

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On Jointly Stationary Feedback-Free Stochastic Processes

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***Abstract*—We consider stationary stochastic vector processes made up of two component processes y and u. Such processes arise, for example, in feedback processes. We consider the task of determining whether there is feedback from one process, say, y, to the other, say, u. A definition is proposed for the absence of feedback in terms of the spectrum $\phi_{yu}(z)$ of the joint process. Comparison with previous results on feedback-free processes shows that the proposed definition has some desirable properties which were absent in previous work. In particular, system structures other than canonical ones are shown to be feedback-free.**

I. Introduction

We study the task of determining the presence (or absence) of feedback between two jointly stationary stochastic processes $\{y\}$ and $\{u\}$. These processes can be the output and the input of a physical linear stochastic time invariant feedback system; they can also originate as two separate subvectors y and u of a joint stationary vector process $x=(y^T, u^T)^T$, for which it is desired to examine the feedback relationships. The first situation is typical of many engineering applications where a linear time invariant system (or plant) is controlled by a linear feedback. The second situation arises in econometric or biological time series, where the question may be asked about the existence of feedback between any two scalar or vector time series.

The notion of feedback between time series is closely related to that of causality. Both notions have been studied by a number of authors [1]–[7], and a precise definition of feedback between stationary processes has been given by Caines and Chan in [4]. (These references also include examples on the testing for the presence of feedback.) The definition given by Caines and Chan is based on the upper block triangular structure of a canonical innovations representation of the joint (y, u) process. In [7] Caines introduced a new definition (based on work in the econometrics literature), terming it "strong feedback-free" and renaming the earlier "weak feedback-free"; any pair of time series with the strong feedback-free property enjoys the weak feedback-free property, but not necessarily conversely. The main result of this paper is that if the joint process has the strong feedback-free property, then normally every square feedback representation (as well as almost every joint process representation) of the (y, u) process has an upper block triangular structure. The term "normally" is understood to mean "except in nongeneric cases," the appropriate notion of genericity being defined in the paper. Thus, the definition, even though it is given in terms of a canonical innovations representation, will be shown to extend normally to all equivalent square representations of the joint (y, u) process. We suggest that this definition is desirable because it corresponds, as we shall show, with the intuitive

Manuscript received August 13, 1980; revised April 24, 1981. Paper recommended by S. I. Marcus, Past Chairman of the Stochastic Control Committee. This work was supported by the Australian Research Grants Committee.

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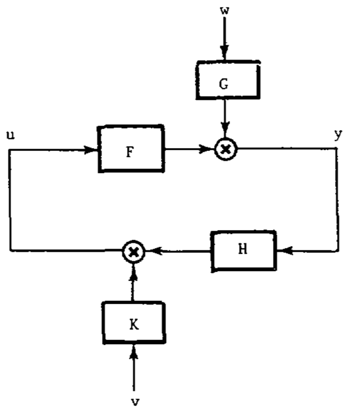

Fig. 1. Standard feedback system.

notion that a process is feedback-free if the noises in the forward path and the feedback path are uncorrelated, and if the transfer function matrix relating y to u in the feedback path is identically zero, even when the system is not associated with a canonical (minimum phase) spectral factor of the spectrum. We shall show by an example that another widely used definition of feedback-free does not imply this property. This paper relies heavily on some results obtained in [8], where the feedback representations of joint stationary stochastic processes (y, u) have been studied, including their stability properties and their relationship to the spectral factors of the joint spectral density matrix $\phi_{yu}(z)$.

II. Definitions of Feedback

We consider two stationary vector stochastic processes $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$. The dynamical behaviour of the joint (y, u) process can often (although not always) be represented by the following feedback system (see [8] and Fig. 1):

$$y_i = F(z)u_i + G(z)w_i \tag{2.1a}$$

$$u_i = H(z)y_i + K(z)v_i. \tag{2.1b}$$

$F(z), G(z), H(z), K(z)$ are causal real rational transfer function matrices, $\{w_i\}$ and $\{v_i\}$ are white noise processes, $w_i \in \mathbb{R}^q, (q \geq p), v_i \in \mathbb{R}^n, (n \geq m)$, and

$$E\left\{\begin{bmatrix} w_i \\ v_i \end{bmatrix}\begin{bmatrix} w_j^T & v_j^T \end{bmatrix}\right\} = Q\delta_{ij}, \qquad Q \geq 0. \tag{2.2}$$

The following standing assumption will be made throughout this paper.

Assumption A: (y, u) is a real full rank bounded stationary stochastic process with rational spectrum.[1]

In [8] conditions on F, G, H, K have been derived for the stationarity of the joint process. If there is a delay somewhere in the loop, then an equivalent representation for the (y, u) process is given by the joint process model[2]

$$\begin{bmatrix} y_i \\ u_i \end{bmatrix} = \begin{bmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{bmatrix}\begin{bmatrix} w_i \\ v_i \end{bmatrix} = W(z)\begin{bmatrix} w_i \\ v_i \end{bmatrix} \tag{2.3}$$

where

$$W(z) = \begin{bmatrix} (I-FH)^{-1}G & (I-FH)^{-1}FK \\ (I-HF)^{-1}HG & (I-HF)^{-1}K \end{bmatrix}. \tag{2.4}$$

The spectrum of the joint process (y, u) is given by

$$\phi_{yu}(z) = W(z)QW^*(z) \tag{2.5}$$

[1]Many of the results will actually carry over to infinite-dimensional processes, i.e., processes without a rational spectrum.

[2]In the sequel we shall often use partitioned matrices. The dimensions will always be assumed to correspond to those of y and u.

where $W^*(z) \triangleq W^T(z^{-1})$, the adjoint of $W(z)$. We recall an important result about spectral factorization (see [9]).

Spectral Factorization Theorem: Let $\phi(z)$ be an $n \times n$ real rational full rank spectral density matrix.

1) Then there exists a unique factorization of the form $\phi(z) = \overline{W}(z)\overline{Q}\overline{W}^*(z)$, in which $\overline{W}(z)$ is $n \times n$ real rational, stable, minimum phase and such that $\overline{W}(\infty) = I$, with $\overline{Q}$ positive definite symmetric.

2) Any other factorization of the form $\phi(z) = W(z)QW^*(z)$ in which $W(z)$ is real rational, and Q is nonnegative definite symmetric, is such that $W(z) = \overline{W}(z)V(z)$, where $V(z)$ is a real rational scaled paraunitary matrix, i.e., $V(z)QV^*(z) = \overline{Q}$. Moreover, $V(z)$ is stable if and only if $W(z)$ is stable.

The unique (canonical) spectral factor $\{\overline{W}(z), \overline{Q}\}$ defined in part 1) of the theorem, will be called the normalized minimum phase spectral factor (NMSF) of $\phi(z)$. (See [8].)

Equations (2.1a) and (2.1b) are normally understood to represent the forward path and the feedback path of a feedback system. It is then reasonable to say that there is no feedback from y to u if $H(z) = 0$ and $\{w\}$ and $\{v\}$ are uncorrelated. This then ensures that the processes $\{u\}$ and $\{w\}$ in (2.1a) are uncorrelated, which is how Akaike [5] and Bohlin [6] define the absence of feedback. If $H = 0$, then (2.4) shows that $W_{21}(z) = 0$

$$W(z) = \begin{bmatrix} G & FK \\ 0 & K \end{bmatrix}. \tag{2.6}$$

Therefore, an equivalent definition for the absence of feedback between y and u would be that the joint process model is upper block triangular, while the noise-covariance matrix Q is block diagonal. However, from the spectral factorization theorem it follows that there are other couples $\hat{W}(z), \hat{Q}$ that produce the same spectrum $\phi_{yu}(z)$ as $W(z), Q$. If the joint model $W(z), Q$ obtained from a physical feedback system has $W_{21}(z) = 0$ and Q block diagonal, this does not necessarily imply that equivalent spectral factors $\hat{W}(z), \hat{Q}$ of the same joint spectral density matrix $\phi_{yu}(z)$ will also have $\hat{W}_{21}(z) = 0$ and $\hat{Q}$ block diagonal. Therefore, a more precise definition is needed. As it turns out, the definition of strong feedback-free of [7] is what is required. The definition requires some notation.

We consider the Hilbert space $\mathcal{H}$ which is the mean-square completion of the space of all stationary processes with finite first- and second-order moments. We denote the joint process by $x = (y^T, u^T)$, and we assume that $\{x\}$ has zero mean, and that the process $\{x\}$ is purely nondeterministic [10]. We denote by $X^k = \sigma\{x_{-\infty}, \cdots, x_k\}$ the subspace of $\mathcal{H}$ generated by the components of $\{x_{-\infty}, \cdots, x_k\}$. The subspaces U^k and Y^k are defined similarly. If r is any random variable, $r|X^k$ (or $r|Y^k, U^k$) will denote the projection of r on X^k.

Specifically, $r|X^k = \sum_{i=-\infty}^{k} H_i x_i$, where the matrices H_i are such that $E\{(r - r|X^k)x_i^T\} = 0$, $i = -\infty, \cdots, k$. $r \perp X^k$ will mean that $E\{rx^T\} = 0$ for all $x \in X^k$. We shall also use $r|Y^k$ or $r|U^k$ to denote the projection of r on the corresponding subspaces Y^k or U^k. We can now state a first definition for a feedback-free process.

Definition 1: Consider the jointly stationary process (y, u) with Assumption A. The process (y, u) is feedback-free (i.e., there is no feedback from y to u) if and only if

$$u_i \perp \left(y_j - y_j|U^{j-1}\right) \qquad i \geq j. \tag{2.7}$$

Comments: 1) The definition is a natural one. There is no feedback from y to u if and only if the process $\{u\}$ is orthogonal to the process obtained from the past y's after removing the effect of the past u's.

2) The second-order statistics of the u and y processes are all that are needed to check (2.7), i.e, (2.7) is a property checkable using the spectrum, not the detailed structure of (2.1) (which in general is not known if only the spectrum is known).

The following theorem can be obtained from [7].

Theorem 1: Consider a real full rank bounded stationary stochastic process (y, u) with rational spectrum $\phi_{yu}(z)$. Then there is no feedback from y to u if and only if any one of the following equivalent conditions hold:

1) $u_i \perp \left(y_j - y_j|U^{j-1}\right) \qquad i \geq j \qquad (2.8)$

2) $u_i \perp \left(y_j - y_j|Y^{j-1}, U^{j-1}\right) \qquad \text{for all } i, j \qquad (2.9)$

3) $u_i|U^{i-1}, Y^i = u_i|U^{i-1}$ (2.10)

4) The NMSF $\{\overline{W}(z), \overline{Q}\}$ of $\phi_{yu}(z)$ has $\overline{W}_{21}(z)=0$ and $\overline{Q}$ block diagonal. (2.11)

Theorem 1 provides four equivalent definitions for a feedback-free process. Definition 2 [i.e., condition (2.9)] states that there is no feedback from y to u if the one-step ahead prediction error process on y is orthogonal to the process u. This condition guarantees that the transfer matrices F and G are identifiable using a prediction error method on (y, u) [11]. Definition 3 is intuitively appealing: there is no feedback from y to u if knowledge of past and present y does not add any information as far as the prediction of u is concerned. The last definition in terms of the NMSF will be useful in deriving some further results on feedback-free processes in the next sections. In the next section we examine some properties of feedback-free processes.

III. Properties of Feedback-Free Systems

We first recall some results on feedback systems established in [8].

Definition 2: Consider the feedback system (2.1)–(2.2) with Assumption A, and the corresponding $W(z)$ and $\phi_{yu}(z)$ defined by (2.4) and (2.5). Let $A^{-1}[B \vdots C]=[F \vdots G]$ be a left coprime polynomial matrix fraction description (MFD) of $[F \vdots G]$, and similar by $D^{-1}[M \vdots N]=[H \vdots K]$. (Note: A, B, C, D, M, N are matrices whose elements are polynomials in z; see, e.g., [12] and [13].) Let r = highest power of z^{-1} in $\det C(z)C^*(z)$ and s = highest power of z^{-1} in $\det N(z)N^*(z)$. Then the system $\{F, G, H, K\}$ is called *generic* if

$$W(z) \text{ has minimal degree, i.e., } \delta[W(z)] = \tfrac{1}{2}\delta[\phi_{yu}(z)]^3 \quad (3.1)$$

$$z^r \det C(z)C^*(z) \quad \text{and} \quad z^s \det N(z)N^*(z) \text{ have no common zeros.} \quad (3.2)$$

Comments:

1) When G and K are square, the zeros of $z^r \det CC^*$ are the zeros of $\det C$ and their inverses, and similarly for the zeros of $\det NN^*$. Notice that the zeros of $\det C$ are zeros of G and poles of F, while the zeros of $\det N$ are zeros of K and poles of H. Therefore, (3.2) is a very natural condition which will almost always hold. The minimal degree condition (3.1) will be satisfied if there is no pole-zero cancellation between a pole of $W(z)$ and a zero of $W^*(z)$. It is essentially implied by (3.2) plus some additional technical assumptions which are of the same nature, namely the absence of common poles and zeros in some of the matrices F, G, H, K. See [8] for more details on generic systems and for the interpretation of (3.2) in the case of nonsquare G or K. In the sequel we shall in most cases assume that G and K are square.

2) We have argued in [8] and it follows from the first comment that almost all feedback systems satisfy the conditions (3.1) and (3.2), which partly explains the name generic.

3) The everyday meaning of generic is, roughly, that nothing is special; there is of course a technical meaning also in algebraic geometry. The meaning we are assigning here is precisely that of Definition 2, but no more, and in some sense is in between the two extremes. By excluding special pole-zero cancellations and the like, (3.1) and (3.2) are implied, and in this sense we might say nothing is special. On the other hand, without Definition 2 a purist might hold that any rational spectrum was by virtue of that rationality special, and consequently nongeneric; Definition 2, by specializing the meaning of generic, rules out such an argument. Just as relevant to this paper is the fact that when $H=0$, (3.1) and (3.2) can still hold, and $\{F, G, 0, K\}$ will then be generic.

4) With A, B, C, D, M, N as in Definition 2, it is easy to see that $W(z)$ admits a "natural" polynomial MFD

$$W(z)=\begin{bmatrix} A & -B \\ -M & D\end{bmatrix}^{-1}\begin{bmatrix} C & 0 \\ 0 & N\end{bmatrix}. \quad (3.3)$$

The genericity condition (3.2) guarantees that $\det C$ and $\det N$ have no common zeros; it then easily follows that this MFD is coprime (see [8]).

[3] $\delta[W(z)]$ is the McMillan degree of $W(z)$ (see, e.g., [14] and [15]).

5) Given a joint model $\{W(z), Q\}$ for a joint process (y, u) there corresponds a unique quadrupole $\{F, G, H, K\}$ to it via

$$F=W_{12}W_{22}^{-1} \qquad G=W_{11}-W_{12}W_{22}^{-1}W_{21}$$
$$H=W_{21}W_{11}^{-1} \qquad K=W_{22}-W_{21}W_{11}^{-1}W_{12}. \quad (3.4)$$

The inverses exist by the full rank assumption on (y, u). Therefore, we shall also say that a joint model $\{W(z), Q\}$ is generic if (3.1) and (3.2) hold, where it is understood that $C(z)$ and $N(z)$ are obtained from MFD's of a feedback model $\{F, G, H, K\}$ defined by (3.4).

In [8] the following theorem was proved.

Theorem 2 [8]: Consider the feedback system (2.1)–(2.2) with Assumption A and the corresponding $W(z)$ and $\phi_{yu}(z)$ given by (2.4) and (2.5). Assume that

$$\text{the system } \{F, G, H, K\} \text{ is generic} \quad (3.5a)$$
$$Q \text{ is block diagonal} \quad (3.5b)$$
$$F(\infty)=H(\infty)=0, \quad G(\infty) \text{ and } K(\infty) \text{ have full rank.} \quad (3.5c)$$

Then any other minimal degree spectral factor $\{\hat{W}(z), \hat{Q}\}$ of $\phi_{yu}(z)$ with $\hat{W}(\infty)$ block diagonal and nonsingular has a block diagonal $\hat{Q}$. In addition, the scaled paraunitary transformation matrix $V(z) = \hat{W}^{-1}(z)W(z)$ is also block diagonal, and if $\hat{F}, \hat{G}, \hat{H}, \hat{K}$ is the feedback system obtained from $\hat{W}(z)$ by (3.4),[4] then $\hat{F}, \hat{G}, \hat{H}, \hat{K}$ is also generic. Moreover, $F=\hat{F}$, $H=\hat{H}$, $G=\hat{G}V_1$, $K=\hat{K}V_2$, where $V_1(z)$ and $V_2(z)$ are scaled paraunitary matrices.

Since there is a one-to-one relation between $\{F, G, H, K\}$ and $W(z)$ by (2.4), the conditions (3.5) of Theorem 2 can also be stated as follows:

$$W(z) \text{ is generic} \quad (3.6a)$$
$$Q \text{ is block diagonal} \quad (3.6b)$$
$$W(\infty) \text{ is block diagonal and has full row rank.} \quad (3.6c)$$

Therefore, Theorem 2 actually says that if any spectral factorization $\{\hat{W}(z), \hat{Q}\}$ has the properties (3.6), then all minimal degree spectral factors $W(z)$ which are block diagonal and nonsingular at $z=\infty$ have a block diagonal Q and are related by block diagonal transformations. This result has important consequences insofar as the identifiability of the closed-loop system $\{F, G, H, K\}$ is concerned. The question of identifiability of feedback systems has been dealt with in [16] and will not be pursued here.

We shall now specialize the results of Theorem 2 to feedback-free processes to show that if a system is generic and feedback-free in a sense defined below, then all its feedback representations have block diagonal Q and $H(z)=0$. [See (2.1)]. First, we establish the relations between the feedback model (2.1) and the joint model (2.3) when $H(z)=0$.

Lemma 1: Consider the feedback system (2.1) with $H(z)=0$. Then there is a bijective relationship (one-to-one and onto) between $\{F, G, K\}$ and $W(z)$

$$W(z)=\begin{bmatrix} G & FK \\ 0 & K\end{bmatrix} \quad (3.7)$$

$$F=W_{12}W_{22}^{-1}, \quad G=W_{11}, \quad K=W_{22}. \quad (3.8)$$

Moreover, the joint process (y, u) is stationary if and only if F, G, and K are stable.

Proof: If $K(z)$ is not square, W_{22}^{-1} is understood to be a right inverse; its existence is guaranteed by the full rank assumption on (y, u). (For a proof, see [16].) Then (3.7) and (3.8) follows immediately from (2.4). Inspection of Fig. 2 shows that (y, u) is stationary if and only if F, G, and K have all their poles in $|z|<1$. (An unstable pole-zero cancellation between F, K would in practice lead to nonstationary y, and is therefore ruled out.) This last result also follows as a special case of the stability results obtained in [8] for 4-block models $\{F, G, H, K\}$.

[4] The existence of a one-to-one relationship between a joint model $W(z)$ and a 4-block feedback model F, G, H, K was proved in [8]; see (2.4) and (3.4).

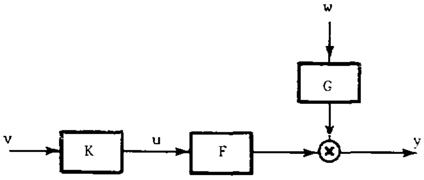

Fig. 2. Feedback-free system.

We now introduce a definition for a *feedback-free system*, as distinguished from a *feedback-free process*. This new definition is inspired by property 4) of Theorem 1.

Definition 3: Consider a system (2.1)–(2.2) obeying Assumption A. Then the *system is* (y, u) *feedback-free if* $H(z)=0$, Q is block diagonal and $F(\infty)=0$.

Comment: The requirement in this definition that $F(\infty)=0$ may seem a departure from intuition. We defend it on two grounds: without it, a number of the following conclusions are no longer valid and second, with $F(\infty)\neq 0$, u_t is in part predictable from y_t, and in a perverse sense this is like feedback.

In the next two results we shall relate the definition of a feedback-free system to that of a feedback-free process. Notice that it follows immediately from Definition 3 and Theorem 1 that a stationary process (y, u) with rational spectrum $\phi_{yu}(z)$ is feedback-free if and only if the feedback system obtained from the NMSF of $\phi_{yu}(z)$ is (y, u) feedback-free. Indeed, by (2.4) and (3.4), $H(z)=0$ is equivalent with $\overline{W}_{21}(z)=0$.

Theorem 3: Consider a system (2.1)–(2.2) obeying Assumption A, and assume that the system is (y, u) feedback-free. Let $W(z)$ be the corresponding joint model and $\phi_{yu}(z)$ the corresponding spectrum. Then the NMSF $\{\overline{W}(z), \overline{Q}\}$ of $\phi_{yu}(z)$ has $\overline{W}_{21}(z)=0$ and $\overline{Q}$ block diagonal, and the (y, u) process is feedback-free.

Proof:[5] Since the system is feedback-free, we may write

$$y_t=\sum_{i\geq 1} F_i u_{t-i}+n_t$$

where $n_t=\sum_{i\geq 0} G_i w_{t-i}$. Furthermore, as the structure of Fig. 2 shows, $n_t \perp u_s$ $\forall s$ and so $n_t \perp U^{t-1}$. Hence

$$n_t=y_t-y_t|U^{t-1}.$$

Because $u_s \perp n_t$ $\forall s \geq t$, $u_s \perp (y_t - y_t|U^{t-1})$ $\forall s \geq t$ and so by Theorem 1 [see (2.8)], the process is feedback-free.

Theorem 3 shows that if a system generating (y, u) is feedback-free, then the process (y, u) is feedback-free. The following example shows that the converse is not always true, even when the system in question has minimal degree, and $F(\infty)=0$.

Example 1: Consider the system

$$y_t=\frac{1}{z+0.5}u_t+\frac{z-0.6}{z+0.8}w_t$$

$$u_t=\frac{z+0.5}{z+0.75}v_t$$

with $Q=I$. This system is feedback-free. Its transfer function matrix is actually the normalized minimum phase spectral factor, as we now check. Constructing left coprime MFD's for $\{F \,\vdots\, G]$ and for K, yields

$$W(z)=\begin{bmatrix}(z+0.5)(z+0.8) & -(z+0.8)\\ 0 & z+0.75\end{bmatrix}^{-1}\cdot\begin{bmatrix}(z+0.6)(z+0.5) & 0\\ 0 & z-0.5\end{bmatrix}. \tag{3.9}$$

This factorization is not coprime. A coprime factorization is

$$W(z)=\begin{bmatrix}5(z+0.8) & 1\\ 0 & z+0.75\end{bmatrix}^{-1}\begin{bmatrix}5(z+0.6) & 6\\ 0 & z-0.5\end{bmatrix}.$$

Notice that $W(z)$ is square, stable, minimum phase, and $W(\infty)=I$. Therefore, $W(z)=\overline{W}(z)$, the NMSF of $\phi_{yu}(z)=W(z)W^*(z)$. Since $W_{21}(z)=0$ and $Q=I$, it follows by Theorem 1 that (y, u) is feedback-free. (Equally, we could have applied Theorem 3 without verifying that $W(z)=\overline{W}(z)$, to obtain this conclusion.) Now one can show that the following is an equivalent spectral factorization of $\phi_{yu}(z)$

$$\tilde{W}(z)=\begin{bmatrix}5(z+0.8) & 1\\ 0 & z+0.75\end{bmatrix}^{-1}\begin{bmatrix}5z+13.913 & 4.034\\ -1.085 & z+0.882\end{bmatrix}$$

$$\tilde{Q}=\begin{bmatrix}0.159 & 0.197\\ 0.197 & 0.813\end{bmatrix}.$$

Obviously, this defines a system $\tilde{S}$. We remark that $\tilde{W}(\infty)=I$, which will ensure $\tilde{F}(\infty)=0$, but $\tilde{W}_{21}(z)\neq 0$ and $\tilde{Q}$ is not block diagonal, even though the process (y, u) is feedback-free. Also, the McMillan degrees of $W(z)$ and $\tilde{W}(z)$ are the same.

The example above is nongeneric: there is a pole-zero cancellation at $z=-0.5$ between F and the noise model K for $\{u\}$. (See Fig. 2.) Therefore, $\det C$ and $\det N$ have a common zero and the standard factorization (3.9) for $W(z)$ is not coprime. One would expect (or hope) that if a system is feedback-free, then all equivalent minimal degree representations $\{W(z), Q\}$ with $W(\infty)$ block diagonal would also produce feedback-free systems (i.e., would have $W_{21}=0$ and Q block diagonal). Although Example 1 shows that this is not so [$\tilde{W}_{21}(z)\neq 0$ and $\tilde{Q}$ is not block diagonal, and yet the process (y, u) is feedback-free, i.e., the NMSF yields a feedback-free system via (3.4)], we show now that the hoped for result is true for all generic systems, provided that we restrict attention to other representations which are square.

Theorem 4: Consider the stationary stochastic system

$$y_t=F(z)u_t+G(z)w_t \tag{3.10a}$$

$$u_t=K(z)v_t. \tag{3.10b}$$

Assume that it is generic, that $F(\infty)=0$ while $G(\infty)$ and $K(\infty)$ have full rank, and that Q is block diagonal (hence it is feedback-free). Let the corresponding $W(z)$ be defined by (3.7) and $\phi_{yu}(z)$ by (2.5). Then all other square minimal degree spectral factors $\{\hat{W}(z), \hat{Q}\}$ of $\phi_{yu}(z)$ with $\hat{W}(\infty)$ block diagonal and nonsingular have $\hat{W}_{21}(z)=0$ and $\hat{Q}$ block diagonal; in addition, the triple $\hat{F}, \hat{G}, \hat{K}$ obtained from $\hat{W}(z)$ through (3.8) is also generic, and the corresponding system is feedback-free.

Proof: The result follows as a special case of Theorem 2. By the assumptions, all the conditions of Theorem 2 are satisfied. Hence, $\hat{Q}$ is block diagonal and

$$\hat{W}(z)=W(z)\begin{bmatrix}V_1(z) & 0\\ 0 & V_2(z)\end{bmatrix}. \tag{3.11}$$

Therefore, $\hat{W}_{21}(z)=0$. Let $A^{-1}[B \,\vdots\, C]$ be a left coprime polynomial MFD of $[F \,\vdots\, G]$, and similarly $D^{-1}N=K$. Then by the genericity assumption, the following "natural" MFD for $W(z)$ is coprime, since C and N do not drop in rank at the same z_0

$$W(z)=\begin{bmatrix}A & -B\\ 0 & D\end{bmatrix}^{-1}\begin{bmatrix}C & 0\\ 0 & N\end{bmatrix}. \tag{3.12}$$

Therefore, by the proof of Theorem 2 (see [8]), $\hat{W}(z)$, being of minimal degree, admits a factorization

$$\hat{W}(z)=\begin{bmatrix}A & -B\\ 0 & D\end{bmatrix}^{-1}\begin{bmatrix}\hat{C} & 0\\ 0 & \hat{N}\end{bmatrix} \tag{3.13}$$

where $\hat{C}(z)$ and $\hat{N}(z)$ satisfy the condition (3.2). Therefore, the system $\hat{F}=A^{-1}B$, $\hat{G}=A^{-1}\hat{C}$, $\hat{K}=D^{-1}\hat{N}$, $\hat{H}=0$ is also generic, and it is obviously feedback-free.

Corollary 1: Consider a stationary full rank feedback-free (y, u) process with rational spectrum $\phi_{yu}(z)$ and assume that the NMSF $\{\overline{W}(z), \overline{Q}\}$ of $\phi_{yu}(z)$ is generic. Then the feedback systems $\{F, G, H, K\}$ obtained from all equivalent minimal degree spectral factors $\{W(z), Q\}$ of $\phi_{yu}(z)$ with $W(\infty)$ block diagonal and nonsingular are all (y, u) feedback-free.

[5] This proof, supplied by a reviewer, replaces a longer proof in the first version of the paper.

Proof: Replace $W(z)$ in Theorem 4 by the NMSF $\overline{W}(z)$ and remember that $\overline{W}(\infty)=I$.

Theorem 4 and Corollary 1 are interesting in that in a generic situation, if a process (y,u) is feedback-free, then any minimal degree spectral factorization with $W(\infty)$ block diagonal and nonsingular (and not just the NMSF) will show this, i.e., will have $W_{21}(z)=0$ and Q block diagonal; therefore, all the corresponding feedback systems are (y,u) feedback-free. In Theorem 3 we proved that if a system (2.1)–(2.2) is (y,u) feedback-free, then the process (y,u) is feedback-free. Corollary 1 gives a converse result, at least for generic systems with square G and K. We summarize these results in the following corollary.

Corollary 2: Consider a joint stationary process (y,u) obeying Assumption A, and represented by either a feedback system (2.1)–(2.2) or a joint model (2.3). Let $\phi_{yu}(z)$ be its spectrum. Assume that the model has minimal degree, that $W(\infty)$ is block diagonal and nonsingular and that the NMSF $\overline{W}(z)$ is also generic. Then the process is feedback-free if and only if every minimal degree spectral factorization $\{\hat{W}(z),\hat{Q}\}$, with $\hat{W}(\infty)$ block diagonal and nonsingular, has $\hat{W}_{21}(z)=0$ and $\hat{Q}$ block diagonal. In particular, the process is feedback-free if and only if the system is (y,u) feedback-free.

Proof—Sufficiency: If every factorization has $\hat{W}_{21}=0$, $\hat{Q}$ block diagonal, then the NMSF has this property and the result follows from Theorem 1.

Necessity: If the process is feedback-free, then the NMSF has $\overline{W}_{21}=0$ and $\overline{Q}$ block diagonal. Since it is generic, all other minimal degree spectral factors with $\hat{W}(\infty)$ block diagonal and nonsingular have $\hat{W}_{21}=0$, $\hat{Q}$ block diagonal by Theorem 4. In particular, the true $W(z)$, Q, corresponding to the physical system, has this property and hence the system is feedback-free.

IV. Comparison with Previous Results

In [4] the following definition was proposed for a feedback-free process by Caines and Chan. This definition was later renamed "weak feedback-free" in [7].

Definition 4 [4]: Consider a real stationary full rank bounded stochastic process (y,u) with rational spectrum $\phi_{yu}(z)$. Then there is no feedback from y to u if and only if the NMSF $\{\overline{W}(z),\overline{Q}\}$ has $\overline{W}_{21}(z)=0$.

Comparing with our definition (see Theorem 1) shows that the only difference is that no requirement is made on $\overline{Q}$, while we require $\overline{Q}$ to be block diagonal. As a consequence, there is no equivalent of our Theorem 3 under the definition in [4]; in other words if a system has $H(z)=0$ (or $W_{21}(z)=0$) and $W(\infty)=I$, it does not follow that the NMSF will have $\overline{W}_{21}(z)=0$, as the following example indicates.

Example 2: Consider the following stationary full rank bounded (y,u) process:

$$\begin{cases} y_i=\dfrac{z+2}{z+0.4}w_i \\ u_i=\dfrac{z+4}{z+0.6}v_i \end{cases} \qquad Q=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

Note that $F(z)=H(z)=0$, $W(\infty)=I$, but Q is not diagonal. Note also that $C=z+2$ and $N=z+4$ have no common zeros; the genericity conditions (3.1) and (3.2) are satisfied. The corresponding $W(z)$ is not minimum phase

$$W(z)=\begin{bmatrix}\dfrac{z+2}{z+0.4} & 0\\ 0 & \dfrac{z+4}{z+0.6}\end{bmatrix}.$$

Let $\phi_{yu}(z)=W(z)QW^*(z)$. The NMSF of $\phi_{yu}(z)$ is

$$\overline{W}(z)=\begin{bmatrix}\dfrac{z+0.575}{z+0.4} & \dfrac{-0.069}{z+0.4}\\ \dfrac{0.348}{z+0.6} & \dfrac{z+0.176}{z+0.6}\end{bmatrix}$$

$$\overline{Q}=\begin{bmatrix}7.838 & 7.250\\ 7.250 & 31.204\end{bmatrix}.$$

Note that $\overline{W}_{21}(z)\neq 0$, even though $W_{21}(z)=0$. We cannot recover a result similar to Theorem 3 by allowing some "denormalization" of $\overline{W}(z)$: thus if we permit $\overline{W}(z)$ to be replaced by $\overline{W}(z)T$ and $\overline{Q}$ by $T^{-1}\overline{Q}T^{-T}$, it is impossible to obtain $[\overline{W}(z)T]_{21}=0$ by any choice of constant nonsingular T.

The next example shows that the converse of Theorem 3, namely Corollary 1, does not hold either with Definition 4. If a process is feedback-free according to Definition 4, and the NMSF is generic, it does not follow that all equivalent representations with block diagonal $W(\infty)$ have an upper block triangular $W(z)$.

Example 3: Consider the process (3.10) with

$$F=\frac{1}{z},\quad G=1,\quad K=\frac{z+0.5}{z+0.4}$$

and

$$Q=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}.$$

The corresponding $W(z)$ is

$$W(z)=\begin{bmatrix}z & -1\\ 0 & z+0.4\end{bmatrix}^{-1}\begin{bmatrix}z & 0\\ 0 & z+0.5\end{bmatrix}.$$

$W(z)$ is stable, minimum phase, $W(\infty)=I$ and $W_{21}(z)=0$. Therefore, the process (y,u) is feedback-free according to Definition 4. In addition, the system is generic. An equivalent factorization $\{\tilde{W}(z),\tilde{Q}\}$ of $\phi_{yu}(z)=W(z)QW^*(z)$ with $\tilde{W}(\infty)=I$ is given by the following:

$$\tilde{W}(z)=\begin{bmatrix}z & -1\\ 0 & z+0.4\end{bmatrix}^{-1}\begin{bmatrix}z & 0\\ -1.5 & z+2\end{bmatrix}$$

$$\tilde{Q}=\begin{bmatrix}1 & 1\\ 1 & 1.25\end{bmatrix}.$$

Notice now that $\tilde{W}_{21}(z)\neq 0$. Once again, there is no way one can change the normalization to obtain $[\tilde{W}(z)T]_{21}=0$.

The failure of Definition 4 to imply a result similar to Theorem 3 and Corollary 1 means that, with this definition, the feedback-free property cannot be detected by inspection from an arbitrary feedback model $\{F,G,H,K\}$. If $H=0$, this does not necessarily mean that the process is feedback-free; conversely if $H\neq 0$, the process could still be feedback-free in the sense of Definition 4. A transformation to the equivalent NMSF is always required to decide whether the process is feedback-free. In other respects, most of the conclusions that can be drawn with Definition 1 are paralleled by those which can be drawn with Definition 4. Theorem 1 has a close parallel—see [7].

V. Detecting Feedback from Estimated Spectra

We have seen in Section III that the absence of feedback between y and u can be detected in different ways. If a model for (y,u) is known and if $H=0$ and Q is block diagonal, then the process is feedback-free by Theorem 3. If in addition the model is generic, then $H=0$ and Q block diagonal is a necessary condition by Corollary 2. If the spectrum $\phi_{yu}(z)$ is known rather than a model, then an NMSF $\{\overline{W},\overline{Q}\}$ can be computed and the process is feedback-free if $\overline{W}_{21}(z)=0$ and $\overline{Q}$ is block diagonal. In most practical cases, however, only an estimate of the spectrum is available. We can then use a continuity result derived in [16].

Theorem 5 [16]: Consider a stationary full rank stochastic process and assume that its spectrum $\phi(z)$ is such that

$$0<c_1I\leq\phi(e^{j\omega})\leq c_2I<\infty \tag{5.1}$$

for some positive constants c_1, c_2 and $\omega\in[-\pi,\pi]$. Regarding ϕ as a function in $\mathcal{C}[-\pi,\pi]$, $\overline{W}(e^{j\omega})$ as a function in $\mathcal{L}_2[-\pi,\pi]$ and $\overline{Q}$ in $\mathbb{R}^{n\times n}$, then the mappings $\phi\to\overline{W}(e^{j\omega})$ and $\phi\to\overline{Q}$ are continuous, where $\overline{W},\overline{Q}$ is the NMSF of ϕ.

The following results then follow from Theorem 5 and the results of Section III.

Corollary 3: Consider a (y,u) feedback-free system represented by (3.15), and obeying Assumption A. Let $\phi_{yu}(z)$ be the spectrum of the (y,u) process; assume that it obeys condition (5.1) and that an NMSF $\{\hat{\overline{W}}(z),\hat{\overline{Q}}\}$ is computed from an approximation $\hat{\phi}_{yu}(z)$ of $\phi_{yu}(z)$. Then given arbitrary $\epsilon>0$, there exists $\delta(\epsilon)$ such that

$$\int_{|z|=1} z^{-1}\operatorname{trace}\left[\hat{\bar{W}}_{21}^{*}(z)\hat{\bar{W}}_{21}(z)\right] dz \leq \epsilon \tag{5.2a}$$

$$\|\hat{\bar{Q}}_{21}\| \leq \epsilon \tag{5.2b}$$

whenever

$$\sup_{z=1} \|\phi_{yu}(z) - \hat{\phi}_{yu}(z)\| \leq \delta. \tag{5.3}$$

Proof: The proof results from Theorems 3 and 5.

The above corollary says that if a system is really feedback-free but one has somewhat inaccurate spectral data, one will find the process is approximately feedback-free. The next corollary says that if a system is approximately feedback-free and one has accurate or somewhat inaccurate spectral data, then one will find the process is approximately feedback-free.

Corollary 4: Consider a system of the form (2.1)–(2.2), satisfying Assumption A, and suppose that F, G, and K are all stable, with no zeros on $|z|=1$ and $F(\infty)$, $H(\infty)$ are zero. Let $\phi_{yu}(z)$ be the spectrum of the associated (y, u) process, assumed to satisfy (5.1), and let $\{\hat{\bar{W}}(z), \hat{\bar{Q}}\}$ be the NMSF obtained from an approximation $\hat{\phi}_{yu}(z)$ to $\phi_{yu}(z)$. Then given $\epsilon > 0$, there exists δ_1, δ_2, such that

$$\sup_{|z|=1} \|\hat{\phi}_{yu}(z) - \phi_{yu}(z)\| \leq \delta_1 \tag{5.4a}$$

$$\sup_{|z|=1} \|H(z)\| \leq \delta_2 \tag{5.4b}$$

$$\|Q_{21}\| \leq \delta_2 \tag{5.4c}$$

imply (5.2).

Proof: Let $\bar{\phi}_{yu}(z)$ be obtained by setting $H(z)=0$, $Q_{21}=Q_{12}'=0$ in (2.1) and (2.2). Then (5.1) holds with ϕ replaced by $\bar{\phi}$ and so there exists $\delta(\epsilon) > 0$ such that (5.2) is implied by

$$\|\bar{\phi}_{yu}(z) - \hat{\phi}_{yu}\| < \delta.$$

Select $\delta_1 = \delta/2$. Also, since the spectrum depends continuously on the system matrices, we can find δ_2 such that (5.4) implies

$$\|\phi_{yu}(z) - \bar{\phi}_{yu}(z)\| < \delta/2. \tag{5.5}$$

Then (5.4a) with $\delta_1 = \delta/2$ and (5.5) imply (5.2), as required.

Finally, we need to ask the following question. Given that measurements show that a process is approximately feedback-free, given that we know the underlying system has $F(\infty) = H(\infty) = 0$, is square, and is generic, can one conclude that it is at least approximately feedback-free? (Naturally, there is no possibility of concluding that it is exactly feedback-free). The answer to this question is more complicated (see [16]); if the true and approximate spectrum are positive definite on $|z|=1$, if the NMSF of the approximate spectrum is generic, and if there is a feedback-free approximation with the same characteristic polynomial as the NMSF, then one can conclude the original system is approximately feedback-free.

VI. Concluding Remarks

We have analyzed a definition for the absence of feedback between two (vector) subcomponents of a stationary stochastic process. Using some new results on feedback processes, we have shown that this new definition has some desirable properties which were not present in an earlier definition proposed by Caines and Chan, despite earlier suggestions that the two definitions and their consequences are very close indeed. However, with the definition used in this paper the absence of feedback can be checked by inspection from any given $\{F, G, H, K\}$ or $W(z)$ model. In addition, in the generic case the block-triangular structure of the (y, u) model extends naturally to all equivalent square, minimal degree factors of the joint spectrum $\phi_{yu}(z)$ which have the right behavior at $z = \infty$.

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Linear Systems with Two-Point Boundary Lyapunov and Riccati Equations

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***Abstract*—This paper extends some well-known system theories for algebraic Lyapunov and Riccati equations. These extended results deal with the existence and uniqueness properties of the solutions to matrix differential equations with two-point boundary conditions and are shown to include conventional results as special cases. Necessary and sufficient conditions are derived under which linear systems are stabilizable with periodic feedback gains derived from the two-point boundary matrix differential equations. An easy iterative method for solving the two-point boundary differential Riccati equation is given with an initial guess which is obtained from the intervalwise receding horizon control. The results in this paper are related to periodic feedback gain controls and also to the quadratic cost problem with a discrete state penalty.**

I. Introduction

The following well-known theoretical result exists in the area of Lyapunov stability for the linear time invariant homogeneous system

$$\dot{x}(t) = Ax(t) \tag{1.1}$$

where $x(t) \in R^n$ and A is an $n \times n$ matrix. The system (1.1) is asymptotically stable if and only if for any C such that $\{A, C\}$ is observable there exists a positive definite matrix K satisfying the Lyapunov matrix equation

$$A'K + KA + C'C = 0. \tag{1.2}$$

Manuscript received July 28, 1980; revised April 8, 1981. Paper recommended by A. Z. Manitius, Past Chairman of the Optimal Systems Committee. This work was supported by the Korean Ministry of Education and the National Science Foundation under Grant ENG-78-25828.

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0018-9286/82/0400-0436$00.75