Cα 4

STOCHASTIC ADAPTIVE CONTROL: RESULTS AND PERSPECTIVE

G.C. Goodwin
Dept. of Electrical & Computer Engineering University of Newcastle, NSW, 2308, Australia.

M. Gevers
Louvain University, Batiment Maxwell, B-1348, Louvain-La-Neuve, Belgium.

D.Q. Mayne
Dept. of Electrical Engineering & Computer Science,
University of California, Davis, CA 95616, USA

V. Wertz
Louvain University, Batiment Maxwell, B-1348, Louvain-La-Neuve, Belgium.

1. INTRODUCTION

A key question asked in stochastic adaptive control theory is whether or not it is possible to stabilize a linear system which is perturbed by unmeasured random disturbances and whose model is only partially known.

One approach to this problem is to use the, so called, Certainty Equivalence principle. In this strategy, one divides the problem into two parts. Firstly, one uses a gradient type estimation algorithm to obtain an on-line estimate of the model of the system and disturbances. Then one acts as if the estimated model were the correct model when designing the feedback control law. We will argue later that this strategy is limited since no attempt is made to quantify the accuracy of the estimated model. Hence it is not possible to use robust control design methods which, inter-alia, require a measure of fidelity of the model. For the moment however, we will adopt this approach as a working strategy.

Clearly, one would wish to say something about the performance of this method. For example, a minimal requirement would be to show that if the system is ideal (no nasty nonlinearities or high frequency unmodelled dynamics) then the design objectives are asymptotically achieved. This turns out to be a surprisingly difficult problem. The reason for this is that the full system (plant + estimator + feedback law) is a nonlinear, time varying, stochastic system and such systems are notoriously difficult to analyze. Thus the associated theory presents difficult and challenging mathematical problems.

To make progress on the problem, simplified versions have been studied. One line of attack has been to remove the random disturbances from the specification whilst retaining the format of the parameter estimation algorithm. The latter step is important since otherwise one could be led to trivial solutions in which the model parameters are estimated in infinitesimal time using open loop measurements.

The simplified deterministic certainty equivalence adaptive control problem still presents a significant mathematical challenge and it was not until 1978 [1] that global stability was established for a moderately general case. This first result covered the deterministic continuous time case. Soon after, simpler methods of analysis were found for the deterministic discrete time case [2]. These were subsequently extended to the corresponding continuous time problem [3]–[4].

These solutions suggest that the stochastic case might not be intractable. Indeed, as early as 1973, it had been shown [5] that, provided the estimated parameters converged to something (not necessarily their true values), that a certainty equivalence form of the stochastic minimum variance control law would asymptotically achieve the design objectives. However, parameter convergence remained an open question. By side–stepping the stability issue it was shown via an associated ODE approach [6] that a positive real condition was a necessary condition for convergence. However, the theory was still incomplete.

Finally in 1980, using Martingale Convergence theory, it was shown in [7] that for the discrete time stochastic adaptive control problem, a certainty equivalence minimum variance control law was capable of stabilizing a partially known system and of asymptotically achieving the design objective.

A remaining open question was the corresponding continuous time stochastic adaptive control problem. One might legitimately ask if this problem is worth studying given that all realistic controllers are certain to be implemented digitally. However, apart from the issue of intellectual completeness, there is also the issue that most existing discrete algorithms do not have a meaningful limit as the sampling rate is increased [8]. This suggests that the methods will have numerical and conceptual difficulties when used with moderately fast sampling rates. This is indeed the case as explored in detail in [8]. This problem is fundamental and applies to the discrete stochastic adaptive control laws studied in [5], [6], [7]. These turn out to depend on high order differences of the plant output and these approach high order derivatives as the sampling rate increases! Thus there is both theoretical and practical motivation to study the continuous time stochastic problem. In turn, this will have consequences for the discrete case since the continuous algorithms are likely to show how a sensible limit can be maintained as the sampling rate increases.

An initial objective of this chapter is to review the essential features of continuous time certainty equivalence stochastic adaptive control as originally presented in [9], [10]. These results specialize to the simpler cases of deterministic systems [1] to [4] and lead to new insights for the discrete time stochastic case [5] to [7].

Subsequently we will point to the limitations of the current theory and suggest some open research problems.

2. PRELIMINARIES

We assume that the system can be exactly modelled by a finite dimensional linear stochastic differential equation expressed in innovations form as:

$$dx_t = Ax_t dt + Bu_t dt + K d\omega_t \tag{2.1}$$

$$dz_t = Cx_t dt + d\omega_t \tag{2.2}$$

where ω_t is a Wiener process with incremental covariance $\sigma^2 dt$, x_t is a state–vector of dimension n, u_t is a scalar control input and z_t is the integral of the output of the system. The matrices A, B, K, C contain unknown but fixed parameters. We shall denote by $\mathcal{F}_t$ the increasing σ – fields generated by $\{\omega_s,\ 0 \le s \le t\}$ and the unknown initial conditions x_0, and we shall assume that $\| x_0 \|^2$ is bounded.

We do not necessarily assume that (2.1) is stable. Our control objective is firstly to guarantee stability of the closed loop system and subsequently to ensure that the system output has certain desirable properties (e.g. follows a given reference input signal with minimal mean square error). We will approach this problem using the Centainty Equivalence Principle described earlier, i.e. we will first attempt to estimate the unknown parameters in A, B, K and C and then, assuming these estimates are correct, we will design a suitable feedback control law.

Towards this end, we will first express the model (2.1), (2.2) in a slightly different form which turns out to be more convenient for parameter estimation. Without loss of generality, we assume that the above model is in observer form where

$$A = \begin{bmatrix} -a_{n-1} & 1 & & \\ -a_{n-2} & & \ddots & \\ \vdots & & & 1 \\ -a_0 & & & 0 \end{bmatrix}; \quad B = \begin{bmatrix} b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix}; \quad K = \begin{bmatrix} k_{n-1} \\ \vdots \\ \vdots \\ k_0 \end{bmatrix} \tag{2.3}$$

$$C = [1 \quad 0 \quad \quad 0] \tag{2.4}$$

To avoid differentiations in the parameter estimator it is helpful to re-express this model in fractional form [4]. We therefore re-parameterize the system as follows. Let

$$E(\varrho) = \varrho^n + e_{n-1}\varrho^{n-1} + \ldots + e_0 \tag{2.5}$$

$$G^T = [g_{n-1}, \ \ldots, \ g_0] \ \text{ with } \ g_i = e_i - a_i; \quad i = 0, \ \ldots, \ n-1 \tag{2.6}$$

and

$$E = \begin{bmatrix} -e_{n-1} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ -e_0 & & 0 & 0 \end{bmatrix} \tag{2.7}$$

where the coefficients are arbitrary subject to $E(\varrho)$ having all its zeros in the open left half plane and $K \neq G$.

Then, by adding and subtracting $Ex_t dt$ to the right hand side, equations (2.1), (2.2) can be rewritten as

$$dx_t = Ex_t dt + Bu_t \, dt + (K - G) d\omega_t + G dy_t \tag{2.8}$$

$$dz_t = Cx_t \, dt + d\omega_t \tag{2.9}$$

Using superposition, equations (2.8), (2.9) can also be expressed as

$$d\phi_t^1 = E\phi_t^1 dt + G dy_t \tag{2.10}$$

$$d\phi_t^2 = E\phi_t^2 dt + B u_t dt \tag{2.11}$$

$$d\phi_t^3 = E\phi_t^3 dt + (K - G) d\omega_t \tag{2.12}$$

$$dz_t = C\left[\phi_t^1 dt + \phi_t^2 dt + \phi_t^3 dt\right] + d\omega_t \tag{2.13}$$

Since z_t is a scalar, we have that $C(sI - E)^{-1}B = B^T\left(sI - E^T\right)^{-1}C^T$ etc. Hence (2.10) to (2.13) can be rewritten as

$$d\phi_t^z = E^T\phi_t^z dt + C^T dz_t \tag{2.14}$$

$$d\phi_t^u = E^T\phi_t^u dt + C^T u_t dt \tag{2.15}$$

$$d\phi_t^\omega = E^T\phi_t^\omega dt + C^T d\omega_t \tag{2.16}$$

$$dz_t = \left[G^T\phi_t^z + B^T\phi_t^u + (K - G)^T\phi_t^\omega\right]dt + d\omega_t \tag{2.17}$$

In (2.14)–(2.16) we choose $\phi_0^z = 0$, $\phi_0^u = 0$ and ϕ_0^ω such that

$$(K - G)^T\phi_0^\omega = Cx_0 \tag{2.18}$$

Equation (2.17) is in the form of a linear regression, i.e.

$$dz_t = \phi_t^T\theta dt + d\omega_t \tag{2.19}$$

where

$$\phi_t^T = \left[(\phi_t^z)^T,\ (\phi_t^u)^T,\ (\phi_t^\omega)^T\right] \tag{2.20}$$

$$\theta^T = \left[G^T,\ B^T,\ (K - G)^T\right] = \left[G^T,\ B^T,\ F^T\right] \tag{2.21}$$

Note that ϕ_t does not depend on the unknown system parameters since E is known. Thus equations (2.14) to (2.16) simply represent a state space form of the usual regression vector as in [11] and [4]. To make the comparison with the regression vector formulations more complete, we note that with $\varrho \triangleq \frac{d}{dt}$ we have, with some abuse of notation and ignoring initial conditions

$$(\phi_t^u)^T = \left[\frac{\varrho^{n-1}}{E(\varrho)}u, \; \ldots, \; \frac{1}{E(\varrho)}u\right] \tag{2.22}$$

and similarly for ϕ_t^z, ϕ_t^ω.

3. PSEUDO LINEAR REGRESSION

Equation (2.19) suggests that the unknown parameter vector θ could be easily estimated by some form of least squares. However, the model is not quite in a form which is suitable for parameter estimation. This is because the component ϕ_t^ω depends on the unmeasured noise source ω_t. Thus, following a standard strategy [11] we define the predicted output by a pseudo regression in which $d\omega_t$ is replaced by the prediction error. Thus we define

$$d\hat{z}_t = \psi_t^T \hat{\theta}_t dt \tag{3.1}$$

where $\hat{\theta}_t$ is an on-line estimate of θ_t obtained from a stochastic gradient algorithm of the form:

$$d\hat{\theta}_t = \frac{\psi_t}{r_t}\left(dz_t - \psi_t^T \hat{\theta}_t dt\right) \tag{3.2}$$

In equation (3.1), r_t is a normalization variable which will be specified in Section 5.

Also, in (3.1) we have

$$\psi_t^T = \left[(\psi_t^z)^T, \; (\psi_t^u)^T, \; (\psi_t^e)^T\right] \tag{3.3}$$

$$d\psi_t^z = E^T\psi_t^z dt + C^T dz_t \qquad (3.4)$$

$$d\psi_t^u = E^T\psi_t^u dt + C^T u_t dt \qquad (3.5)$$

$$d\psi_t^e = E^T\psi_t^e dt + C^T de_t \qquad (3.6)$$

$$de_t = dz_t - d\hat{z}_t = dz_t - \psi_t^T\hat{\theta}_t dt \qquad (3.7)$$

It is assumed that $\psi_0 = 0$ and that $\hat{\theta}_0$ is $\mathcal{F}_0$ – measurable . With this choice of initial conditions $\psi_t^z = \phi_t^z$ and $\psi_t^u = \phi_t^u$ for $t \geq 0$.

The effect of a using pseudo regression is to disturb the direction of the gradient vector ϕ_t .

Heuristically one might expect that it would be necessary for the angle between the true gradient ϕ_t and the pseudo gradient ψ_t to be accute on average. In turn, this requires that the filter $E(\varrho)$ used in defining ψ_t be a reasonable approximation to the optimal Kalman Filter for the system which is implicitly used in defining ϕ_t . To make this clearer, let us consider the difference, η_t , between the true system quantity $\phi_t^T\theta$ and the estimated quantity $\psi_t^T\hat{\theta}_t$:

$$\eta_t = \phi_t^T\theta - \psi_t^T\hat{\theta}_t \qquad (3.8)$$

From (2.19), (3.7), (3.8) we have

$$de_t - d\omega_t = \eta_t dt \qquad (3.9)$$

Notice that η_t is the "deterministic part" of the prediction error.

Now let

$$\gamma_t \triangleq \psi_t^e - \phi_t^\omega \qquad (3.10)$$

Then from (3.6), (3.8), (2.16) we have

$$d\gamma_t = E^T\gamma_t dt + C^T\eta_t dt, \quad \text{with} \quad \gamma_0 = -\phi_0^\omega \qquad (3.11)$$

Noting that

$$\psi_0^z = \phi_0^z = 0 \quad \text{and} \quad \psi_0^u = \phi_0^u = 0 \tag{3.12}$$

then

$$(K-G)^T\gamma_t = -(\phi_t - \psi_t)^T\theta \tag{3.13}$$

In particular:

$$(K-G)^T\gamma_0 = -Cx_0 \tag{3.14}$$

Notice that γ_0 cannot be made zero if x_0 is unknown, but we will assume that $\| \gamma_0 \|^2$ is bounded; this is consistent with our assumption on x_0 and (3.14).

Denoting

$$\tilde{\theta}_t \triangleq \hat{\theta}_t - \theta \tag{3.15}$$

we then have from (3.13), (3.11) that

$$\eta_t = -(K-G)^T\gamma_t - \psi_t^T\tilde{\theta}_t \tag{3.16}$$

where

$$d\gamma_t = (A-KC)^T\gamma dt + C^T\left(-\psi_t^T\tilde{\theta}_t\right)dt \tag{3.17}$$

We thus see that η_t is related to $-\psi_t^T\tilde{\theta}_t$ by the following transfer function equation

$$D(\varrho)\eta_t = E(\varrho)\left[-\psi_t^T\tilde{\theta}_t\right] \tag{3.18}$$

where $D(\varrho)$ is the characteristic polynomial of the optimal Kalman filter for the system, i.e.

$$D(\varrho) = \varrho^n + (k_{n-1} + a_{n-1})\varrho^{n-1} + \ \dots\ (k_0 + a_0) = \det(\varrho I - A + KC) \tag{3.19}$$

and where $E(\varrho)$ is as in (2.5).

We will see later that the required "accute angle condition" between the regression vectors is captured by assuming that the transfer function $\frac{D}{E}$ in (3.18) satisfies the following restriction:

Assumption 1:

$D(\varrho)$ is strictly Hurwitz and the filter $E(\varrho)$ is chosen such that

1) $\operatorname{Re} \sigma_i(E(\varrho)) \leq -a < 0, \quad i = 1, \ldots, n$ where $\sigma_i(E)$

2) $\frac{D}{E}$ is input strictly passive, i.e. $\exists \epsilon > 0$ and $K > 0$ such that

$$\forall T > 0, \int_0^T y_\tau u_\tau d\tau \geq \epsilon \int_0^T u_\tau^2 d\tau - K \| \gamma_0 \|^2 \tag{3.20}$$

where y_t is the output of the filter D/E driven by u_t.

Noting that $d\theta = 0$ and using the definition of e_t and η_t given in (3.7), (3.9), we observe from (3.15) and (3.2) that $\tilde{\theta}_t$ is the solution of the following stochastic differential equation

$$d\tilde{\theta}_t = \frac{\psi_t}{r_t}\eta_t dt + \frac{\psi_t}{r_t} d\omega_t \tag{3.21}$$

In the sequel we will require that $\tilde{\theta}_t$ be bounded. We guarantee this by introducing a projection scheme as described below. We first introduce the following assumption.

Assumption 2:

There exists a known parameter value θ_c and a positive number R_1 such that the true value θ lies inside $\mathfrak{C}_1$ where

$$\mathfrak{C}_1 = \{\Theta : \| \Theta - \theta_c \| \leq R_1\}$$

∇∇∇

Let R_2 be another positive number larger than R_1 and define another convex set $\mathcal{C}_2$ analogous to the one given in Assumption $\mathcal{C}_1$. We then modify the parameter estimator to ensure that $\| \hat{\theta}_t - \theta_c \| < R_2$ for all t. We do this by using the following projection scheme:

<u>Parameter estimator with projection</u>

Let τ be a time for which the solution of (3.2) is such that $\| \theta_\tau - \theta_c \| = R_2$. Denote the corresponding value of $\hat{\theta}_\tau$ by $\hat{\theta}_{\tau_-}$. At time τ, the estimate $\hat{\theta}_\tau$ is then defined as

$$\hat{\theta}_\tau \triangleq \theta_c + \frac{R_1}{R_2}\left(\hat{\theta}_{\tau_-} - \theta_c\right) \tag{3.22}$$

For $t \geq \tau$ equation (3.2) is then integrated with initial condition $\hat{\theta}_\tau$ defined by (3.23). This makes for $\hat{\theta}_t$ right continuous at the projection times and ensures $\hat{\theta}_t \in \mathcal{C}_2$ for all t.

4. FEEDBACK CONTROL

We consider a general class of control laws in state feedback form:

$$u_t = -\left[\hat{\ell}_{n-1}, \; \ldots, \; \hat{\ell}_0\right] \psi_t^u - \left[\hat{p}_{n-1}, \; \ldots, \; \hat{p}_0\right] \psi_t^z + z^* \tag{4.1}$$

where $\hat{\ell}_{n-1}, \ldots, \hat{\ell}_0, \hat{p}_{n-1}, \ldots, \hat{p}_0$ are Lipschitz functions of the estimated parameter vector $\hat{\theta}$. This is equivalent to the feedback law

$$\hat{Q}(\varrho)u_t = -\hat{P}(\varrho)z_t + E(\varrho)z_t^* \tag{4.2}$$

where

$$\hat{Q}(\varrho) = E(\varrho) + \hat{L}(\varrho); \;\; \hat{L}(\varrho) = \hat{\ell}_{n-1}\varrho^{n-1} + \ldots + \hat{\ell}_0; \;\; \hat{P}(\varrho) = \hat{p}_{n-1}\varrho^{n-1} + \ldots + \hat{p}_0 \tag{4.3}$$

Note that the control law transfer function is $\frac{-\hat{P}}{\hat{Q}}$ which is strictly proper. Also z_t^* denotes a bounded reference signal. For the moment, we make no further assumptions about $\hat{L}$, $\hat{P}$.

From the model (2.1), (2.2), the general controller (4.1), the definition of ψ_t ((3.3) to (3.6)), and the definition of the errors ((3.7) to (3.9)) we can write:

$$d\psi_t = A_t\psi_t dt + B_1(\eta_t dt + d\omega_t) + B_2 z_t^* dt \tag{4.4}$$

where

$$A_t = \left[\begin{array}{ccc|ccc|ccc}
-\hat{a}_{n-1} & \cdots\ \cdots & \cdots -\hat{a}_0 & \hat{b}_{n-1} & \cdots\ \cdots & \cdots\ \hat{b}_0 & \hat{f}_{n-1} & \cdots\ \cdots & \cdots\ \hat{f}_0 \\
 & & 0 & & & & & & \\
 & I_{n-1} & \vdots & & & & & & \\
\cdots & & 0 & \cdots & & \cdots & \cdots & & \cdots \\
\hline
-\hat{p}_{n-1} & \cdots\ \cdots & \cdots -\hat{a}_0 & -\hat{q}_{n-1} & \cdots\ \cdots & \cdots -\hat{q}_0 & 0 & \cdots\ \cdots & \cdots\ 0 \\
 & & & & & 0 & & & \\
 & & & & I_{n-1} & \vdots & & & \\
\cdots & & \cdots & \cdots & & 0 & \cdots & & \cdots \\
\hline
0 & \cdots\ \cdots & \cdots\ 0 & 0 & \cdots\ \cdots & \cdots\ 0 & -\hat{e}_{n-1} & \cdots\ \cdots & \cdots -\hat{e}_0 \\
 & & & & & & & & 0 \\
 & & & & & & & I_{n-1} & \vdots \\
\cdots & & \cdots & \cdots & & \cdots & \cdots & & 0
\end{array}\right] \tag{4.5}$$

$$B_1^T = [10 \ \ldots \ 010 \ \ldots \ 0] \tag{4.6}$$

$$B_2^T = [0 \ \ldots \ 010 \ \ldots \ 0] \tag{4.7}$$

where B_1^T has 1's in the 1st and $(2n + 1)^{st}$ positions and B_2^T has 1 in the $(n + 1)^{st}$ position.

A key point about (4.5) is that A_t is a Lipschitz function of $\tilde{\theta}_t$ provided $\hat{\ell}_{n-1}, \ldots, \hat{\ell}_0, \hat{p}_{n-1}, \ldots, \hat{p}_0$ are Lipschitz in $\hat{\theta}$.

5. PROPERTIES OF THE ESTIMATION ALGORITHM

An important first step is to establish existence and uniqueness of the solution of the full set of equations describing the system and estimation algorithm. Without this step, one cannot proceed – indeed one can easily show $1+1 = 3$ if one is not careful [12]. Combining equations (3.15), (3.21) and (4.4), the full set of equations is

$$d\psi_t = \left[A_t(\tilde{\theta}_t)\psi_t - B_1\psi_t^T\tilde{\theta}_t - B_1 \int_0^t h_{t-\tau}\psi_\tau^T\tilde{\theta}_\tau d\tau \right] dt$$

$$+ B_1 d\omega_t + B_2 \dot{y}_t^* dt$$

$$d\tilde{\theta}_t = \left[-\frac{\psi_t\psi_t^T}{r_t(\psi.)}\tilde{\theta}_t - \int_0^t h_{t-\tau}\frac{\psi_t\psi_\tau^T}{r_t(\psi.)}\tilde{\theta}_\tau d\tau \right] dt$$

$$+ \frac{\psi_t}{r_t(\psi.)} d\omega_t \tag{5.1}$$

where $A_t(\tilde{\theta}_t) \equiv A_t(\hat{\theta}_t)$ and h_t is the impulse response of the strictly proper part of the transfer function E/D in (3.18).

Preliminary inspection of (5.1) indicates that existence and uniqueness of the solution will depend on being able to bound the coefficient $\left(\frac{\psi_t\psi_t^T}{r_t}\right)$. Thus we define the normalization variable r_t as follows:

$$r_t \triangleq \sup_{0\le\tau\le t} \psi_\tau^T\psi_\tau + \int_0^t \psi_\tau^T\psi_\tau d\tau + c_0; \quad c_0 > 0 \tag{5.2}$$

where c_0 is any positive deterministic number.

We then have the following result

Lemma 5.1: The composite set of equations (5.1) has a unique solution with continuous sample paths a.s. up to the random time T of the first explosion. (That is T is the first time that either a component of ψ or $\tilde{\theta}$ becomes infinite or $T = \infty$.

Proof: We first note from (4.5) that $A_t(\tilde{\theta}_t)$ is Lipschitz in $\tilde{\theta}_t$ due to the assumed form of the dependence of $\hat{\ell}_{n-1}, \ldots, \hat{\ell}_0, \hat{p}_{n-1}, \ldots, \hat{p}_0$ on $\hat{\theta}$. This implies that the coefficient vectors multiplying dt and $d\omega_t$ in (5.1) are locally Lipschitz with respect to the supremum norm on sample paths [9]. (That is given a compact set in the space of ψ_t, $\tilde{\theta}_t$, the functions are Lipschitz with constant depending on the choice of the set).

The result then follows from Theorems (14.18) and (14.20) of [13].

∇∇∇

Note that Lemma 5.1 does not use the projection of the parameter estimates as described at the end of Section 3. When this additional facet of the algorithm is included, we can strengthen Lemma 5.1 as follows:

Lemma 5.2: With the addition of the projection scheme (3.22) to equation (5.1) then $\hat{\theta}_t$ remains in $\mathcal{C}_2$ for all t and the composite set of equations (5.1) has a unique solution a.s. with sample paths $(\psi_t, \tilde{\theta}_t)$ which are continuous except at the projection times.

Proof: Up to the time of the first projection, θ_t is bounded and thus (5.1) is a linear time varying equation with bounded coefficients and hence ψ_t cannot become unbounded in a finite time. Hence the first projection occurs strictly before the explosion time T of Lemma 5.1 (unless both T and the first projection time are infinite).

After projection, we can apply Lemma 5.1 again and repeat the same argument. Hence due to the linearity of (5.1) for given $\psi_\cdot$, $\tilde{\theta}_t$ exists, is unique and it is bounded by the projection. Then (5.1) is a linear equation with bounded coefficients, and hence ψ_t exists a.s. for all t.

∇∇∇

Lemma 5.2 provides the basic existence and uniqueness result necessary to establish the following result giving properties of the parameter estimator. This result does not depend on a-priori boundedness of the system states. (Note, however, that $\tilde{\theta}_t$ and $\frac{\|\psi_t\|^2}{r_t}$ are bounded by the structure of the algorithm.)

Theorem 5.1: For the general class of feedback control laws described in Section 4 and under Assumptions 1 and 2, the following properties hold for the model ((2.1), (2.2)) and the estimator (3.2), (5.2) with the projection scheme (3.22):

$$\text{(i)} \quad \lim_{t\to\infty} \sup \int_0^t \frac{\eta_\tau^2}{r_\tau} d\tau \leq K_1 < \infty \qquad \text{a.s.} \tag{5.3}$$

where K_1 is a random variable (realization dependent)

$$\text{(ii) For all finite } \Delta, \lim_{t\to\infty} \sup_{0\leq T\leq\Delta} \| \hat{\theta}_{t+T} - \hat{\theta}_t \| = 0 \qquad \text{a.s.} \tag{5.4}$$

(iii) There exists a finite random time t_R beyond which no further parameter projections occur.

Proof: (a) Starting from (3.21) and using Ito's rule (see e.g. [14]), we have that, between projections:

$$d\left(\tilde{\theta}_t^T\tilde{\theta}_t\right) = 2\tilde{\theta}_t^T \frac{\psi_t}{r_t}(\eta_t dt + d\omega_t) + \sigma^2 \frac{\| \psi_t \|^2}{r_t^2} dt$$

$$= -\frac{2}{r_t}\left(-\tilde{\theta}_t^T\psi_t\eta_t - \epsilon\eta_t^2\right)dt - 2\epsilon\frac{\eta_t^2}{r_t}dt + 2\tilde{\theta}_t^T\frac{\psi_t}{r_t}d\omega_t + \sigma^2\frac{\| \psi_t \|^2}{r_t^2}dt \tag{5.5}$$

for some $\epsilon > 0$.

Note that from (3.22) and using Assumption 2, at the times of projection we have $\| \tilde{\theta}_t \|^2 \leq \| \tilde{\theta}_{t-} \|^2 - (R_2 - R_1)^2$.

Defining $\xi_t \triangleq -\tilde{\theta}_t^T\psi_t\eta_t - \epsilon\eta_t^2$ where ϵ is as in Assumption 1, integrating (5.5) and accounting for projections yields

$$\tilde{\theta}_t^T\tilde{\theta}_t \leq \tilde{\theta}_s^T\tilde{\theta}_s - 2\int_s^t \frac{\xi_\lambda}{r_\lambda}d\lambda - 2\epsilon\int_s^t \frac{\eta_\lambda^2}{r_\lambda}d\lambda + 2\int_s^t \tilde{\theta}_\lambda^T\frac{\psi_\lambda}{r_\lambda}d\omega_\lambda$$
$$+ \int_s^t \sigma^2\frac{\| \psi_\lambda \|^2}{r_\lambda^2}d\lambda - N_{t,s}(R_2 - R_1)^2 \tag{5.6}$$

where $N_{t,s}$ is the number of times that projections occur between times s and t.

Consider now the integral $\int_0^t \frac{\| \psi_\lambda \|^2}{r_\lambda^2} d\lambda$. We have, using (5.2),

$$\int_0^t \frac{\| \psi_\lambda \|^2}{r_\lambda^2} d\lambda \leq \int_0^t \frac{dr_\lambda}{r_\lambda^2} = \frac{1}{r_0} - \frac{1}{r_t} \leq \frac{1}{r_0} = \frac{1}{c_0} \tag{5.7}$$

(This operation makes sense due to Bonnet's and Du Bois–Reymond's formulae [15] which allow the integral to be considered as a Riemann integral.)

We now define $\mathcal{H}_t$ as the solution of the Ito integral

$$d\mathcal{H}_t = \frac{2\tilde{\theta}_t^T \psi_t}{r_t} d\omega_t \; ; \qquad \mathcal{H}_0 = \frac{\sigma^2}{c_0} + \frac{2K \| \tilde{\gamma}_0 \|^2}{r_0} + \tilde{\theta}_0^T \tilde{\theta}_0 \tag{5.8}$$

This integral makes sense thanks to (5.7) and Lemma 5.2. Since c_0, r_0, γ_0 and $\tilde{\theta}_0$ are $\mathcal{F}_0$ – measurable , $(\mathcal{H}_t, \ \mathcal{F}_t)$ is a Martingale. Moreover it follows from (5.6) and (5.7) that $\mathcal{H}_t$ satisfies

$$\mathcal{H}_t \geq \tilde{\theta}_t^T \tilde{\theta}_t + 2 \int_0^t \frac{\xi_\lambda}{r_\lambda} d\lambda + 2\epsilon \int_0^t \frac{\eta_\lambda^2}{r_\lambda} d\lambda + \frac{\sigma^2}{c_0} - \int_0^t \sigma^2 \frac{\| \psi_\lambda \|^2}{r_\lambda^2} d\lambda + \frac{2K \| \gamma_0 \|^2}{r_0}$$
$$+ N_{t,0}[R_2 - R_1]^2 \tag{5.9}$$

The second term on the right hand side o f (5.8) can be easily shown [9] to be positive using (3.20). Hence using (5.7) it follows that $\mathcal{H}_t$ is positive. Thus $(\mathcal{H}_t, \ \mathcal{F}_t)$ is a positive Martingale and hence

$$\lim_{t \to \infty} \mathcal{H}_t = \mathcal{H} < \infty \quad \text{a.s.} \tag{5.10}$$

Using (5.10) and noting (5.7) and Assumption 1 we conclude that (5.3) holds for some finite random variable K_1.

This establishes (i). Also, since $R_2 > R_1$, then from (5.9), (5.10) $N_{t,0}$ is bounded a.s. Hence (iii) follows.

(b) Result (ii) follows in a straightforward way from (i) and (ii) using the Schwartz inequality. Details are given in [9].

∇∇∇

Comment 5.1

A key feature of our estimator is our choice of r_t (see (5.2)). It guarantees that $\frac{\| \psi_t \|^2}{r_t} \leq 1 \quad \forall t$ and this is crucial in establishing Lemma 5.1, 5.2 and Theorem 5.1. If we were to define $r_t = \int_0^t \psi_\tau^T \psi_\tau d\tau$ which is the direct analogue of the discrete case [7], then proofs would require the assumption that $\frac{\| \psi_t \|^2}{r_t}$ is almost surely bounded [16]. This assumption is rather unrealistic given that ψ_t contains signals driven by white noise.

6. LINKING ESTIMATOR PROPERTIES TO CLOSED LOOP PROPERTIES

In establishing the estimator properties given in Theorem 5.1 we made only oblique appeal to the form of the certainty equivalence feedback law. In particular, it was not assumed that the law would stabilize the system if the true parameters were known. Thus, to make further progress we will need to strengthen the assumption on the feedback law made in Section 4.

The proof given below depends upon the fact that the certainty equivalence control law stabilizes the frozen estimated model. This is true of a wide class of algorithms. However, to be specific, we will illustrate the analysis procedure by considering two cases; namely adaptive pole assignment and adaptive model reference control. In this section we will study the pole assignment algorithm. The model reference problem will be taken up later. In the pole assignment case, the polynomials $\hat{L}$ and $\hat{P}$ defining the control law are computed from the estimated $\hat{A}(\hat{\theta}_t)$ and $\hat{B}(\hat{\theta}_t)$ as follows.

Let

$$\hat{\theta} = \left[\hat{g}_{n-1}, \ \ldots, \ \hat{g}_0, \ \hat{b}_{n-1}, \ \ldots, \ \hat{b}_0, \ \hat{f}_{n-1}, \ \ldots, \ \hat{f}_0\right]$$

and define (c.f. (2.6), (2.21))

$$\hat{a}_i = e_i - \hat{g}_i; \quad i = 0, \ \ldots, \ n-1.$$

$$\hat{A}(\varrho) = \varrho^n + \hat{a}_{n-1}\varrho^{n-1} + \ \ldots \ \hat{a}_0$$

$$\hat{B}(\varrho) = \hat{b}_{n-1}\varrho^{n-1} + \ \ldots \ + \hat{b}_0$$

$$\hat{F}(\varrho) = \hat{f}_{n-1}\varrho^{n-1} + \ \ldots \ + \hat{f}_0$$

Then, for a given possibly time varying A^* of degree 2n, solve the following equation for $\hat{Q}$ and $\hat{P}$

$$\hat{Q}\hat{A} + \hat{P}\hat{B} = A^* \quad \text{with} \quad \operatorname{Re}\lambda_i(A^*) \le -\beta < 0, \quad i = 1, \ \ldots, \ 2n \tag{6.1}$$

where $\lambda_i(A^*)$ are the eigenvalues of the polynomial A^*.

Finally compute

$$\hat{L}(\varrho) = \hat{Q}(\varrho) - E(\varrho)$$

Equation (6.1) can also be written

$$M(\hat{\theta}) \begin{bmatrix} \hat{q}_0 \\ \cdot \\ \cdot \\ \hat{q}_{n-1} \\ \hat{p}_0 \\ \cdot \\ \cdot \\ \hat{p}_{n-1} \end{bmatrix} = \begin{bmatrix} a_0^* \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{2n-1}^* \end{bmatrix} \tag{6.2}$$

The polynomial A^* is a design polynomial available to the user. The only restriction we require on the polynomial $A^*(t)$ is:

<u>Assumption 3</u>

$\forall t$, the polynomial $A^*(t)$ is continuous and bounded, has a uniform stability margin (i.e. $\mathrm{Re}\,\lambda_i(A^*(t)) \le -\beta < 0,\ i = 1, \ldots, 2n$) and $\lim_{t\to\infty} \sup_{0\le T\le\Delta} \| a^*(t+T) - a^*(t) \| = 0$ for some $\Delta > 0$, where a^* is the vector of coefficients of A^*.

$\nabla\nabla\nabla$

From Section 4, we require that $\hat{L},\ \hat{P}$ be Lipschitz functions of $\hat{\theta}$ for all time. We note that the projection facility (3.22) ensures that $\hat{\theta} \in \mathcal{C}_2$ for all time. Then, the Lipschitz condition will be automatically satisfied provided all models corresponding to $\hat{\theta} \in \mathcal{C}_2$ are uniformly stabilizably.

This assumption will be eliminated in the case of model reference control of stably invertible systems studied later. For the case of pole assignment we introduce:

Assumption 4:

Assumption 2 is satisfied and there exists a known positive constant ϵ such that for all θ_t in $\mathcal{C}_2$,

$$\det M(\theta_t) \ge \epsilon$$

$\nabla\nabla\nabla$

Subject to Assumption 4, the projection scheme ensures that $\det M(\theta_t) \ge \epsilon$ for all t. In view of (6.2) this will ensure that $\hat{L},\ \hat{P}$ are Lipschitz functions of $\hat{\theta}$ as required.

To link the estimated properties to the closed loop properties we will make use of the following Lemma.

Lemma 6.1 (Continuous Time Key Technical Lemma): Consider a realisation produced by the model (2.1)–(2.2). Suppose the estimator is such that property (i) of Theorem 5.1 holds and the controller is such that the following growth condition is satisfied:

$$\frac{r_t}{t} \le C + \frac{K_2}{t}\int_0^t \eta_\tau^2 d\tau \qquad (6.3)$$

where $r_t,\ \eta_t$ are defined before and $C,\ K_2$ are finite positive constants. Then

(i) $\lim_{t\to\infty} \sup \frac{r_t}{t} \le K_3 < \infty$ (6.4)

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \eta_\tau^2 d\tau = 0$ (6.5)

Proof: (i) From (6.3) and the nonnegativity of η_τ^2 and r_τ it follows that

$$\frac{r_t}{t} \le C + K_2 \int_0^t \frac{\eta_\tau^2}{r_\tau}\frac{r_\tau}{\tau} d\tau$$

The result then follows from the Bellman–Grønwall lemma (see e.g [18]) using (i) of Theorem 5.1.

(ii) Suppose first that $\lim_{t\to\infty} r_t = \infty$. Then, by the Kronecker lemma [7], [9]:

$$\lim_{t\to\infty} \frac{1}{r_t}\int_0^t \eta_\tau^2 d\tau = 0 \tag{6.6}$$

Hence, using (6.4) and (6.6)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \eta_\tau^2 d\tau = \lim_{t\to\infty} \frac{r_t}{t}\frac{1}{r_t}\int_0^t \eta_\tau^2 d\tau = 0 \tag{6.7}$$

Alternatively, if $\lim_{t\to\infty} r_t \le K_4 < \infty$ then

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t}\int_0^t \eta_\tau^2 d\tau &\le \lim_{t\to\infty} \frac{1}{t}\frac{K_4}{r_t}\int_0^t \eta_\tau^2 d\tau \\ &\le \lim_{t\to\infty} \frac{K_4}{t}\int_0^t \frac{\eta_\tau^2}{r_\tau} d\tau = 0 \end{aligned}$$

using (i) of Theorem 5.1. The last inequality follows from the monotonicity of r_τ.

∇∇∇

Our key remaining task is to establish the growth condition (6.3). We will use the BIBO properties of the closed loop state space model (4.4). We first establish that the homogeneous part of (4.4) is exponentially stable.

Lemma 6.2: Consider the homogeneous time varying differential equation

$$\frac{d}{dt}\psi_t = A_t\psi_t$$

with A_t given by (4.5). Assume that the $\hat{a}_i$, $\hat{b}_i$, $\hat{f}_i$ are estimated using the parameter estimator of Section 3 including the projection scheme (3.22) and that Assumptions 1 to 4 hold. Then (6.1) is exponentially stable a.s..

Proof: We have shown in Theorem 5.1 that there exists a random time t_R beyond which no further projections occur. The result then follows using Lemma 3 of [17] since the pole assignment condition (6.1) ensures that the eigenvalues of each frozen matrix A_t are in the strict left half plane. Details are given in [9].

∇∇∇

We can now establish the following convergence result.

Theorem 6.1: Consider the system ((2.1), (2.2)), with the parameter vector θ satisfying Assumption 2, the parameter estimator of Section 3 with projection, an observer polynomial E satisfying Assumption 1, and an adaptive pole assignment control law of the form (4.2), (6.1) satisfying Assumptions 3 and 4. Then for arbitrary finite initial conditions and an arbitrary, piecewise continuous, uniformly bounded reference input z_t^*, the following results hold

(i) $\lim_{t\to\infty} \sup \frac{1}{t}\int_0^t \| \psi_\tau \|^2 d\tau < \infty$ a.s.

(ii) $\lim_{t\to\infty} \sup \frac{1}{t}\int_0^t u_\tau^2 d\tau < \infty$ a.s.

(iii) $\lim_{t\to\infty} \sup \frac{1}{t}\int_0^t |Cx_\tau|^2 d\tau < \infty$ a.s.

Proof: The proof uses a standard argument on robustness of bounded solutions of exponentially stable linear time varying systems. We use Lemma 6.2 to establish from (4.4) that the growth condition (6.3) holds. We then use Lemma 6.1 in an obvious way – see [9].

∇∇∇

7. MODEL REFERENCE STOCHASTIC ADAPTIVE CONTROL

The adaptive pole assignment result of the previous section established a form of closed loop stability ((i), (ii), (iii) of Theorem 6.1) but did not establish any form of asymptotic convergence to a desired tracking objective. We thus consider an alternative feedback law of the model reference type [3], [4]. An important preliminary observation from [3], [4] that the issue of plant relative degree is crucial to avoid differentiators in the control law. We will thus strengthen our assumptions to include knowledge of this index.

To relate our results more closely to those in the deterministic literature, particularly those results in [4] which are extended here to the stochastic case, we will use the following equivalent form of the model (2.1) (2.2):

$$\overline{A}(\varrho)y_t = \overline{B}(\varrho)u_t + \overline{C}(\varrho)\omega_t \tag{7.1}$$

where

$$\overline{A}(\varrho) \triangleq \varrho^n + a_{n-1}\varrho^{n-1} + \cdots + a_0 \tag{7.2}$$

$$\overline{B}(\varrho) \triangleq b_m\varrho^m + \cdots + b_0 \,, \; b_m > 0 \tag{7.3}$$

$$\overline{C}(\varrho) \triangleq \varrho^n + c_{n-1}\varrho^{n-1} + \cdots + c_0 \tag{7.4}$$

and ϱ denotes (d/dt). Clearly y_t and ω_t replace dz_t/dt and dw_t/dt respectively. The representation (7.1) is formal and should be regarded as shorthand for (2.1) and (2.2). The transcription from (2.1) and (2.2) to (7.1) and vice versa is simple if an appropriate canonical form is employed.

In (7.1), $\overline{C}(\varrho)$ is the Kalman observer polynomial and is monic and Hurwitz.

Assumptions 1 to 4 in the previous analysis are replaced by::

A1: The degrees n and m of the polynomials $\overline{A}$ and $\overline{B}$ are known.

A2: Upper and lower bounds are known for all parameters. Also the sign of b_m is known as well as a lower bound ($\overline{b}_{\min}$) on its magnitude.

A3: There are no unmodelled dynamics

A4: The reference model is exponentially stable and the plant is minimum phase (i.e. $\bar{B}$ is Hurwitz). The input to the reference model, ξ, is piecewise continuous and uniformly bounded.

We will use $\nu(A)$ to denote the degree of A and without loss of generality we will take b_m to be positive (b_m negative simply requires u to be redefined as $-u$).

In view of assumption A2, we can then define two convex regions $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows:

$$\mathcal{C}_1 \triangleq \left\{\theta : \theta^i_{\min} \le \theta_i \le \theta^i_{\max}\right\} ; \; \theta^i_{\min} = \bar{b}_{\min} \text{ for } \theta_i \equiv b_m$$

$$\mathcal{C}_2 \triangleq \left\{\theta : \theta^i_{\min} - b_{\min} \le \theta_i \le \theta^i_{\max} + b_{\min}\right\} ; \; b_{\min} = \frac{\bar{b}_{\min}}{2}$$

Note that $\theta \in \mathcal{C}_1 \subset \mathcal{C}_2$ and that $b_m > b_{\min}$ for $\theta \in \mathcal{C}_2$. Also, note that unlike Assumption 4 of Section 6, we do not now require any relative primeness assumptions inside $\mathcal{C}_2$.

To form a basis for the adaptive control algorithm it is first necessary to define the control objective in the non-adaptive case. Our basic aim is to have a 'minimum variance' type algorithm as in [2]. However, the simplistic stochastic discrete time objective of [2] does not have a continuous time counterpart since the direct analogue requires some of the closed loop poles to be placed at $-\infty$. Thus, we will instead, combine the essential idea from the stochastic discrete time case [2] of using an optimal Kalman observer, with the essential idea from the continuous time deterministic case [3], [4] of defining the feedback objective via a reference model. This gives a stochastic model reference control law.

The reference model is defined to be

$$y^* = \mathcal{M}\xi \tag{7.5}$$

where $\mathcal{M}$ is exponentially stable and proper. Since the relative degree of the closed loop system cannot be less than that of the plant, we further decompose $\mathcal{M}$ as

$$\mathcal{M} = M/E_2 \tag{7.6}$$

where M is any proper exponentially stable transfer function and E_2 is any monic Hurwitz polynomial with $\nu(E_2) = n - m$.

Since our aim is to achieve the closed loop transfer function given in (7.5), then the plant numerator $\overline{B}$ must be cancelled in forming the control law. Also we require that the "observer" be the optimal Kalman observer $\overline{C}$. Hence the closed loop characteristic polynomial is required to be of the form: $A^{*} = \overline{C}E_2\overline{B}$. Thus, if we consider linear feedback control laws of the form:

$$Lu = -Gy + Dr \tag{7.7}$$

where $\nu(L) = n$; $\nu(G) = n-1$ and D is an appropriate exponentially stable transfer function, then the desired closed loop characteristic polynomial is achieved if G and L satisfy the following pole-assignment identity

$$E_2CB = L\overline{A} + G\overline{B} \tag{7.8}$$

In (7.8) and subsequent equations we will suppress the explicit dependence on the operator ϱ when this is clear from the context.

Since $\overline{B}$ is a factor of two of the terms in (7.8), we can write $L = F\overline{B}$. Then cancelling $\overline{B}$ in (7.8) gives

$$E_2\overline{C} = F\overline{A} + G \tag{7.9}$$

where

E_2 is the denominator of the reference model and is any monic Hurwitz polynomial with $\nu(E_2) = n - m$;

$\overline{C}$ is the Kalman observer as in equation (7.1);

F is a monic polynomial; $\nu(F) = n - m$;

G is a (not–necessarily monic) polynomial; $\nu(G) = n - 1$.

Note that (7.9) can always be solved for F and G, given E_2, $\overline{C}$ and $\overline{A}$ whether or not $\overline{A}$ and $\overline{B}$ are relatively prime.

We then have the following result:

Lemma 7.1: Consider the feedback control law (7.7) where $L = F\overline{B}$, $D = CM$ and F and G satisfy the identity (7.9). Then subject to assumption A4, the closed loop system is exponentially stable and (modulo exponentially decaying terms) y tracks the output of the reference model, y^*, with error $(F/E_2)\omega$ i.e.

$$y = (1/E_2)\overline{\xi} + (F/E_2)\omega \tag{7.10}$$

where $\overline{\xi} \triangleq M\xi$.

Proof: Immediate by substitution.

∇∇∇

In the next section we will construct a certainty equivalence adaptive form of the above control law.

8. DIRECT FORM OF THE ALGORITHM

To obtain a direct adaptive controller it is necessary to express (7.1) in a form such that the estimated parameters yield directly the control parameters. We proceed as follows:

To simplify the subsequent analysis we introduce the following notation. For every integer j, P^j is a row vector of polynomials (in ϱ) defined by:

$$P^j \triangleq (1, \varrho, \varrho^2, \ \dots \ , \varrho^j) \tag{8.1}$$

For any polynomial Γ of the form:

$$\Gamma(\varrho) \triangleq \gamma_p \varrho^p + \gamma_{p-1}\varrho^{p-1} + \ \dots \ + \gamma_0 \tag{8.2}$$

the coefficient vector $\mathrm{c}(\Gamma) \in \mathbb{R}^{p+1}$ is defined by

$$\mathrm{c}(\Gamma) \triangleq (\gamma_0, \gamma_1, \ \dots \ , \gamma_p) \tag{8.3}$$

Finally, for any signal z and any polynomial Γ, the signal z_Γ is defined by:

$$z_\Gamma \triangleq (1/\Gamma)z \tag{8.4}$$

In the sequel, we will define a number of regression vectors ψ, ϕ etc. depending on y, u in various ways. For convenience, the notation is summarized below.

<u>Notation for ϕ and ψ vectors</u>:

ψ_0, ϕ_0	full regression vectors containing ω (see (8.9), (8.10))
ψ, ϕ	full regression vectors with ω replaced by $\hat{\omega}$ (see (8.18), (8.19))
$\overline{\psi}, \overline{\phi}$	regression vectors with ω replaced by $\hat{\omega}$ and the last $n-m$ terms set to zero (see (8.22), (9.3))
ψ_c, ϕ_c	regression vectors containing ω and the last $n-m$ terms set to zero (see (8.13))

In all cases $\psi_* = (\phi_*/E_2)$

We have the following result:

Lemma 8.1:

(i) The model (7.1) can be rearranged into the following (non-minimal) form:

$$E_2 y = \left[\overline{G} y_{E_1} + \overline{F} G y_E + F\overline{F} B u_E - \overline{G} F \omega_E + (F - E_2)\omega\right] + E_2\omega \tag{8.5}$$

where E_1 is any Monic Hurwitz polynomial, $\nu(E_1) = n$; E_2 is as in (7.6); F and G satisfy (7.9) and $\overline{F}, \overline{G}$ satisfy

$$E \triangleq E_1 E_2 = \overline{F} C + \overline{G} \tag{8.6}$$

with $\overline{F}$ a monic polynomial; $\nu(\overline{F}) = n - m$; $\overline{G}$ a (not necessarily monic) polynomial; $\nu(\overline{G}) = n - 1$.

(ii) The model (8.5) can be expressed in the linear regression form (similar to (2.19)); i.e.

$$E_2 y = \phi_0^T \theta_0 + E_2 \omega \tag{8.7}$$

or

$$y = \psi_0^T \theta_0 + \omega \tag{8.8}$$

where

$$\phi_0^T \triangleq \left(P^{n-1} y_{E_1}, P^{2n-m-1} y_E, \overline{P}^{2n-m} u_E, P^{2n-m-1} \omega_E, P^{n-m-1} \omega\right) \tag{8.9}$$

$$\psi_0 \triangleq (1/E_2)\phi_0 \tag{8.10}$$

$$\theta_0^T \triangleq \left(\mathrm{c}(\overline{G}), \mathrm{c}(\overline{F}G), (\mathrm{c}(Q), b_m), \mathrm{c}(-\overline{G}F), \mathrm{c}(F - E_2)\right) \tag{8.11}$$

and

$$\overline{P}^{2n-m} \triangleq (P^{2n-m-1}, E) \tag{8.12}$$

so that $\overline{P}^{2n-m} u_E = (P^{2n-m-1} u_E, u)$.

It follows that θ_0, ϕ_0 and ψ_0 are p-dimensional vectors where $p \triangleq 8n - 4m + 1$ and $\phi_0^q = u$ and $\theta_0^q = b_m$ where $q \triangleq 5n - 2m + 1$.

(iii) The equivalent model (8.8) is in a form whereby the control law can be expressed as a direct function of θ_0. To do this, we define ϕ_c to be equal to ϕ_0 save that the last n–m (non causal) terms are set to zero:

$$\phi_c^T \triangleq (P^{n-1} y_{E_1}, P^{2n-m-1} y_E, \overline{P}^{2n-m} u_E, P^{2n-m-1} \omega_E, 0) \tag{8.13}$$

Then the stochastic model reference control law (7.7) is equivalent to

$$\phi_c^T \theta_0 = \overline{\xi} \tag{8.14}$$

which yields a closed loop response as in (7.10)

Proof: (i) Multiplying (7.1) by $F\overline{F}$ and substituting $E_2\overline{C} - G$ for $F\overline{A}$ and $E - \overline{G}$ for $\overline{F}\overline{C}$ yields the modified system equation:

$$E_2\overline{F}\overline{C}y = \overline{F}Gy + F\overline{F}Bu + F(E - \overline{G})\omega \tag{8.15}$$

Substituting $E - \overline{G}$ for $\overline{F}C$ in the left hand side of (8.3) yields (8.5) after some simple algebra.

(ii) Noting that $\overline{F}F\overline{B} = b_mE + \overline{Q}$ where $\nu(\overline{Q}) = 2n - m - 1$, (8.5) may be expressed as (8.7).

(iii) From (8.13), (8.9) we have

$$\phi_0^T = E_2\psi_0^T = \phi_c + (0,0,0,0,P^{n-m-1}\omega_{E_2}) \tag{8.16}$$

so that; from (8.5) and (8.7):

$$\begin{aligned} E_2y &= E_2\left[\psi_0^T\theta_0 + \omega\right] = \phi_0^T\theta_0 + E_2\omega \\ &= \left[\overline{G}y_{E_1} + \overline{F}Gy_E + \overline{F}FBu_E - \overline{G}F\omega_E + F\omega\right] \\ &= \phi_c^T\theta_0 + F\omega \end{aligned} \tag{8.17}$$

Hence substituting the control law (8.14) into (8.17) immediately gives (7.10) i.e.

∇∇∇

The above result shows how the system model can be rearranged into a form in which the control law is a direct function of the model parameters. However, the regressor vector ψ_0 cannot be used for estimation since ω is not known. Therefore, as is in Section 3, we introduce a pseudo-regressor ψ obtained by replacing ω with $\hat{\omega}$. Thus we define

$$\psi \triangleq (1/E_2)\phi \tag{8.18}$$

where

$$\phi \triangleq \left(P^{n-1}y_{E_1}, P^{2n-m-1}y_E, \overline{P}^{2n-m}u_E, P^{2n-m-1}\hat{\omega}_E, P^{n-m-1}\hat{\omega}\right) \tag{8.19}$$

$$\hat{\omega} \triangleq y - \hat{y}, \tag{8.20}$$

$$\hat{y} \triangleq \psi^T \hat{\theta} = \hat{\overline{G}} y_E + (\overline{F}\hat{G}) y_{EE_2} + (F\hat{\overline{F}}B) u_{EE_2} - (\hat{\overline{G}}F)\hat{\omega}_{EE_2} + (F - \hat{\ } E_2)\hat{\omega}_{E_2} \tag{8.21}$$

and $\hat{\theta}$ is the current estimate of θ.

Similarly, we define the certainty equivalence stochastic model reference control law as follows: We first define $\overline{\phi}$ by replacing ω, θ_0 by $\hat{\omega}, \hat{\theta}$ respectively in the definition of ϕ_c (see (8.13)):

$$\overline{\phi}^T \triangleq \left(P^{n-1} y_{E_1}, P^{2n-m-1} y_E, \overline{P}^{2n-m} u_E, P^{2n-m-1} \hat{\omega}_E, 0 \right) \tag{8.22}$$

Then, the certainty equivalence control law is defined as in (8.14) but with ϕ_c, θ_0 replaced by $\overline{\phi}, \hat{\theta}$ respectively:

$$\overline{\phi}^T \hat{\theta} = \overline{\xi} \tag{8.23}$$

We then have the following result on certainty equivalence adaptive control:

Lemma 8.2:

(i) Using the regression vector ψ to define $\hat{y}$ as in (8.21) gives

$$y - \omega = \hat{y} + \eta = \psi_0^T \theta_0 \tag{8.24}$$

where

$$\eta \triangleq \hat{\omega} - \omega \tag{8.25}$$

is the "noise reduced prediction error".

(ii) η is related to the parameter error $\tilde{\theta}$ as follows:

$$(F\overline{FC}/EE_2)\eta = -\psi^T \tilde{\theta} \tag{8.26}$$

where

$$\tilde{\theta} \triangleq \hat{\theta} - \theta_0 \tag{8.27}$$

(iii) Provided $\hat{\theta}$ is constrained so that $\hat{\theta} \in \mathcal{C}_2$ then the control law (8.23) is well posed and involves proper functions of the input and output.

(iv) Use of the certainty equivalence control law (8.23) gives a closed loop system satisfying:

$$E_2 y = [\bar{\xi} + F\omega] + \left[(\overline{G}F/E)\eta - \bar{\phi}^T\tilde{\theta}\right] \tag{8.28}$$

$$E_2\overline{B}u = \overline{A}(\bar{\xi} - \bar{\phi}^T\tilde{\theta}) + [\overline{AG}F/E]\eta - G\omega \tag{8.29}$$

Proof: Straightforward by substitution – see [10].

∇∇∇

Comparing (7.10) with (8.28) we see that the adaptive controller, when compared with 'ideal' controller, introduces the extra errors $\bar{\phi}^T\tilde{\theta}$ and $(\overline{G}F/E)\eta$. One main concern will be to quantify these errors.

9. ESTIMATION ALGORITHM REVISITED

The estimation algorithm is exactly as in (3.2) with ψ_t etc. redefined as above. We also replace the projection scheme given at the end of Section 3 by:

Modified Projection Scheme:

Let τ be the time for which the solution of (3.2) is such that $\hat{\theta}_t$ lies on the boundary of $\mathcal{C}_2$ (defined in section 7). Denote the corresponding value of $\hat{\theta}_\tau$ by $\hat{\theta}_{\tau-}$. At time τ, the estimate $\hat{\theta}_\tau$ is then defined as the point in $\mathcal{C}_1$ closest to $\hat{\theta}_{\tau-}$. For $t \geq \tau$, equation (3.2) is then integrated with initial condition $\hat{\theta}_\tau$. This makes $\hat{\theta}_t$ right continuous at the projection times and ensures that $\hat{\theta}_t \in \mathcal{C}_2$ for all t.

The initial value of $\hat{\theta}_0$ is chosen so that $\hat{\theta}_0 \in \mathfrak{e}_1$.

For the case of stochastic model reference adaptive control we need to modify the normalization variable r_t to also bound the derivatives of $\overline{\psi}$. This will be needed in our subsequent analysis. We thus define r_t as follows: for all $j \in 1, 2 \; \ldots \; , n-m$, μ_t^j is defined by

$$\mu_t^j \triangleq \sup_{\tau \in [0,t]} \left\{ \| \varrho^{j-1} \overline{\psi}_\tau \|^2 \right\} + C_0 \tag{9.1}$$

whereas μ_t^0 is given by:

$$\mu_t^0 \triangleq \sup_{\tau \in [0,t]} \left\{ \| \psi_\tau \|^2 \right\} + \int_0^t \| \psi_\tau \|^2 \, d\tau + C_0 \tag{9.2}$$

and $\overline{\psi}$ is defined to be $(1/E_2)\overline{\phi}$, i.e. using (8.26);

$$\overline{\psi}^T \triangleq (1/E_2) \left\{ P^{n-1} y_{E_1} \; ; \; p^{2n-m-1} y_E \; ; \; \overline{p}^{2n-m} u_E \; ; \; p^{2n-m-1} \hat{\omega}_E \; ; \; 0 \right\} \tag{9.3}$$

Also C_0 in (9.1) and (9.2) is an arbitrary positive constant. Finally, the normalization variable r_t is defined by:

$$r_t \triangleq \max \left\{ \mu_t^j | j \in \{0, 1, \; \ldots \; , n-m\} \right\} \tag{9.4}$$

Clearly, r_t, is not less than μ_t^0 which we observe is the normalization employed in Section 3.

To establish convergence properties of the adaptive control law, we need the following additional assumption which is analogous to Assumption 1 of Section 3:

A5: The transfer function relating η to $-\psi^T \tilde{\theta}$, i.e. ,$(F\overline{F}C/EE_2)$ is strictly passive, i.e. there exists a $\beta > 0$ such that (ignoring the term due to initial conditions):

$$\int_0^t \bar{y}_s \bar{u}_s ds \geq \int_0^t \beta \bar{u}_s^2 ds$$

for all $t > 0$ and all $\bar{u}$ where $\bar{y} \triangleq (F\overline{FC}/EE_2)\bar{u}$.

We then immediately have the properties established in Lemma 5.1, Lemma 5.2 and Theorem 5.1.

10. CLOSED LOOP PROPERTIES

We will again employ Lemma 6.1. To do this we must first establish the growth condition given in (6.3). This is relatively straightforward using the inverse stability of the plant (assumption A4) – see [10] for details.

Finally, we obtain an extended form of theorem 6.1, namely:

Theorem 10.1: Consider the system (2.1) and (2.2) (equivalently, (7.1)), the parameter estimator of Sections 3 and 9, and the certainty equivalence controller of (8.23). Assume that assumptions A1 – A5 are satisfied, and that the reference input ξ is piecewise continuous and uniformly bounded. Then the algorithm converges in the sense that

(i) The regression vector is sample m.s. bounded almost surely; i.e.

$$\lim_{t \to \infty} \sup (1/t) \int_0^t \| \psi_s \|^2 ds < \infty \quad \text{a.s.} \tag{10.1}$$

(ii) The noise reduced output is sample m.s. bounded almost surely; i.e.

$$\lim_{t \to \infty} \sup (1/t) \int_0^t \| \overline{C} x_s \|^2 ds = \lim_{t \to \infty} \sup (1/t) \int_0^t (y_s - \omega_s)^2 ds < \infty \quad \text{a.s.} \tag{10.2}$$

(iii) The input is sample m.s. bounded almost surely; i.e.

$$\limsup_{t\to\infty}(1/t)\int_0^t u_s^2 ds < \infty \text{ a.s.} \tag{10.3}$$

(iv) The following asymptotic tracking performance is achieved:

$$\limsup_{t\to\infty}(1/t)\int_0^t (y_s - y_s^{\#})^2 ds = 0 \text{ a.s.} \tag{10.4}$$

where $y_s^{\#}$ is the desired reference model response together with a noise term, i.e.

$$y^{\#} = (1/E_2)\bar{\xi} + (\hat{F}/E_2)\omega \tag{10.5}$$

where $\hat{F}$ is the estimate of the polynomial F.

(Equations (10.4), (10.5) should be compared with (7.10) for the case of known parameters.) ❑

Proof: Basically as for the proof of theorem 6.1 – see [10] for details

∇∇∇

The above theorem establishes both stability ((10.1) to (10.3)) as well as asymptotic convergence to the desired tracking objective ((10.4), (10.5)).

A key assumption in the above analysis has been the requirement that the relative degree be known. We have recently shown [19] that this assumption can be removed by running parallel estimators corresponding to each possible relative degree and then using a switching strategy to decide which estimator to use for the purpose of defining the feedback control law.

11. PERSPECTIVE

The convergence analyses given above establish key existence results for stochastic adaptive control. Their significance is that they establish stability of the closed loop which is clearly an essential property. However, the form of stability is weak (L_2 functions can still be unbounded). Moreover, they fall well short of what is needed in practical applications. The difficulty is that the beauty of the theory lies in its critical dependence on the assumptions. Paradoxically this also equates to a practical weakness. Indeed, a key practical requirement is likely to be that one have insensitivity to assumptions but this is clearly contrary to the requirements of the current precisely "honed" theory.

An issue which is yet to be addressed satisfactorily is that of nasty effects in the plant (nonlinearities, unmodelled dynamics, nonstationary noise etc.) These effects need to be accounted for in the design of practical stochastic adaptive control algorithms.

An initial step has been taken in [20] where we show that it is possible to quantify the errors due to both noise and undermodelling in the estimated model with finite data. In principle this permits the following actions:

- use of robust control design methods which account for model inadequacy.
- optimal determination of model order. (Indeed, we argue in [20] that this will never be the true model order but a lower order model which gives an optimal trade-off between bias errors (due to undermodelling) and variance errors (due to noise)).

However, these are very preliminary results and much more work is needed to overcome the restrictions of the Certainty Equivalence approach. We suggest that a useful area for future research would be to study adaptive control algorithms which, inter alia, incorporate both an estimated nominal model together with a quantification of its accuracy accounting for noise, undermodelling, finite data and possible time variations. These algorithms will almost certainly look very different from those outlined above.

12. REFERENCES

[1] Feuer A. and S. Morse (1978). Adaptive control of single input single output linear systems. *IEEE Trans. Auto. Control*, AC-23, pp.557-570.

[2] Goodwin, G.C., P.J. Ramadge and P.E. Caines (1980). Discrete time multivariable adaptive control. *IEEE Trans. Auto. Control,* AC-25, pp.449-456.

[3] Morse, A.S. (1980). Global stability of parameter–adaptive control systems. *IEEE Trans. Auto. Control*, AC–25, pp.433–439.

[4] Goodwin, G.C. and D.Q. Mayne (1987). A parameter estimation perspective of continuous time model reference adaptive control. *Automatica,* 23, pp.57–70.

[5] Astrom, A.S. and B. Wittenmark (1973). On self tuning regulators. *Automatica,* 9, pp.195–199.

[6] Ljung, L. Analysis of recursive stochastic algorithms. *IEEE Trans. Auto. Control,* AC–22, pp.551–575.

[7] Goodwin, G.C., P.J. Ramadge and P.E. Caines (1981). Discrete time stochastic adaptive control. *SIAM J. Control and Optimization,* 19, pp.829–853.

[8] Middleton, R.H. and G.C. Goodwin (1990). Digital Control and Estimation: A Unified Approach. Prentice–Hall.

[9] Gevers, M., G.C. Goodwin and V. Wertz (1987). Continuous time stochastic adaptive control. To appear, *SIAM Journal on Control and Optimization,* 1991.

[10] Goodwin, G.C. and D.Q. Mayne (1991). Continuous time stochastic model reference adaptive control. To appear *IEEE Trans. Auto. Control.*

[11] Goodwin, G.C. and K.S. Sin (1984). Adaptive Filtering Prediction and Control, Prentice Hall.

[12] Morse, S. (1990). Private Correspondence.

[13] Jacod, J. (1979). Calcul Stochastique et Problemes de Martingales, 714, Lecture Notes in Mathematics, Springer Verlag, Berlin.

[14] Wong, E. (1971). Stochastic Processes in Information and Dynamical Systems. McGraw Hill.

[15] Deheuvels (1980). L'integrale. Presses Universitaires de France, Mathematiques.

[16] Chen, H.F. Recursive Estimation and Control for Stochastic Systems. Wiley, New York.

[17] Kreisselmeier, G. (1985). An approach to stable indirect adaptive control. *Automatica*, 21, 4, pp.425–433.

[18] Desoer, C.A. and M. Vidyasagar (1975). Feedback Systems: Input–Output Properties. Academic Press.

[19] Morse, A.S., D.Q. Mayne and G.C. Goodwin (1990). Identifier–based, switched–parameter algorithms for the adaptive stabilization of linear systems. *The Sixth Yale Workshop on Adaptive and Learning Systems.*

[20] Goodwin, G.C., M. Gevers and B. Ninness. Optimal model order selection and estimation of model uncertainty for identification with finite data. Submitted to *IEEE Trans. on Auto. Cont. Special Issue on System Identification for Control Design.*