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Discrete-Time Fake Riccati Equations for Kalman Filtering and Receding-Horizon Control

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I. INTRODUCTION

The DRE (Difference Riccati Equation) is a basic ingredient of LQ (Linear Quadratic) optimal control. In particular, the steady-state solutions of the DRE play a key role in the design of stabilizing control laws by means of *infinite-horizon* LQ optimization. This motivated a vast amount of research throughout the last decades in order to clarify the stabilizing properties of the solutions of the ARE (Algebraic Riccati Equation).

Conversely, it has always been difficult to prove stability for LQ strategies based on *finite-horizon* optimization. This issue is far from being purely speculative, because *predictive control* [1], which is widely applied in the industrial process control community, is essentially based on *receding-horizon* LQ control, in which the controller results from a finite-horizon optimization. Before the mid eighties, the only stability results were due to Kwon and Pearson [2] and were restricted to a special type of receding-horizon controller with zero-state terminal constraints.

A main difference between the infinite-horizon controller and the receding-horizon one is that the former is associated with the solution of an algebraic Riccati equation whereas the latter is associated with the solution of a difference Riccati equation. Now, contrary to what happens for the ARE, little or nothing was known concerning the stabilizing properties of the DRE. A major breakthrough was made in 1985, when it was recognized that closed-loop stability of RH control schemes could be proven by suitably manipulating the DRE in order to convert it into an ARE, provided that certain monotonicity properties were fulfilled [3]. Due to its origin, this new Riccati equation was named *Fake Algebraic Riccati Equation* or FARE (see [4] and references therein). The technique of converting a DRE into a FARE in order to produce monotonicity results can clearly be attributed to M.A. Poubelle who invented the term Fake Algebraic Riccati Technique: see [5]. It is interesting to note that fake Riccati techniques were originally developed in order to study the dual problem, namely under what conditions an asymptotically stable time-invariant filter can be obtained by freezing the gain of the (time-varying) Kalman filter [3].

In the subsequent years, the fake Riccati techniques were applied to the analysis of existing predictive control schemes [6]. The key ingredient of predictive control schemes is that predictions of the output *over a finite horizon* are taken into account in the optimization problem. Several points were raised against predictive control in view of its inability to guarantee closed-loop stability and it was even suggested to abandon predictive control in favour of infinite-horizon LQ optimization.

There was also a reaction by the predictive control community with the development of novel predictive controllers incorporating some ideas of the fake Riccati theory in order to ensure closed-loop stability. In particular, CRHPC (Constrained Receding Horizon Control) [7], [8], SIORHC (Stabilizing I/O Receding Horizon Control) [9], and the controller of Rawlings and Muske [10] all owe their stability properties to the monotonic behaviour of the associated DRE.

The latest developments include the extension of the fake Riccati techniques to the case of periodic receding-horizon control, that is the stabilization of time-invariant plants by means of periodic feedback laws computed through the optimization of a finite-horizon cost functional [11], [12], [8].

The aim of the present chapter is to offer a comprehensive review of the theory of fake Riccati equations. In Section II the problem of establishing closed-loop stability of RH controllers is introduced. In Section III the theory of the fake (algebraic and periodic) Riccati equation is presented. The practical implementation of RH stabilizing controllers is addressed in Section IV. Section V is devoted to a concise presentation of the dual problem, that is the stability of the so-called frozen Kalman filter. Some concluding remarks (Section VI) end the paper.

II. PROBLEM STATEMENT

Consider the linear time-invariant discrete-time system

$$x(t+1) = Ax(t) + Bu(t) \quad , \tag{1}$$

where $x(t) \in \Re^n$ is the state and $u(t) \in \Re^m$ is the input. Throughout the paper it will be assumed that (A,B) is a stabilizable pair. Associated with (1) we introduce the LQ cost function

$$J(x(t),N) = x(t+N)'P_0x(t+N) + \sum_{j=0}^{N-1} \{x(t+j)'Qx(t+j)+u(t+j)'Ru(t+j)\} \, , \tag{2}$$

where $P_0 \geq 0$, $Q \geq 0$ and $R > 0$. It is well known that the problem of minimizing $J(x(t),N)$ with respect to the input sequence $u(j)$, $j \geq t$, is solved by the state-feedback control law

$$u(t+j) = K(N-1-j)x(t+j) \tag{3.a}$$

$$K(j) = -[B'P(j)B+R]^{-1}B'P(j)A \, , \; j = 0, \, 1, \, \ldots, \, N-1, \tag{3.b}$$

where $P(\cdot)$ is the solution of the DRE (Difference Riccati Equation)

$$P(t+1) = A'P(t)A + Q - A'BP(t)[B'P(t)B + R]^{-1}B'P(t)A \tag{4}$$

with initial condition $P(0) = P_0$.

A classical way to devise a stabilizing control law for system (1) is through the solution of an *infinite-horizon* LQ problem, that is by minimizing $J_\infty(x(t)) \doteq lim_{N\rightarrow\infty}J(x(t),N)$. More precisely, the (candidate) stabilizing feedback K_∞ is the gain corresponding to the asymptotic value of $P(t)$ as $t \rightarrow \infty$. Note that, under the additional assumption that (A,Q) is detectable, such a gain is well defined because, for all $P_0 \geq 0$, $P(t)$ converges to the unique nonnegative definite solution P_∞ of the ARE (Algebraic Riccati Equation)

$$P = A'PA + Q - A'BP[B'PB + R]^{-1}B'PA \qquad (5)$$

The stabilizing property of the gain $K_\infty = -[B'P_\infty B + R]^{-1}B'P_\infty A$ is stated in the following (well-known) theorem.

Theorem 1: Let (A,B) be stabilizable and (A,Q) detectable. Then, the (unique) nonnegative definite solution of (5) is stabilizing, i.e. $A+BK_\infty$ is asymptotically stable. ■

We mention that the condition on the detectability of the pair (A,Q) can be replaced by the slightly weaker condition that (A,Q) has no unobservable modes on the unit circle: see [13].

In practice, computing P_∞ from the recursion (4) is rather inefficient due to the possible slow convergence of the solution of the DRE, so that the direct solution of the ARE (5) is recommended. However, the numerical solution of such an ARE is computationally demanding even when efficient algorithms are employed [14].

An example in which computational requirements may be critical is provided by real-time adaptive control where the optimal feedback law has to be recomputed whenever a new estimate of the plant model becomes available. This motivates the search of alternative approaches for the design of more easily computable (and hopefully stabilizing) control laws.

In particular, we will consider the RH (Receding Horizon) control scheme which is based on a smart "abuse" of the finite-horizon cost functional (2). At each time instant t, a myopic point of view is adopted: the performance of the system is optimized only until N steps ahead, that is $J(x(t),N)$ is minimized, and the corresponding optimal control is applied at time t. At time $t+1$, instead of proceeding with the control strategy designed at the previous step, the N-step optimization window is moved forward by one step and a new finite-horizon optimization problem over N steps concerning $J(x(t+1),N)$ is considered. The same "moving horizon" strategy is then repeated at any subsequent time step. It is easy to see that for a linear time invariant system this RH scheme is just equivalent to using the constant control law $u(t) = K(N-1)x(t)$.

Receding-horizon control, suitably restated in input-output form, has enjoyed a significant success under the name of predictive control [1], and is widely applied in many industrial process control problems. A major advantage of RH (and predictive) control is the ease of computation: in a state-space setting, $K(N-1)$ is immediately obtained by simply iterating N steps of the recursion (4). When restated in an input-output context, the RH algorithm is just as efficient because only linear algebraic equations are involved [1]. Further, the control criterion is finite dimensional, which permits simple extension to nonlinear and constrained problems.

At this point, we should also mention the main drawback of RH control: even when (A,Q) is detectable, there is no guarantee that $A+BK(N-1)$ has all its eigenvalues strictly inside the unit circle. Now, under detectability of (A,Q), it is known that the solution $P(t)$ of the DRE (4) converges to the stabilizing solution P_- of the ARE (5). Hence, there will always exist a (sufficiently large) value of N such that $A+BK(N-1)$ is stable. Unfortunately, this result is only asymptotic and does not ensure that, for a given value of N, the RH controller will be stabilizing. Conversely, the central issue of this chapter will be the review of some recent results that have revolutionized the stability analysis of RH controllers by demonstrating that closed-loop stability can be established in a non-asymptotic way.

Before proceeding, we present a second control strategy based on a receding horizon philosophy. In this scheme, named PRH (Periodic Receding Horizon) control, the first T values $(T \leq N)$ of the control sequence minimizing $J(x(t),N)$ are applied over the interval $[t,t+T-1]$. At time $t+T$, a new control sequence minimizing $J(x(t+T),N)$ is computed and the first T values applied over $[t+T,t+2T-1]$. Then, the procedure is iterated over the subsequent time intervals. It is easily seen that this PRH control strategy amounts to using the periodic control law

$$u(t+j) = \tilde{K}_N(j)x(t) \qquad (6.a)$$

$$\tilde{K}_N(j) = K(N-1-j) \quad , \; j = 0, \; 1, \ldots, T-1 \qquad (6.b)$$

$$\tilde{K}_N(t+T) = \tilde{K}_N(t) \quad , \; \forall t \; . \qquad (6.c)$$

Observe that, given the solution $P(t)$ of the DRE (4) with $0 \leq t < N$, then $\tilde{K}_N(\cdot)$ is based on the last T values of $P(\cdot)$. As discussed later in this chapter (Section IV, Example 4), the PRH strategy may offer some advantages over the standard RH one for the achievement of closed-loop stability.

III. FAKE RICCATI EQUATIONS

A. THE FAKE ALGEBRAIC RICCATI EQUATION

The main idea in order to establish the closed-loop stability of the RH controller is to exploit the infinite-horizon stability result reported in Theorem 1. To this end, recall that, in view of (3.b), $K(N-1)$ is the optimal gain associated with $P(N-1)$. Now, by defining

$$Q_{N-1} = Q + P(N-1) - P(N) \qquad (7)$$

(4) can be rewritten as

$$P(N-1) = A'P(N-1)A + Q_{N-1} - A'BP(N-1)[B'P(N-1)B+R]^{-1}B'P(N-1)A. \qquad (8)$$

Hence, $P(N-1)$ can be seen as the solution of the ARE (8), and $K(N-1)$ as the associated gain. The only difference between (5) and (8) is that Q has been replaced by Q_{N-1}. Due to this "adjustment" of the state-weighting matrix, (8) is named *Fake Algebraic Riccati Equation* (FARE) [3].

Now, provided that $Q_{N-1} \geq 0$ and (A,Q_{N-1}) is detectable, the stability of $A+BK(N-1)$ is guaranteed by the following result.

Theorem 2: Let $P(\cdot)$ be a nonnegative solution of the DRE (4), let Q_{N-1} be defined by (7), and assume that: (i) (A,B) is stabilizable, (ii) $Q_{N-1} \geq 0$, (iii) (A,Q_{N-1}) is detectable. Then, the RH closed-loop matrix $A+BK(N-1)$ is asymptotically stable.

Proof: Under the stated assumptions $P(N-1)$ is a nonnegative solution of the FARE (8), where (A,B) is stabilizable and (A,Q_{N-1}) is detectable. In view of Theorem 1, such an ARE admits a unique nonnegative definite solution which is also stabilizing. Hence, $A+BK(N-1)$ is asymptotically stable. ■

At this point, our main concern is to derive guidelines for the choice of the design parameters P_0 and N in order to guarantee that $Q_{N-1} \geq 0$ and (A,Q_{N-1}) is detectable. For this purpose, it is useful to recall a couple of properties concerning the solutions of the DRE.

Lemma 1 [15]: Let $P_1(\cdot)$ and $P_2(\cdot)$ be the solutions of two DRE's of the type (4) with the same A and B matrices, but possibly different Q matrices, say Q_1 and Q_2, and with initial conditions $P_1(0) = P_1 \geq 0$ and $P_2(0) = P_2 \geq 0$, respectively. Then, the matrix $\tilde{P}(t) = P_2(t) - P_1(t)$ satisfies the following Riccati equation

$$\tilde{P}(t+1) = \tilde{A}_1(t)'\tilde{P}(t)\tilde{A}_1(t) + \tilde{Q} - \tilde{A}_1(t)'\tilde{P}(t)B[B'\tilde{P}(t)B + \tilde{R}(t)]^{-1}B'\tilde{P}(t)\tilde{A}_1(t) \quad (9)$$

where

$$\tilde{A}_1(t) = A - B[B'P_1(t)B + R]^{-1}B'P_1(t)A$$
$$\tilde{R}(t) = B'P_1(t)B + R$$
$$\tilde{Q} = Q_2 - Q_1.$$

■

Theorem 3 [3]: Let $P(\cdot)$ be the solution of the DRE (4). If $P(t+1) \leq P(t)$ for some t, then $P(t+k+1) \leq P(t+k)$, $\forall k \geq 0$.

Proof: Let $P_1(t) = P(t+1)$, $P_2(t) = P(t)$, and $\tilde{P}(t) = P(t)-P(t+1)$. Then, by Lemma 1, $\tilde{P}(\cdot)$ satisfies (9). Since $\tilde{P}(\cdot)$ satisfies a difference Riccati equation, it is well known that $\tilde{P}(t) \geq 0$ implies $\tilde{P}(t+k) \geq 0$, $\forall k \geq 0$. ■

The above result provides a sufficient condition for ensuring the nonnegative definiteness of Q_{N-1} by means of a proper choice of P_0. In fact, if P_0 is such that $P(1) \leq P_0$, then $P(t+1) \leq P(t)$, $t \geq 0$, and $Q_{N-1} = Q+P(N-1)-P(N) \geq 0$, $\forall N > 0$.

At this stage, the detectability of the pair (A,Q_{N-1}) has to be taken into account. In general, it is very difficult to analyze how the choice of P_0 and N affects the detectability of such a pair. However, if we assume that the solution $P(\cdot)$ of (4) is monotonic nonincreasing, the issue is substantially simplified.

Lemma 2: Assume that (A,Q) is detectable and $P(1) \leq P(0)$. Then, (A,Q_{N-1}) is detectable $\forall N \geq 0$.

Proof: By Theorem 3, the inequality $P(1) \leq P(0)$ implies $P(N) \leq P(N-1)$. Since $Q_{N-1} = Q + P(N-1) - P(N)$, the thesis immediately follows. ■

In most cases the pair (A,Q) will indeed be detectable. In particular, this happens when the plant is open-loop stable or when, as in predictive control, $Q = C'C$ with (A,C) completely observable. Then, provided that P_0 guarantees nonincreasing monotonicity, the fake Riccati approach can be applied irrespective of the value of the design parameter N.

In conclusion, we can give the following sufficient condition for the closed-loop stability of the RH controller.

Theorem 4: Let $P(\cdot)$ be the solution of the DRE (4) and assume that: (i) $P(1) \leq P(0)$; (ii) (A,Q) is detectable. Then, $A+BK(N-1)$ is asymptotically stable, $\forall N > 0$.

Proof: In view of Lemma 2, $Q_{N-1} \geq 0$ and (A,Q_{N-1}) is detectable. Then, the thesis directly follows from Theorem 2. ■

For completeness, we mention that the closed-loop stability of the RH controller can be ensured also under weaker assumptions as stated below.

Theorem 5 [3]: Let $P(\cdot)$ be the solution of the DRE (4) with (A,B) stabilizable and assume that: (i) $P(1) \leq P(0)$, (ii) $(A,Q+P(0)-P(1))$ is detectable. Then, $A+BK(N-1)$ is asymptotically stable for all finite $N > 0$.

The last result is interesting because it shows that fake Riccati techniques can be applied also to RH controllers based on cost functions of the "minimum energy" type, i.e. with $Q = 0$.

B. THE FAKE PERIODIC RICCATI EQUATION

In the previous section, we have studied the closed-loop stability of RH control by associating its state-feedback gain with a suitably defined (fake) algebraic Riccati equation. In a similar way, the analysis of the stability properties of periodic RH control will be performed by associating the periodic gain with a suitably defined (fake) periodic Riccati equation.

To this end, let $P(\cdot)$ be the solution of the DRE (4) with initial condition $P(0) = P_0$, and define the periodic matrices

$$\begin{aligned}
\tilde{P}_N(t) &= P(t) & &, \ N-T \leq t < N \\
\tilde{P}_N(t+T) &= \tilde{P}_N(t) & &, \ \forall t \geq N-T \\
\tilde{Q}_N(t) &= \begin{cases} Q & , \ t \neq kT+N-1 \\ Q+P(N-T)-P(N) & , \ t = kT+N-1 \end{cases} & &, k = 0, 1, \ldots
\end{aligned}$$

In view of the above definitions, it is easy to verify that $\bar{P}_N(\cdot)$ is the solution of the following PRE (Periodic Riccati Equation):

$$\bar{P}_N(t+1) = A'\bar{P}_N(t)A + \bar{Q}_N(t) - A'B\bar{P}_N(t)[B'\bar{P}_N(t)B+R]^{-1}B'\bar{P}_N(t)A\ , \tag{10}$$

with initial condition $\bar{P}_N(N\text{-}T) = P(N\text{-}T)$, where $P(N\text{-}T)$ is the solution of (4) with initial condition $P(0) = P_0$. Indeed, for $t = N\text{-}T$ to $N\text{-}2$, equation (10) is just equivalent to (4), so that $\bar{P}_N(t) = P(t)\ ,\ N\text{-}T \leq t < N$. The periodicity of the solution of (10), i.e. $\bar{P}_N(N) = \bar{P}_N(N\text{-}T)$, is then guaranteed by the way $\bar{Q}_N(t)$ has been defined.

The only difference between (4) and (10) lies in the state weighting matrices Q and $\bar{Q}_N(\cdot)$. Due to this modification, (10) is named *Fake Periodic Riccati Equation* (FPRE).

Note that the PRH feedback $\bar{K}_N(\cdot)$ is just the gain associated with $\bar{P}_N(\cdot)$. In this way, the stability analysis of the PRH control scheme is reduced to assessing whether $\bar{P}_N(\cdot)$ is a stabilizing solution of (10), i.e. whether the periodic matrix $A\text{-}B[B'\bar{P}(\cdot)B+R]^{-1}B'\bar{P}(\cdot)A$ is asymptotically stable. In this respect, it useful to recall that, given a periodic matrix $F(t) = F(t+T)$, where T is a positive integer, the periodic system

$$x(t+1) = F(t)x(t)$$

is asymptotically stable if and only if the transition matrix over one period $\Phi_F(T,0) = F(T\text{-}1)F(T\text{-}2)\ldots F(1)F(0)$ has all its eigenvalues strictly inside the unit circle, see [16]. By means of counterexamples, it can be shown that having all the eigenvalues of $F(t)$ strictly inside the unit circle, $\forall t \in [0,T\text{-}1]$, is *neither necessary nor sufficient* for the asymptotic stability of the periodic matrix $F(\cdot)$.

The subsequent analysis will mainly rely on the following theorem concerning periodic Riccati equations.

Theorem 6 [17]: Consider the PRE (Periodic Riccati Equation):

$$P(t+1) = A'P(t)A + Q - A'P(t)B[B'P(t)B + R]^{-1}B'P(t)A\ ,$$

where $A(\cdot)$, $B(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are periodically varying matrices of period T, and $Q(t) \geq 0$, $R(t) > 0$, $\forall t$. If (A,B) is stabilizable and (A,Q) is detectable, then: (i) the PRE always admits a unique nonnegative definite T-periodic solution $\bar{P}(\cdot)$; (ii) the periodic closed-loop matrix $A\text{-}B[B'\bar{P}(\cdot)B+R]^{-1}B'\bar{P}(\cdot)A$ is asymptotically stable.

In Theorem 6, reference is made to the detectability of the periodic pair (A,Q). The definition of uniform detectability for discrete-time linear time-varying systems has been introduced in [18]. In parallel, several equivalent characterizations of detectability for periodic discrete-time systems have been developed during the eighties, see e.g. [16] and references quoted there. As shown in [17], these detectability notions, besides being equivalent to each other, are also equivalent to the uniform detectability notion. Below, the so-called modal characterization of detectability is stated.

Definition 1: The T-periodic pair (A,Q) is detectable if and only if there does not exist λ, $|\lambda| \geq 1$, and $x \in \Re^n$, $x \neq 0$, such that

$$\Phi_A(t+T,t)x = \lambda x$$
$$Q(t+k)\Phi_A(t+k,t)x = 0 \quad , \quad 0 \leq k < T \, ,$$

where $\Phi_A(t+k,t)$ denotes the transition matrix of $A(\cdot)$ over the interval $[t+k,t]$. ■

It is easy to verify that, when the pair (A,Q) is time-invariant, the above definition reduces to the usual detectability notion.

Next, we show how Theorem 6 can be used to establish the closed-loop stability of the PRH controller.

Theorem 7: Let $P(\cdot)$ be the solution of the DRE (4) with $R > 0$ and assume that: (i) (A,B) is stabilizable, (ii) $Q+P(N-T)-P(N) \geq 0$, (iii) the periodic pair $(A,\tilde{Q}_N(\cdot))$ is detectable. Then, the PRH closed-loop matrix $A+B\tilde{K}_N(\cdot)$ is asymptotically stable.

Proof: Under the stated assumptions, $\tilde{P}_N(\cdot)$ is a nonnegative definite solution of the FPRE (10). In view of Theorem 6, such an FPRE admits a unique nonnegative periodic solution which, in addition, is stabilizing. ■

In the following, our primary concern will be to guarantee the fulfilment of assumptions (ii) and (iii) of Theorem 6. In this respect, an interesting tool is offered by the so-called *cyclomonotonicity* property of the DRE.

Definition 2: The symmetric matrix function $P(t)$, $t \geq \tau$, is said to be a *cyclomonotonic* nonincreasing sequence of period T, if $P(\tau+kT+T) \leq P(\tau+kT)$, $k \geq 0$. If, in addition,

$P(t+T) \leq P(t)$, $\forall t \geq \tau$, then $P(\cdot)$ is said to be a *strongly cyclomonotonic* nonincreasing sequence.

Theorem 8 [11]: Let $P(\cdot)$ be the solution of the DRE (4). If $P(\tau+T) \leq P(\tau)$ for some τ and T, then $P(t+T) \leq P(t)$, $\forall t \geq \tau$, i.e. $P(\cdot)$ is a strongly cyclomonotonic nonincreasing sequence.

Proof: The proof hinges on Lemma 1 and is substantially analogous to the proof of the monotonicity property given in Theorem 3. ■

Now, in view of the above result, if P_0 is such that $P(T) \leq P_0$, it follows that $P(t+T) \leq P(t)$, $\forall t \geq 0$, and we are free to choose any $N \geq T$ in the design of the PRH controller without fear of losing the nonnegative definiteness of $\tilde{Q}_N(\cdot)$.

For what concerns the detectability of the periodic pair $(A,\tilde{Q}_N(\cdot))$, we have the following results. The first one is rather straightforward and its proof is therefore omitted.

Lemma 3: Let (A,Q) be a detectable time-invariant pair with $Q = Q' \geq 0$, and consider a symmetric nonnegative definite T-periodic matrix $D(t) = D(t)' \geq 0,\ \forall t$. Then, the periodic pair $(A,Q+D(\cdot))$ is detectable.

Lemma 4: Assume that (A,Q) is detectable and $P(T) \leq P_0$. Then, the periodic pair $(A,\tilde{Q}_N(\cdot))$ is detectable $\forall N \geq 0$.

Proof: In view of Theorem 8, $P(T) \leq P(0)$ implies $P(t+T) \leq P(t)$, $\forall t \geq 0$. Then, recalling the definition of $\tilde{Q}_N(\cdot)$, it follows that $\tilde{Q}_N(t) \geq Q$, $\forall t \geq 0$, so that Lemma 3 can be applied. ■

As already pointed out for the standard RH scheme, in most cases the pair (A,Q) is detectable and one need only bother about the cyclomonotonicity of $P(\cdot)$. Then, closed-loop stability of the PRH controller is guaranteed by the following result.

Theorem 9: Consider the DRE (4) with (A,B) stabilizable and assume that (A,Q) is detectable and $P(T) \leq P_0$. Then, the PRH closed-loop matrix $A+B\tilde{K}_N(\cdot)$ is asymptotically stable.

Proof: Immediate in view of Theorem 8, Lemma 4 and Theorem 7. ■

In analogy with the previous subsection, there is also a more general result that covers the situation when (A,Q) is not detectable.

Theorem 10 [11]: Consider the DRE (4) with (A,B) stabilizable and let $\breve{K}_N(\cdot)$ be defined as in (6.b)-(6.c). Let $P(\cdot)$ be the solution of the DRE (4) and assume that: (i) $P(T) \leq P_0$, (ii) the pair $(A,\breve{Q}_T(\cdot))$ is detectable. Then, letting $N = kT$, the PRH closed-loop matrix $A+B\breve{K}_N(\cdot)$ is asymptotically stable, $\forall k \geq 1$.

IV. STRATEGIES FOR ACHIEVING MONOTONICITY

A. HOW TO ACHIEVE MONOTONICITY

In the previous section it has been shown that, under standard assumptions, the nonincreasing monotonicity of the solution of the DRE is sufficient to ensure the closed-loop stability of the RH control scheme. In this subsection, we address the problem of properly choosing the initial condition P_0 so as to obtain a monotonic behaviour. In this respect, it is useful to recall the following lemma.

Lemma 5 [19]: Suppose that (A,B) is stabilizable and let P^+ be the maximal symmetric solution of the ARE (5). Furthermore, assume that the following ARE has a stabilizing solution P_1:

$$P = A'PA + Q_1 - A'PB[B'PB + R]^{-1}B'PA \; .$$

If $Q_1 \geq Q$, then $P_1 \geq P^+$. ■

Now, consider the FARE (8) and assume that $P(N) \leq P(N\text{-}1)$ and $A+BK(N\text{-}1)$ is stable. Since $Q_{N-1} \geq Q$, it follows that $P(N\text{-}1) \geq P^+$. Therefore, in order to achieve stability through monotonicity, the initial condition P_0 of the DRE (4) must be larger than the maximal solution of the corresponding ARE (5). Unfortunately, this necessary condition for monotonicity is far from being sufficient.

Fallacious Conjecture 1: If $P_0 > P^+$, then $P(1) \leq P(0)$.

Counterexample 1 [3]: Letting $R = 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 10 \end{bmatrix},$$

it turns out that

$$P^+ = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1+\sqrt{3} \end{bmatrix} < P_0 .$$

However, it is easy to see that

$$P(1) = \begin{bmatrix} 1.9 & 0 \\ 0 & 101 \end{bmatrix} \not\leq P(0) .$$

■

As shown below, monotonicity of $P(\cdot)$ is not guaranteed even under the additional assumption that both P^+ and P_0 are stabilizing.

Fallacious Conjecture 2: If $P_0 > P^+$, and both P_0 and P^+ are stabilizing, then $P(1) \leq P(0)$.

Counterexample 2 [20]: Let $R = 1$ and

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Taking

$$P_0 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 12 \end{bmatrix} > P^+ = \begin{bmatrix} \frac{4}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 0 \\ \frac{4}{3} & 0 & \frac{16}{3}+\sqrt{21} \end{bmatrix},$$

it turns out that both P_0 and P^+ are stabilizing. However,

$$P(1) = \begin{bmatrix} 21.6094 & 10 & -2.25 \\ 10 & 5 & 0 \\ -2.25 & 0 & 13 \end{bmatrix} \not\leq P_0 .$$

Moreover, it can be seen that $P(1)$ is not stabilizing. ■

Under stabilizability conditions on the pair (A,B) it is known that, if $P_0 \geq P^+$, the solution $P(\cdot)$ of the DRE (4) asymptotically converges towards P^+ [13]. Therefore, a promising strategy is to start with a "very large" P_0, hoping that the convergence property forces $P(\cdot)$ "downwards" in a monotonic manner. In practice, one may take $P_0 = \alpha I$ where α is a sufficiently large positive scalar. Again, this attempt is frustrated by a counterexample.

Fallacious Conjecture 3: Let $P_0 = \alpha I$. Then, there always exists α such that $P(1) \leq P(0)$.

Counterexample 3: Let $R = 1$ and

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, P_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \alpha > 0.$$

It is easy to verify that $P(1) \not\leq P_0$, $\forall \alpha > 0$. ■

Having shown that the most immediate (and naive) strategies fail to achieve monotonicity, it is time to present two effective approaches: the first one traces back to the works of Kwon and Pearson [2], whereas the second one is more recent [10]. Rather interestingly, in both cases the proof of monotonicity can be based on dynamic programming arguments.

The first scheme is based on constrained optimization. Consider system (1) with initial state $x(0) = x_0$, and the finite-horizon cost function

$$J(x_0,L) = \sum_{j=0}^{L-1} \{x(j)'Qx(j) + u(j)'Ru(j)\} \tag{11.a}$$

subject to the zero-state terminal constraint

$$x(t+L) = 0 . \tag{11.b}$$

Assuming that (A,B) is controllable, a sufficient condition for the solvability of the constrained minimization problem (11) is that $L \geq n$. Then, letting $J^o(x_0,L)$ denote the optimal value of $J(x_0,L)$ subject to (11.b), it is easy to see that $J^o(x_0,L)$ is a quadratic function of x_0 [21]. Therefore, there always exists a nonnegative definite matrix Π_L such that $x_0'\Pi_L x_0 = J^o(x_0,L)$, $\forall x_0$. The following result shows that the initialization $P_0 = \Pi_L$ guarantees monotonicity.

Theorem 11: Assume that (A,B) is controllable and let the initial condition of the DRE (4) be $P_0 = \Pi_L$, with $L \geq n$. Then, $P(1) \leq P_0$.

Proof: By the basic principle of dynamic programming, $\forall x_0$,

$$\begin{aligned} J^o(x_0,L+1) &= \min_{u(0)} \{J^o(x(1),L) + x_0'Qx_0 + u(0)'Ru(0)\} \\ &= \min_{u(0)} \{x(1)'\Pi_L x(1) + x_0'Qx_0 + u(0)'Ru(0)\} = x_0'P(1)x_0 \ . \end{aligned}$$

A moment's reflection shows that $J^o(x_0,L+1) \leq J^o(x_0,L)$. Therefore, $P(1) \leq P_0$ and the thesis follows.

Corollary 1: Assume that (A,B) is controllable, (A,Q) is detectable, and let the initial condition of the DRE (4) be $P_0 = \Pi_L$, with $L \geq n$. Then, $A+BK(N-1)$ is stabilizing $\forall N > 0$. ■

The last result suggests a stabilizing strategy but leaves open the problem of calculating Π_L. If we assume that A is nonsingular, this can be done through a rather insightful scheme. Indeed, it can be shown [2], [6] that $\Pi_L = S(L)$ where $S(\cdot)$ is the solution of the DRE

$$\begin{aligned} S(t+1) &= A^{-1}S(t)A'^{-1} + BR^{-1}B' \\ &\quad - A^{-1}S(t)A'^{-1}Q^{1/2}[Q^{1/2}A^{-1}S(t)A'^{-1}Q^{1/2} + I]^{-1}Q^{1/2}A^{-1}S(t)A'^{-1} \ , \end{aligned} \tag{12}$$

with initial condition $S(1) = BR^{-1}B'$. Now, the DRE (12) is strictly related to the DRE

(4). In fact, if P_0 is nonsingular and $S(1) = P_0^{-1}+BR^{-1}B'$, then $S(t+1) = P(t)^{-1}+BR^{-1}B'$, $t \geq 0$, where $P(t)$ is the solution of the DRE (4) with initial condition $P(0) = P_0$ [6]. Therefore, in some sense, Π_L can be regarded as the solution at time L of the DRE (4) with $P_0^{-1} = 0$ (an "infinitely large" but structured initial condition!).

The nonreversible case ($det\, A = 0$) is more involved and is not discussed here. The interested reader can obtain the recursive formulas for the computation of Π_L by dualizing the formulas for Kalman filtering with zero initial information given in [22].

It is worth pointing out that the constrained RH controller of Kwon and Pearson, is the basis of some recent predictive control schemes with guaranteed stability [7], [8], [9]. In the context of predictive control, the constrained problem (11) is given an input-output formulation by imposing zero terminal constraints on a certain number of future inputs and outputs. The explicit computation of Π_L is usually skipped, because the optimal input can be directly computed through the solution of a quadratic optimization problem with equality constraints.

The second approach for achieving monotonicity is based on the following infinite-horizon cost functional

$$J_I(x_0,L) = \sum_{j=0}^{\infty} x(j)'Qx(j) + \sum_{j=0}^{L-1} u(j)'Ru(j)\,, \tag{13.a}$$

subject to the constraint

$$u(j) = 0\,,\ j \geq L\,. \tag{13.b}$$

Let r denote the number of eigenvalues of A that do not lie inside the open unit disk. Then, assuming that (A,B) is stabilizable, a sufficient condition for the solvability of the optimization problem (13) is that $L \geq r$. Letting $J_I{}^o(x_0,L)$ denote the optimal value of $J_I(x_0,L)$, it is easy to see that $J_I{}^o(x_0,L)$ is a quadratic function of x_0. Therefore, there always exists a nonnegative definite matrix Σ_L such that $x_0'\Sigma_L x_0 = J_I{}^o(x_0,L)$, $\forall x_0$.

Theorem 12: Assume that (A,B) is stabilizable and let the initial condition of the DRE (4) be $P_0 = \Sigma_L$, with $L \geq r$. Then, $P(1) \leq P_0$.

Proof: The proof is completely analogous to the proof of Theorem 11.

Corollary 2: Assume that (A,B) is stabilizable, (A,Q) is detectable, and let the initial condition of the DRE (4) be $P_0 = \Sigma_L$ with $L \geq r$. Then, $A+BK(N-1)$ is stabilizing $\forall N > 0$. ■

As for the numerical computation of Σ_L, this is an easy task when A is stable. In such a case $r = 0$, so that we can take $L = 0$. Then,

$$\Sigma_L = \sum_{j=0}^{\infty} A'^j Q A^j \ ,$$

which is more efficiently computed as the solution of the Lyapunov equation

$$\Sigma_L = A'\Sigma_L A + Q.$$

The major merits of the stabilizing controller of Rawlings and Muske are twofold. First, it does not require controllability and applies to the more general class of stabilizable plants. Second, it does not impose unnecessary deadbeat constraints on the modes of the system that are already stable. Hence, compared to the controller of Kwon and Pearson, it should lead to a smaller control effort. As a drawback, however, the computational machinery seems more involved, due to the infinite length of the optimization horizon. As a final remark, we observe that for totally unstable plants the two approaches, in spite of the different formulation, are completely equivalent (in that case a necessary condition for $J_I(x_0,L)$ to be finite is that $x(L) = 0$).

B. HOW TO ACHIEVE CYCLOMONOTONICITY

In the previous subsection it has been shown through some counterexamples that the condition $P_0 \geq P^+$, though necessary, is not sufficient to ensure nonincreasing monotonicity of the solution of the DRE (4). In particular the simple choice $P_0 = \alpha I$ with α sufficiently large, fails to achieve this scope (Counterexample 3). This motivated the introduction of two schemes with guaranteed monotonicity involving either constrained or infinite-horizon optimization. Conversely, as shown below, cyclomonotonicity enjoys the potential advantage of being attainable by means of simpler schemes.

Lemma 6: Assume that (A,B) is stabilizable, $P_0 > 0$, and $P_0 > P^+$, where P^+ denotes the maximal solution of the ARE (5). Then, there exists an integer $T \geq 0$ such that $P(t+T) \leq P(t)$, $t \geq 0$.

Proof: Under the stated assumptions, the solution $P(\cdot)$ of the DRE (4) asymptotically converges to the strong solution P^+ of the ARE (5) [13, Theorem 4.2]. Now, letting $\Delta(t) = P(t) - P^+$, then for all $x \in \Re^n$

$$x'[P_0 - P(t)]x = x'[P_0 - P^+]x - x'\Delta(t)x \geq \lambda_{min} \|x\|_2^2 - \|\Delta(t)\|_2 \|x\|_2^2 \ ,$$

where $\lambda_{min} > 0$ is the minimum eigenvalue of $P_0 - P^+$. Since

$$\lim_{t \to \infty} \Delta(t) = 0 \ ,$$

there exists a time point T such that $P_0 - P(T) > 0$, i.e. $P(T) < P_0$. Indeed, by the very definition of limit, for any $\varepsilon > 0$, there exists a finite $T = T(\varepsilon)$ such that $t \geq T$ implies $\|\Delta(t)\|_2 < \varepsilon$. Then, letting $\varepsilon = \lambda_{min}$, we have $P_0 - P(T(\varepsilon)) > 0$, and the result follows from Theorem 8. ■

The last result, together with Theorem 9 can be exploited to obtain a constructive procedure for computing a stabilizing periodic receding-horizon controller.

Corollary 3: Assume that (A,B) is stabilizable, (A,Q) is detectable, $P_0 > 0$ and $P_0 > P^+$. Then, there exists an integer T such that $P(T) < P(0)$. Moreover, the PRH closed-loop matrix $A+B\breve{K}_N(\cdot)$ is asymptotically stable. ■

In view of the above corollary, one just needs to integrate the DRE (4) and store the optimal gains until a time point T is reached such that $P(T) < P(0)$. The following example illustrates the use of cyclomonotonicity.

Example 4: Let $R = 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, P_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

with $a > 0$ and $b > 0$. The maximal solution of the ARE (5) is

$$P^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the pair (A,Q) has unobservable modes on the unit circle, the ARE (5) does not have a stabilizing solution [13]. Consequently, the infinite-horizon LQ controller is not stabilizing.

The solution of the DRE (4) is

$$P(2i) = \begin{bmatrix} \frac{a}{ia+1} & 0 \\ 0 & \frac{b}{ib+1} \end{bmatrix}, P(2i+1) = \begin{bmatrix} \frac{b}{ib+1} & 0 \\ 0 & \frac{a}{(i+1)a+1} \end{bmatrix}.$$

For $a = 1$ and $b = 1/3$, there exists no integer value t such that $P(t+1) \leq P(t)$ so that the results based on monotonicity cannot be invoked. However,

$$P(0) - P(2) = \begin{bmatrix} \frac{a^2}{a+1} & 0 \\ 0 & \frac{b^2}{b+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{12} \end{bmatrix} > 0.$$

Hence, the solution $P(\cdot)$ is cyclomonotonic of period 2. Since $P(0) - P(2) > 0$, the assumptions of Theorem 9 are satisfied and the periodic gain

$$\bar{K}_2(t)' = \begin{cases} [\,0 \quad -0.25\,]\,, & t = 2i \\ [\,0 \quad -0.5\,]\,, & t = 2i+1 \end{cases} \qquad i \geq 0$$

is stabilizing. ■

Although it is apparent that cyclomonotonicity is more easily attainable than monotonicity, further research will be needed to assess whether this is worth the increased complexity involved by the use of a periodic feedback.

V. THE "FROZEN" KALMAN PREDICTOR

The problem of guaranteeing closed-loop stability of receding-horizon control has its dual in the problem of guaranteeing the stability of the so-called "frozen" Kalman predictor [3]. Consider the linear stochastic discrete-time system

$$
\begin{aligned}
x(t+1) &= Ax(t) + w(t) \\
y(t) &= Cx(t) + v(t),
\end{aligned}
$$

where the signals $w(\cdot)$ and $v(\cdot)$ are zero-mean white noises independent of each other with $Var[w(t)] = Q$ and $Var[v(t)] = R$. The initial condition $x(0) = x_0$ is a zero-mean normal random variable independent of $w(\cdot)$ and $v(\cdot)$ with $Var[x_0] = P_0$.

The one-step-ahead Kalman predictor is given by

$$
\begin{aligned}
x(t+1|t) &= (A-K(t)C)\, x(t|t-1) + K(t)y(t) \\
K(t) &= AP(t)C'[CP(t)C' + R]^{-1} \\
P(t+1) &= AP(t)A' + Q - AP(t)C'[CP(t)C' + R]^{-1}CP(t)A',
\end{aligned}
\tag{14}
$$

with $P(0) = P_0$.

Now, assume that at a given time point $t = N-1$, we stop updating the DRE (14) and "freeze" the Kalman gain $K(N-1)$. The frozen predictor is time-invariant and its stability depends on the position in the complex plane of the eigenvalues of $A-K(N-1)C$.

It is immediately seen that the problem of choosing P_0 and N so as to guarantee stability of the frozen Kalman predictor is dual to the problem of assessing the closed-loop stability of the RH control scheme presented in Section II. Consequently, all the stability results presented in the previous sections can be dualized to this filtering problem.

In particular, the stabilizing controller of Kwon and Pearson admits an interesting interpretation in the filtering context. Indeed, assuming $P_0^{-1} = 0$ is equivalent to considering a Kalman predictor starting with zero information concerning the initial state [22]. Then, in view of Corollary 1, closed loop stability of the frozen predictor is always guaranteed (provided that $N > n$).

VI. CONCLUDING REMARKS

In this chapter the theory of the fake Riccati equation has been reviewed, illustrating its application to the stability analysis of RH control schemes. In particular, it has been shown that, for the time being, the most effective way to achieve stability relies on the nonincreasing monotonicity of the solution of the DRE.

It is worth mentioning that the application of fake Riccati techniques is not restricted to time-invariant systems. In particular, periodic receding-horizon strategies find their natural application in the control of periodic and multirate sampled-data systems [11], [23].

We end the chapter with some final comments on the potential advantages of RH control over infinite-horizon optimal control. As mentioned in Section II, in the LQ case there is an obvious computational advantage due to the possibility of skipping the numerical solution of the ARE (5). However, as pointed out in [6], the availability of efficient algorithms together with the increased performance of digital hardware tend to reduce the impact of these numerical aspects. Does this imply that RH control is becoming obsolete? If we restrict our attention to the LQ case, the answer may perhaps be affirmative. In practice, however, the controller design has to cope with the presence of nonlinear constraints affecting the inputs (actuator saturations, for instance), the outputs, or even the states. In some cases, it may also happen that the plant model is substantially nonlinear. Then, a RH control strategy hinging on (nonlinear) finite-horizon optimization is incomparably more viable than (nonlinear) infinite-horizon optimization.

This chapter addressed the stability properties of RH controllers in the idealized case of linear plant and quadratic cost function, without taking into account plant nonlinearities and/or inequality constraints affecting the system variables. The advantages of RH control over infinite-horizon LQ may be questionable in this specific case. Nevertheless, the analysis developed throughout the chapter, is the necessary starting point towards the study of more complex RH strategies for nonlinear and constrained plants, a challenging research topic of practical relevance in industrial applications [24], [25], [26], [27].

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