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Brief Paper

On Plant and LQG Controller Continuity Questions*

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Key Words—LQG control; identification for control; continuity questions.

Abstract—Using the dual Youla parametrizations of controller-based coprime factor plant perturbations and plant-based coprime factor controller perturbations, we study the LQG plant-controller continuity question. Indeed, we show that it is possible to calculate a new optimal LQG controller from a previous one when the plant is slightly changed, and to quantify the change in the controller as a function of the change in the plant. In addition, we compute the degradation in the achieved LQG cost when the LQG controller is computed on the basis of a perturbation of the real plant. As a by-product, we characterize the set of all plants that have the same optimal LQG controller. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

Our motivation for this paper originates from the recent schemes for iterative identification and control design, in which models and model-based controllers are successively updated on the basis of new data collected on the real plant operating in feedback with the most recent controller: see Lee *et al.* (1992) Schrama (1992) and Zang *et al.* (1995) for a representative sample of these iterative design schemes and Gevers (1993) for a tutorial presentation of the ideas. An implicit but unproven assumption underlying these schemes is that a small change in the plant model should result in a small change in the controller, and hence a small change in the actual closed-loop system. This in turn should result in a slightly modified identified plant model.

Our main contribution in this paper is to shed some light on this continuity question in the case of a Linear Quadratic Gaussian (LQG) control criterion. In this paper, we have opted for an approach that uses the dual Youla parametrizations of controller-based coprime factor plant perturbations and plant-based coprime factor controller perturbations. This approach is motivated by the fact that in many of the iterative identification and control schemes presented in the literature, the identification step is performed using the identification method developed in (Hansen, 1989) in view of closed-loop experiment design. In that method, the dual Youla parametrization is used to parametrize the unknown plant, and the closed-loop identification is reduced to an open loop identification of the Youla parameter. By using a Youla parametrization-based approach for the minimization of the control criterion, we embed the identification and control steps in a uniform framework in which the controller and the model are computed as a perturbation of, respectively, the previous controller and the previous model.

In the sequel, the following concepts are used extensively.

Proposition 1.1 (Vidyasagar, 1985). Suppose P_0 and C_0 have fractional representations $P_0 = N_P D_P^{-1}$ and $C_0 = N_C D_C^{-1}$, where N_P, D_P, N_C, D_C belong to **S**, the ring of proper stable transfer functions. Assume that the following Bezout equation holds:

$$N_C N_P + D_C D_P = 1. \tag{1}$$

For any arbitrary stable (linear) operator S, define

$$N_S = N_C - D_P S, \qquad D_S = D_C + N_P S. \tag{2}$$

1. Then $C(S) = N_S D_S^{-1}$ is a stabilizing controller for $P_0 = N_P D_P^{-1}$.
2. *Any* controller that stabilizes P_0 has a fractional representation (2) for some $S \in \mathbf{S}$. The dual result is as follows. For any arbitrary stable (linear) operator Q, define

$$N_Q = N_P - Q D_C, \qquad D_Q = D_P + Q N_C. \tag{3}$$

1. Then $P(Q) = N_Q D_Q^{-1}$ is stabilized by $C_0 = N_C D_C^{-1}$.
2. *Any* plant stabilized by C_0 has a fractional representation (3) for some $Q \in \mathbf{S}$.

Our basic one-degree-of-freedom control loop is that of Fig. 1 and our control design criterion is the following regulation LQG index (expressed here in discrete time):

$$J_{\mathrm{LQG}} = \lim_{N \to \infty} \frac{1}{N} E\left\{ \sum_{t=1}^{N} (y_t^2 + \lambda u_t^2) \right\}, \tag{4}$$

where y_t is the plant output, u_t is the control signal. The disturbance signal v_t is assumed zero mean stationary with spectral density function ϕ_v. In the sequel, we consider a disturbance rejection problem, i.e. $r_t = 0$.

We now summarize a solution of the minimization problem (4) using the Youla parametrization. This solution borrows from a collection of results from (Desoer *et al.*, 1980; Francis, 1982; Vidyasagar *et al.*, 1982; Youla *et al.*, 1976). Note that a more elegant solution is obtained using a standard plant approach; we refer to (Francis, 1982) for further details.

Let P_0 and C_0 be as described in Proposition 1.1. Replace C_0 in Fig. 1 by an arbitrary controller $C(S)$ defined in Proposition (1.1). For this $(P_0, C(S))$ configuration we have

$$y_t = (D_C + N_P S) D_P v_t, \qquad u_t = (N_C - S D_P) D_P v_t.$$

Using Parseval's theorem, we obtain the following expression for equation (4):

$$J_{\mathrm{LQG}} = \frac{1}{2\pi} \int \mathrm{d}\omega \left\{ |D_C + N_P S|^2 + \lambda |N_C - D_P S|^2 \right\} |D_P|^2 \phi_v. \tag{5}$$

It is standard that a stable minimizing S can be found analytically by means of spectral factorizations and projections, i.e. by taking stable parts. Indeed, by completing the square, the LQG control criterion can be rewritten as

$$J_{\mathrm{LQG}} = \| \mathscr{A} S + \mathscr{A}^{-*} \mathscr{B} \|_2^2 + \frac{1}{2\pi} \int \mathrm{d}\omega \left\{ \mathscr{C} - (\mathscr{A}^* \mathscr{A})^{-1} \mathscr{B}^* \mathscr{B} \right\} \tag{6}$$

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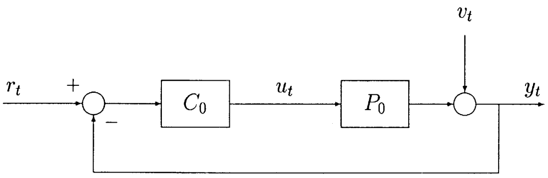

Fig. 1. One degree of freedom control loop.

with

$$\mathscr{A}\mathscr{A}^* = [|N_P|^2 + \lambda |D_P|^2]\, |D_P|^2 \phi_v, \tag{7}$$

$$\mathscr{B} = [N_P^* D_C - \lambda D_P^* N_C]\, |D_P|^2\, \phi_v, \tag{8}$$

$$\mathscr{C} = [|D_C|^2 + \lambda |N_C|^2]\, |D_P|^2\, \phi_v, \tag{9}$$

where $\mathscr{A}$ is minimum phase, stable and of relative degree zero. The minimizing S is clearly given by $S_{\text{opt}} = -\mathscr{A}^{-1}[\mathscr{A}^{-*}\mathscr{B}]_{\text{st}}$ where $[\,]_{\text{st}}$ denotes the stable part. Note that the constant term in the partial fraction expansion of $\mathscr{A}^{-*}\mathscr{B}$ must be so partitioned between $[\mathscr{A}^{-*}\mathscr{B}]_{\text{unst}}$ and $[\mathscr{A}^{-*}\mathscr{B}]_{\text{st}}$ that $[\mathscr{A}^{-*}\mathscr{B}]_{\text{unst}}$ has $z = 0$ as a zero. The preceding remark is used extensively in Section 3. We refer the reader to Vidyasagar (1985) for more details. The optimal control cost is

$$J_{\text{LQG}}^{\text{opt}} = \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ |\,[\mathscr{A}^{-*}\mathscr{B}]_{\text{unst}}\,|^2 + \frac{\lambda}{|N_P|^2 + \lambda |D_P|^2}\, |D_P|^2\, \phi_v \right\}. \tag{10}$$

If C_0 is the optimal controller for P_0, then $S_{\text{opt}} = 0$ minimizes J_{LQG} over all $S \in \mathbf{S}$.

Using the Youla parametrizations and the LQG control design criterion (4), we solve the following problems. Here, C_0 denotes the optimal (and hence stabilizing) controller for P_0.

1. Assume that the optimal LQG controller C_0 for a plant P_0 is known and consider a new plant P_1 that is stabilized by C_0 and that is obtained by a perturbation of size Q away from P_0. We then compute the optimal LQG controller C_1 for P_1 as a perturbation of size $\bar{S}$ away from C_0, where $\bar{S}$ is computed from P_0, C_0 and Q. This allows us to relate the size of a change Q in the plant to the size of the corresponding change $\bar{S}$ in the optimal LQG controller. We are especially interested in the simple formula resulting when Q is small where the size of Q is measured using either the H_∞ or the H_2 norm. Here, P_0 and P_1 could be seen as two successive plant models in an iterative design scheme, with C_0 and C_1 the corresponding optimal controllers. Alternatively, P_0 could also be the true plant, with P_1 a model that is close to it.
2. Under the same assumptions, we compute the increase in the LQG cost (i.e. the performance degradation) that results from applying the controller C_1, optimal for P_1, to the initial plant P_0. This increase is expressed as a function of the size of the perturbation Q of P_1 away from P_0. Again, our main focus is on small Q. The question addressed here is how much LQG cost increase is incurred by applying to the real plant P_0, say, an optimal controller C_1 computed on the basis of a plant model P_1 that is close to P_0.
3. As a by-product, we characterize the set of all plants P_1 that have the same optimal LQG controller, C_0, as the original plant P_0, i.e. we characterize the set of perturbations Q that are such that the Youla parameter $\bar{S}$ is zero over the frequency axis.

The outline of our paper is as follows. In Section 2 we compute how much change is induced in a controller by a change in a plant model, while in Section 3 we characterize the set of all plants that have the same optimal LQG controller. In Section 4 we express the degradation in the LQG cost that results from computing the LQG controller on the basis of a model that is a perturbed version of the actual plant. The validity of the theoretical results is checked in Section 5. We conclude in Section 6.

2. *Plant and corresponding controller perturbations*

In this section we examine the change that results in an optimal LQG controller when a plant model is changed from some initial model P_0 to a model P_1 that is expressed as a controller-based perturbation of P_0. Consider first a plant model P_0 and its corresponding optimal controller C_0, both factorized as before. Let now P_1 be some plant that is stabilized by C_0. It is obvious that P_1 is contained in the set

$$\mathscr{P} = \{P_1(Q) = (N_P - QD_C)\,(D_P + QN_C)^{-1} \text{ with } Q \in \mathbf{S}\}, \tag{11}$$

of all models stabilized by C_0. The set of all controllers stabilizing a given $P_1(Q) \in \mathscr{P}$ is then given by

$$\mathscr{C} = \{C_1(\bar{S}, Q) = [N_{C_1}(\bar{S}, Q)][D_{C_1}(\bar{S}, Q)]^{-1} \text{ with } \bar{S}, Q \in \mathbf{S}\} \tag{12}$$

where $N_{C_1}(\bar{S}, Q) = N_C - \bar{S}(D_P + QN_C)$ and $D_{C_1}(\bar{S}, Q) = D_C + \bar{S}(N_P - QD_C)$. Let C_1 be any controller in the set $\mathscr{C}$. Using Parseval's relation, we get the following LQG index:

$$J_{\text{LQG}}(P_1, \phi_{v_1}, C_1) = \frac{1}{2\pi}\int \mathrm{d}\omega \{|D_C + (N_P - D_C Q)\,\bar{S}\,|^2. + \lambda\,|N_C - (D_P + N_C Q)\bar{S}\,|^2\}|D_P + QN_C|^2\, \phi_{v_1}. \tag{13}$$

2.1. *Computation of $\bar{S}_{\text{opt}}$ as a function of Q.* In this subsection, we characterize the optimal controller C_1^{opt}, i.e. we compute $\bar{S}_{\text{opt}}$ that minimizes J_{LQG} and express it as a function of Q and the coprime factorizations of the plant P_0 and its corresponding optimal controller C_0. Thus, $\bar{S}_{\text{opt}}$, which expresses C_1^{opt} as a perturbation of C_0, is to be defined as a function of Q, which expresses P_1 as a perturbation of P_0.

Recall that $\mathscr{A}$ and $\mathscr{B}$, related to the plant P_0 and its optimal controller C_0, are given by

$$\mathscr{A}\mathscr{A}^* = [|N_{\text{P}}|^2 + \lambda\,|D_P|^2]|D_P|^2\, \phi_v$$

and

$$\mathscr{B} = [N_P^* D_C - \lambda D_P^*\, N_C]|D_P|^2\, \phi_v.$$

Two situations can occur when the system is perturbed: either the perturbation influences only the plant model, and the noise model remains unchanged or both the plant model and the noise model are influenced. We consider the case where ϕ_{v_1} varies with Q in such a way that $|D_P + QN_C|^2\, \phi_{v_1}(Q)$ is independent of Q, i.e.

$$|D_P + QN_C|^2\, \phi_{v_1}(Q) = |D_P|^2\, \phi_{v_1}(0) \;= |D_P|^2\, \phi_v. \tag{14}$$

This is typical of an ARMAX model structure, i.e. equation (14) makes sure that the perturbed system (P_1, ϕ_{v_1}) remains an ARMAX system if the original system (P_0, ϕ_v) is an ARMAX system. By an ARMAX model structure, we mean an autoregressive moving-average model structure with exogeneous input. Note that the case $\phi_{v_1}(Q) = \phi_v$ which is typical of an output error (OE) model structure leads to derivations that are more involved. We are now in a position to calculate the perturbed version of $\mathscr{B}$,

$$\begin{aligned}\bar{\mathscr{B}} &= [(N_P^* - Q^* D_C^*)D_C - \lambda(D_P^* + Q^* N_C^*)N_C]|D_P|^2\, \phi_v. \\ &= \mathscr{B} - Q^*[|D_C|^2 + \lambda\,|N_C|^2]|D_P|^2\, \phi_v.\end{aligned} \tag{15}$$

There will be a corresponding change from $\mathscr{A}$ *to* $\bar{\mathscr{A}}$, i.e.

$$\bar{\mathscr{A}}\bar{\mathscr{A}}^* = [|N_P - QD_C|^2 + \lambda\,|D_P + QN_C|^2]|D_P|^2 \phi_v. \tag{16}$$

The minimizing Youla parameter $\bar{S}_{\text{opt}}$ is given by

$$\begin{aligned}\bar{S}_{\text{opt}} &= -\bar{\mathscr{A}}^{-1}[\bar{\mathscr{A}}^{-*}\bar{\mathscr{B}}]_{\text{st}} \\ &= \bar{\mathscr{A}}^{-1}[\bar{\mathscr{A}}^{-*}Q^*[|D_C|^2 + \lambda\,|N_C|^2]|D_P|^2\, \phi_v]_{\text{st}}\end{aligned} \tag{17}$$

because $\mathscr{B}$ is unstable by optimality of C_0 *and* $\bar{\mathscr{A}}^{-*}$ is unstable by definition.

2.2. *A continuity question.* In this subsection, we investigate the continuity properties of LQG controllers with respect to plant perturbations, i.e. we deal with small perturbations Q where the size of Q is measured using either the H_∞ or the H_2 norm. The operators we deal with are the spectral factorization operator and the projection operator, i.e. taking stable parts. These operators are closely related (at least in the scalar case) and their continuity properties have been studied extensively in the literature. Indeed, it has been shown that the spectral factorization operator and the projection operator are continuous with respect to the L_2 norm. In contrast, both operators are discontinuous with respect to the L_∞ norm, i.e. small H_∞ perturbations in Q can lead to arbitrarily large H_∞ perturbations in $\bar{S}_{\text{opt}}$. We refer the reader to Anderson (1985) and Anderson and Green (1988). However, the continuity property with respect to the H_∞ norm can be recovered if the perturbed system satisfies any one of the following conditions:

$$\begin{cases} \text{it has bounded derivative,} \\ \text{its McMillan degree is bounded or} \\ \text{it has a prescribed degree of stability } \sigma > 0. \end{cases} \tag{18}$$

The reader is, respectively, referred to Anderson (1985), Anderson and Green (1988) and De Bruyne *et al.* (1995).

It now follows that

$$\bar{\mathscr{A}} = \mathscr{A} + o(Q) \Rightarrow \bar{\mathscr{A}}^{-1}Q \simeq \mathscr{A}^{-1}Q + o(Q^2). \tag{19}$$

Therefore, we can conclude that up to first order approximations

$$\bar{S}_{\text{opt}} \simeq \mathscr{A}^{-1}[\mathscr{A}^{-*}Q^*[|D_C|^2 + \lambda|N_C|^2]|D_P|^2\phi_v]_{\text{st}}. \tag{20}$$

We conclude that if P_1 is expressed as a controller-based perturbation of P_0 for some perturbation Q with $\|Q\|$ sufficiently small, then the optimal controller C_1 for P_1 can be expressed as a plant-based perturbation of C_0 optimal for P_0 for some perturbation $\bar{S}_{\text{opt}}(Q)$ with $\|\bar{S}_{\text{opt}}(Q)\|$ small computed from equation (17). Here $\|.\|$ denotes either the H_∞ or the H_2 norm. Recall that the restrictions (18) apply if the size of Q is measured using the H_∞ norm.

3. *Parametrization of all plants that have the same LQG or MV controller as* P_0

In this section, we characterize the set of all plants (P_1, ϕ_{v_1}) that admit the same optimal controller, C_0, as (P_0, ϕ_v).

Note that the so called "inverse optimal control problem" is closely related to the problem that is examined in this section. This problem considers whether there exists an optimal control criterion for which a given controller C_0 has the property that it is the optimal control law associated with a given plant P_0. One of many relevant references is given in Fuji and Narazaki (1984).

Recall that ϕ_{v_1} is subject to equation (14). This set is parametrized by all $Q \in \mathbf{S}$ that have $\bar{S} = 0$, i.e. that solve

$$[\bar{\mathscr{A}}^{-*}Q^*[|D_C|^2 + \lambda|N_C|^2]|D_P|^2\phi_v]_{\text{st}} = 0. \tag{21}$$

Equation (21) is a very implicit characterization, not self-evidently allowing any nonzero solutions. In the next subsections, we will display a nontrivial solution set more explicitly. From equation (13) and by optimality of C_0 for P_1, i.e. the fact that $Q \in \mathbf{S}$ solves equation (21) and $\bar{S}_{\text{opt}} = 0$, we observe that the control cost that is associated with the pair (P_1, C_0) is now independent of Q, i.e.

$$J_{\text{LQG}}(P_1, C_0) = J_{\text{LQG}}(P_0, C_0) = \frac{1}{2\pi}\int d\omega\{|D_C|^2 + \lambda|N_C|^2\}|D_P|^2\phi_v.$$

Let us specialize our results to the case where the plant is described by an ARMAX model

$$A(z)y_t = B(z)u_t + C(z)e_t, \tag{22}$$

where e_t is white noise of zero mean and unit variance, $A(z)$, $B(z)$ and $C(z)$ are polynomials in z of degree n, $n-d$ and n, respectively, that have no common factor, with $A(z)$, $B(z)$ coprime, $d \geq 1$, and $C(z)$ having all its zeros inside the unit circle. Note that this system has a delay d. Equation (20) is normalized so that the leading coefficients of the polynomials $A(z)$ and $C(z)$ are unity, i.e. $A(z)$ and $C(z)$ are monic. $B(z)$ can be factorized as $B(z) = B_-(z)B_+(z)$ where $B_+(z)$ has all its zeros strictly inside the unit circle and $B_-(z)$ is monic and has all its zeros on or outside the unit circle.

3.1. *The LQG disturbance rejection problem.* The spectral factorization solution method for the infinite horizon LQG regulation problem consists in first computing the stable minimum phase spectral factor $G(z)$ of

$$G(z)G^*(z) = \lambda A(z)A^*(z) + z^d B(z)B^*(z). \tag{23}$$

It can be shown that if $\deg A > \deg B$ then there always exists a unique polynomial $G(z)$ with $\deg G(z) = n$ and positive coefficient of the highest degree term. The next step consists in solving the following Bezout equation for the polynomials $X(z)$ and $Y(z)$:

$$G(z)C(z) = A(z)X(z) + B(z)Y(z). \tag{24}$$

A unique solution pair is obtained by finding the polynomials $X(z)$, $Y(z)$ such that

$$G^*(z)V(z) = X^*(z)B(z) - \lambda Y^*(z)A(z) \tag{25}$$

and

$$A^*(z)V(z) = C^*(z)B(z) - Y^*(z)G(z) \tag{26}$$

hold with $\deg V(z) < n$. Here $X^*(z) = z^n X(z^{-1})$ and $Y^*(z) = z^n Y(z^{-1})$. The resulting optimal controller is

$$u_t = -\frac{Y(z)}{X(z)}y_t. \tag{27}$$

We refer the reader to Aström and Wittenmark (1990) for further details.

Proposition 3.1. Consider some ARMAX system (A, B, C) and its optimal LQG controller C_0 with their respective fractional representations

$$N_P = \frac{B(z)}{G(z)}, \quad D_P = \frac{A(z)}{G(z)}, \quad N_C = \frac{Y(z)}{C(z)}, \quad D_C = \frac{X(z)}{C(z)} \tag{28}$$

where G, X and Y are determined from A, B, C via equations (23) and (24). Let

$$P_1 = (N_P - QD_C)(D_P + QN_C)^{-1} \tag{29}$$

be a perturbation of P_0 and consider the noise spectrum

$$\phi_{v_1} = \frac{|D_P|^2\phi_v}{|D_P + QN_C|^2} \tag{30}$$

with

$$Q = G^*Q_1 \tag{31}$$

where Q_1 is any element of $\mathbf{S}$ that has relative degree at least $n+1$. Then, for fixed λ, any system (P_1, ϕ_{v_1}) with Q defined as in equation (31) has the same optimal LQG controller, $C_0 = N_C D_C^{-1}$, as the unperturbed system (P_0, ϕ_v). Also, for fixed λ, any ARMAX system (P_1, ϕ_{v_1}) that has optimal LQG controller $C_0 = N_C D_C^{-1}$ can be expressed as equations (29) and (30), for some Q defined in equation (31).

Proof. Note that the fractional representations (28) fulfill the Bezout identity (1), and that each transfer function is stable and proper. Also, the choice (30) assures that assumption (14) is satisfied. The first term in equation (15) is given by

$$\mathscr{B}(z) = \left[\frac{z^d B^*X - A^*Y}{G^*C}\right]\left|\frac{C}{G}\right|^2 = \left[\frac{z^l G V^*}{G^*C}\right]\left|\frac{C}{G}\right|^2 = \frac{z^l V^* C^*}{G^*G^*},$$

where $V(z)$ is defined by equation (25) and $l = n - \deg V \geq 1$. Since $\bar{\mathscr{A}}$ has relative degree zero and is unstable by definition, we have $[\bar{\mathscr{A}}^{-1}\mathscr{B}]_{\text{st}} = 0$. The second term in equation (15) is

$$-Q^*[|D_C|^2 + \lambda|N_C|^2]|D_P|^2\phi_v = -Q^*\frac{|X|^2 + \lambda|Y|^2}{|C|^2}\frac{|A|^2}{|G|^2}\frac{|C|^2}{|A|^2} = -Q^*\frac{|X|^2 + \lambda|Y|^2}{GG^*}.$$

The choice (31) ensures that Q has relative degree at least 1. It now follows that the second term in equation (15) is unstable and has a zero at $z = 0$. This implies that $[\mathscr{A}^{-*}\mathscr{B}]_{st} = 0$, i.e. we have that $\bar{S}_{opt} = 0$. □

3.2. *The minimum variance disturbance rejection problem.* The minimum variance (MV) disturbance rejection control law is given by

$$u_t = -\frac{K(z)}{B_+(z)F(z)}\, y_t\,, \tag{32}$$

where $K(z)$ and $F(z)$ are polynomials that satisfy the Bezout equation

$$z^{d-1}C(z)B_-^*(z) = A(z)F(z) + B_-(z)K(z) \tag{33}$$

in which the polynomial $F(z)$ has the degree $d + \deg B_- - 1$ and $\deg k < n$. We refer the reader to Aström and Wittenmark (1990).

Proposition 3.2. Consider some ARMAX system (A, B, C) and its optimal MV controller C_0 with their respective fractional representations

$$N_P = \frac{B(z)}{C(z)}, \qquad D_P = \frac{A(z)}{C(z)},$$

$$N_C = \frac{K(z)}{z^{d-1}B_+(z)B_-^*(z)}, \qquad D_C = \frac{F(z)}{z^{d-1}B_-^*(z)}, \tag{34}$$

where F and K are determined using equation (33). Let

$$P_1 = (N_P - QD_C)(D_P + QN_C)^{-1} \tag{35}$$

be a perturbation of P_0 and consider the noise spectrum

$$\phi_{v_1} = \frac{|D_P|^2\phi_v}{|D_P + QN_C|^2} \tag{36}$$

with

$$Q = B_-Q_1\,, \tag{37}$$

where Q_1 is any element of **S** that has relative degree at least $d + \deg B_-$. Then any system (P_1, ϕ_{v_1}) with Q defined as in equation (37) has the same optimal MV controller, $C_0 = D_C^{-1}N_C$, as the unperturbed system (P_0, ϕ_v). Also, any ARMAX system (P_1, ϕ_{v_1}) that has optimal MV controller $C_0 = N_C D_C^{-1}$ can be expressed as equations (35) and (36) for some Q defined in equation (37).

Proof. Note that the fractional representations (24) fulfill the Bezout identity (1) and that each transfer function is stable and proper. Also, the choice (36) assures that assumption (14) is satisfied. The first term in equation (15) is given by

$$\mathscr{B}(z) = [N_P^* D_C - \lambda D_P^* N_C]\,|D_P|^2\phi_v = \frac{z^d B^*(z)}{C^*(z)}\frac{F(z)}{z^{d-1}B_-^*(z)}$$

$$= \frac{zB_+^*(z)F(z)}{C^*(z)}\cdot$$

We have $[\mathscr{A}^{-1}\mathscr{B}]_{st} = 0$. The second term in equation (15) is

$$-Q^*|D_C|^2|D_P|^2\phi_v = -Q^*\frac{|F|^2}{z^{d-1}B_-^*B_-}\cdot$$

The choice (27) ensures that Q has relative degree at least d. It now follows that the second term in equation (15) is unstable and has a zero at $z = 0$. This implies that $[\mathscr{A}^{-*}\mathscr{B}] = 0$, i.e. we have that $\bar{S}_{opt} = 0$. □

4. *Plant and corresponding control cost perturbations*

Let $P_0 = N_P D_P^{-1}$ be the real plant and $C_0 = N_C D_C^{-1}$ its optimal controller. Let us consider a model P_1 contained in the set $\mathscr{P}$ defined in equation (11), i.e. P_1 is a perturbation of P_0 also stabilized by C_0. If C_1 is the optimal controller for P_1, one can try to find out how this controller performs on the real plant P_0. One way to do that is to compare the optimal loop (P_0, C_0) and the achieved loop (P_0, C_1) by examining the respective costs. Note that we could equivalently compare the optimal loop (P_0, C_0) and the achieved loop (P_1, C_0). This amounts to find out how the original controller C_0 performs on the perturbed plant P_1.

The controller C_1 belongs to the set $\mathscr{C}$ defined in equation (12). The expression for the achieved cost is now easily derived:

$$J_{LQG}(P_0, C_1) = \frac{1}{2\pi}\int \mathrm{d}\omega \left\{\frac{|D_{C_1}(\bar{S}, Q)|^2 + \lambda|N_{C_1}(\bar{S}, Q)|^2}{|1 - Q\bar{S}|^2}\right\}|D_P|^2\phi_v\,.$$

Since C_1 is optimal for P_1, $\bar{S}$ is equal to $\bar{S}_{opt}$. If we assume that $\|Q\|$ is small, then we have shown in the previous section that (under reasonable conditions if the H_∞ norm is used) $\|\bar{S}_{opt}\|$ is small which implies in turn that $\|Q\bar{S}_{opt}\|$ is small. Again, $\|\cdot\|$ denotes either the H_∞ norm or the H_2 norm.

Dropping second-order terms, we obtain the following approximate expression for the achieved cost:

$$J_{LQC}(P_0, C_1) \simeq \frac{1}{2\pi}\int \mathrm{d}\omega\,\{|D_C + N_P\bar{S}|^2 + \lambda|N_C - D_P\bar{S}_{opt}|^2\}|D_P|^2\phi_v\,.$$

By expanding the integrand, the following approximate expression for the cost is obtained:

$$J_{LQG}(P_0, C_1) \simeq J(P_0, C_0) + \frac{1}{2\pi}\int \mathrm{d}\omega\,\{\mathscr{B}^*\bar{S}_{opt}(Q) + \mathscr{B}\bar{S}_{opt}^*(Q)\} + \|\mathscr{A}\,\bar{S}_{opt}(Q)\|_2^2\,, \tag{38}$$

where $\mathscr{A}$ and $\mathscr{B}$ were defined earlier for the pair (P_0, C_0). This shows that the increase in the control cost that results from applying the controller C_1, optimal for P_1, to the initial plant P_0 is small provided that the perturbation Q away from P_0 is small in the H_2 sense.

5. *Numerical illustration*

To illustrate the theoretical results of Section 3, let us take an ARMAX system described by equation (22) with $A(z) = z$, $B(z) = b$, $C(z) = z + h$. The optimal LQG controller (27) is given by

$$u_t = -\frac{bhz}{(b^2+\lambda)z + \lambda h}\, y_t \tag{39}$$

and the plant and controller factorizations are given by equation (28) with

$$G(z) = \sqrt{b^2+\lambda}\,z, \qquad X(z) = \frac{(b^2+\lambda)\;z + \lambda h}{\sqrt{b^2+\lambda}},$$

$$Y(z) = \frac{bhz}{\sqrt{b^2+\lambda}}\cdot$$

Take $Q = \sqrt{b^2+\lambda}\;Q_1$ where $Q \in$ **S** has relative degree 2. Then, for fixed λ, any ARMAX system (P_1, ϕ_{v_1}) that has optimal LQG controller (39) can be expressed as

$$P_1(Q) = \frac{b/z - Q\,[(b^2+\lambda)z + \lambda h]/(z+h)}{1 + Q\,bhz/(z+h)},$$

$$\phi_{v_1}(Q) = \frac{|(z+h)/z|^2}{|1 + Q\,bhz/(z+h)|^2}\cdot$$

6. *Conclusions*

In this paper, we have used a Youla-parametrization-based approach to compute an infinite horizon LQG controller from a stabilizing controller using coprime factorizations in order to show that, under reasonable conditions if the size of Q is measured using the H_∞ norm, a small coprime factor perturbation away from a given plant will produce a small coprime factor perturbation away from the optimal controller corresponding to that plant. Also, the increase in the LQG cost that results from applying the perturbed controller to the real plant will be small as long as the plant/model perturbation is small in the H_2 sense. As a by-product, we have characterized the set of all plants that have the same optimal LQG controller.

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