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Central Extensions in Closed-loop Optimal Experiment Design

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Abstract— **We consider optimal experiment design for parametric prediction error system identification of linear time-invariant multi-input multi-output systems in closed-loop when the true system is in the model set. The optimization is performed jointly over the controller and the spectrum of the external input. Previously we tackled this problem by parametrizing the set of admissible controller – external input pairs by a finite set of matrix-valued trigonometric moments and derived a description of the set of admissible finite-dimensional moment vectors by a linear matrix inequality. Here we present a way to recover the controller and the power spectrum of the external input from the optimal moment vector. To this end we prove that the central extension of the finite moment sequence yields a feasible solution. This yields the joint power spectrum of the input and the noise vector as an explicit rational function and allows to construct the optimal "controller – external input pair" directly from the optimal moment vector.**

I. INTRODUCTION

Optimal experiment design for system identification has seen an intense development in the last decade. This advance was initiated by the appearance of modern convex optimisation methods in the nineties. Most of the recent work in optimal experiment design focusses on casting experiment design problems as semidefinite programs. One of the pioneering contributions introducing semidefinite programming into system identification was [19]; see also [11].

However, converting optimisation problems into semi-definite programs is often far from trivial. If a semi-definite description cannot be obtained, one usually tries to relax the problem in order to construct a semi-definite approximation. Often such a relaxation is easily at hand, but nothing about its quality is known. The description of an optimal experiment design problem as a semi-definite program is principally determined by the choice of the *design variables*, i.e., those quantities whose values the solver has to optimize. They have to enter both the constraints and the cost criterion of the experiment design problem *linearly*. Moreover, the set of *admissible* vectors of design variables has to be semidefinite representable. This means that the condition on the vector of design variables to correspond to a realizable experiment design has to be equivalent to the satisfaction of a linear matrix inequality (LMI), possibly involving additional auxiliary variables.

Here we consider optimal experiment design for parametric closed-loop identification of discrete linear time-invariant (LTI) systems: the joint optimal design of both controller and external input for the identification experiment is sought.

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The external input usually enters into the cost criteria and the constraints in the form of its power spectrum, the controller in the form of its transfer function. Both can easily be converted into a joint power spectrum of signals present in the control loop during identification. These spectra are infinite-dimensional objects. Their infinitely many degrees of freedom have to be condensed into a finite-dimensional vector of design variables.

In [10],[9] we provided a semi-definite description of the set of input-controller pairs constituting valid experimental setups in the framework of the *partial correlation approach* (see e.g. [8], [11]). In this framework, the design variables are the values of a finite number of *linear functionals* on the joint power spectrum in question. These values are called (generalized) *moments* [14], [17] of the spectrum. The linear functionals are chosen in such a way that both the constraints and the cost function depend only on a finite number of moments. Plugging his particular constraints and cost criteria into the scheme outlined in [9], the user obtains the optimal truncated moment sequence by solving a semi-definite program. Geometrically, the optimization is performed over a finite-dimensional *projection* of the infinite-dimensional cone of possible joint power spectra. The optimal truncated moment sequence will then in general correspond to an infinite set of spectra rather than a single spectrum.

In this contribution, which can be seen as a continuation of [10],[9], we focus on the problem of how to recover a particular joint power spectrum, or equivalently, an "external input spectrum – controller pair", from the optimal truncated moment sequence. This kind of problem is known under the name *trigonometric moment problem* or *Carathéodory extension problem*. The case of scalar-valued moments has been well studied in the last century [3], [21], [2], [17], [14], [1]. The scalar theory can be generalized to the case of matrix-valued moments [4], [5], [16], [6]. The key result for solving the Carathéodory extension problem is the Carathéodory-Fejer theorem. This theorem implies that a given finite sequence of moments is indeed generated by a positive power spectrum if and only if it satisfies a certain LMI [15, Chapter VI, Theorem 4.1]. The set of all possible extensions of a finite moment sequence is parameterized by an infinite sequence of complex numbers (in the scalar case) or matrices (in the matrix case), the *Schur parameters* [1], [5]. The particular extension corresponding to the case when all Schur parameters vanish is called *central extension* [6]. The power spectrum corresponding to this extension can be expressed in closed-form as a rational function with coefficients depending in an explicit manner on the problem data, i.e., on the optimal truncated moment sequence.

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The classical Carathéodory-Fejer theorem holds *only* if no restrictions are imposed on the spectrum other than to produce the truncated sequence of moments under consideration, and positivity. However, in closed-loop optimal experiment design, where the controller is part of the design variables, constraints have to be imposed on the matrix-valued joint power spectrum under consideration. These constraints reflect the fact that the controller must produce a stable closed loop. They translate into additional constraints on the infinite moment sequence of the spectrum. In [10],[9] we have shown that the Carathéodory-Fejer theorem *also* holds for the type of structured trigonometric moment problem arising in closed-loop optimal experiment design. Namely, if a finite sequence of moments satisfies the additional constraints, then the LMI condition given by the Carathéodory-Fejer theorem not only insures the existence of a general extension of this moment sequence, but the existence of an extension which also satisfies the constraints.

Our main contribution here is to show that these constraints are satisfied already by the central extension. This means that the explicit joint power spectrum obtained from the optimal moment sequence by the central extension corresponds to a feasible experimental design. Thus once the optimal truncated moment sequence has been obtained by solving the semi-definite program, an optimal joint power spectrum can be immediately written down in closed form, shortcutting the complicated recovery step in [9]. As a side result, feasibility of the central extension actually implies the validity of the Carathéodory-Fejer theorem for the structured trigonometric moment problem, which also significantly shortens the proof of this result. For this reason, and in order to make the present contribution self-contained, we also provide the new proof of the Carathéodory-Fejer theorem.

Note that rather than solving a particular problem, we present a scheme to solve a class of problems. Accordingly, the focus of our contribution will not be on the constraints and the cost function defining a particular optimal experiment design problem instance, but on an algorithm for the construction of an optimal input-controller pair from the optimal moment sequence. We allow the system to have multiple inputs and outputs (MIMO), but impose the condition that the true system is within the model structure.

The remainder of the paper is structured as follows. In the next section we define the class of experiment design problems to be solved. In Section 3 we introduce the concept of central extensions. In Section 4 we show the feasibility of the central extension for optimal closed-loop experiment design. In Section 5 we devise a solution algorithm for the proposed class of problems, and in Section 6 we illustrate the new theory with an example.

II. PROBLEM FORMULATION

We intend to perform parametric prediction error identification of a MIMO LTI system in closed loop. The system dynamics is given by the relation

$$y = G_0(q)u + H_0(q)e, \tag{1}$$

where the signal u is of dimension l_1, and e, y are of dimension l_2. Here G_0 is the plant transfer function, H_0 the noise transfer function, q the forward-shift operator, e a vector-valued zero mean white noise with (co-)variance $\lambda_0 I_{l_2}$, I_k being the $k \times k$ identity matrix[1]. The transfer function matrices $G_0(z), H_0(z)$ are embedded in a model structure $G(z;\theta), H(z;\theta)$ and correspond to some true parameter value θ_0, $G_0(z) = G(z;\theta_0)$, $H_0(z) = H(z;\theta_0)$. We assume the noise model H_0 to be stable and inversely stable.

The parameter vector θ_0 is to be identified by an experiment, which consists in closing the loop according to

$$u = -K(q)y + r, \tag{2}$$

where r is of dimension l_1, and collecting a set of input-output data u, y. The design variables at our disposal are thus the external vector-valued input signal r and the $l_1 \times l_2$ matrix-valued feedback controller K. The configuration of the identification experiment is schematically depicted in Fig. 1. The estimator $\hat{\theta}$ of the true parameter value θ_0 is then evaluated as the minimizer of some prediction error criterion. Our goal is to design an experiment by choosing an external input r and a controller K such that some cost function of r, K is minimized and some constraints on the pair (r, K) are satisfied.

Following [12], we first move from the quantities r, K to the spectra Φ_u, Φ_{ue}, which, as long as we work in the frequency domain and use formulas that are asymptotic in the number of data, yield an equivalent description of the experimental conditions. The power spectrum Φ_r of r and the controller K determine Φ_u, Φ_{ue} by the formulas

$$\begin{aligned} \Phi_u(\omega) &= \lambda_0(I_{l_1} + KG_0)^{-1}KH_0H_0^*K^*(I_{l_1} + KG_0)^{-*} \\ &\quad +(I_{l_1} + KG_0)^{-1}\Phi_r(\omega)(I_{l_1} + KG_0)^{-*}, \qquad (3) \\ \Phi_{ue}(\omega) &= -\lambda_0(I_{l_1} + KG_0)^{-1}KH_0, \qquad (4) \end{aligned}$$

where the transfer functions on the right-hand side are evaluated at $z = e^{j\omega}$. By A^{-*} we denote the inverse of the complex conjugate transpose of a matrix A. On the other hand, Φ_r and K can be recovered from Φ_u, Φ_{ue} by the formulas

$$\begin{aligned} \Phi_r &= (I_{l_1} + KG_0)(\Phi_u - \lambda_0^{-1}\Phi_{ue}\Phi_{ue}^*)(I_{l_1} + KG_0)^*, \\ K &= -\Phi_{ue}(\lambda_0 H_0 + G_0\Phi_{ue})^{-1}. \end{aligned} \tag{5}$$

Parametrizing the experimental conditions by the joint power spectrum

$$\Phi_{\chi_0} = \begin{pmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{ue}^* & \lambda_0 I_{l_2} \end{pmatrix} \tag{6}$$

of the signals u, e instead of the quantities r, K has the advantage that the feasible set becomes convex, which is a prerequisite for a semi-definite representation [12]. The matrix Φ_{χ_0} is of size $l \times l$ with $l = l_1 + l_2$.

Within the framework of the partial correlation approach, the ultimate design variables are a finite set of moments

[1]For simplicity, we have assumed a white noise (co-)variance $\lambda_0 I_{l_2}$; however, our results apply equally well for any symmetric positive definite (co-)variance matrix Σ.

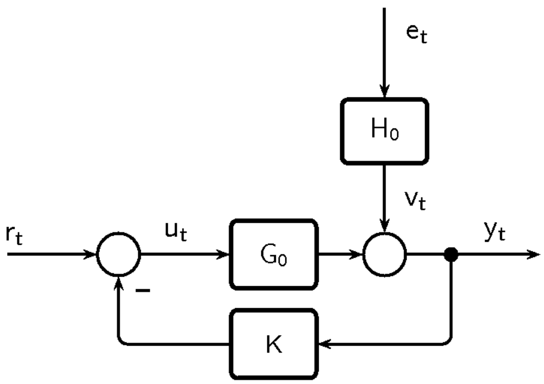

Fig. 1. Experimental setup

of the joint power spectrum Φ_{χ_0}, which we define as follows. Consider a polynomial $d(z) = \sum_{k=0}^{m} d_k z^k$ of degree $m, m \geq 0$ such that the coefficients d_k are real, obey $d_0 \neq 0$, $d_m \neq 0$, and the polynomial $d(z)$ has all roots outside of the closed unit disk. Define $l \times l$ matrices

$$m_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1}{|d(e^{j\omega})|^2} \Phi_{\chi_0}(\omega) e^{jk\omega}\, d\omega \qquad (7)$$

for integral k. The matrices m_k defined by (7) are called the *generalized moments* of the spectrum Φ_{χ_0}. Note that the moments m_k are real and obey the relation $m_k = m_{-k}^T$.

Accordingly, the cost criterion and the constraints of the optimal experiment design problem have to be expressible in a tractable manner in terms of these moments, as specified by Assumption 1 below. Such is the case for integral criteria and constraints, but not for frequency-wise criteria or constraints. For such criteria, a suboptimal solution based on a finite dimensional spectrum parametrization and on the use of a finite-dimensional approximation of a Youla parameter was proposed in [12].

Assumption 1: There exist integers $N \geq 0$, $n \geq m \geq 0$ such that the constraints of the experiment design problem can be written as a linear matrix inequality

$$\exists\ x_1, x_2, \dots, x_N :\ \mathcal{A}(m_0, m_1, \dots, m_n, x_1, \dots, x_N) \succeq 0$$

in the elements of the matrices m_k, $k = 0, \dots, n$ and N auxiliary variables $x_k, k = 1, \dots, N$, and the cost function of the experiment design problem is given by a linear function

$$f_0(m_0, \dots, m_n, x_1, \dots, x_N) = \sum_{k=0}^{n} \langle C_k, m_k \rangle + \sum_{k=1}^{N} c_k x_k,$$

where C_k are fixed matrices, c_l are fixed reals, and $\langle A, B \rangle = trace(AB^T)$ is the usual scalar product in the space of matrices.

In [10],[9] we presented a semi-definite description of the set of finite moment sequences $(m_0, \dots, m_n)$ corresponding to valid experiment designs. This allows to obtain the optimal truncated moment sequence $(m_0, \dots, m_n)$ by solving a semi-definite program.

Under some mild assumptions the asymptotic in the number of data average per data sample information matrix of the experiment is given by [20]

$$\overline{M} = \frac{1}{2\pi\lambda_0} \sum_{k=1}^{l_2} \int_{-\pi}^{+\pi} F_k(e^{j\omega}) \Phi_{\chi_0}(\omega) F_k^*(e^{j\omega})\, d\omega, \qquad (8)$$

where the l-th row of the matrix F_k is given by the k-th row of the matrix $[H_0^{-1} G'_{\theta^l}(\theta_0),\ H_0^{-1} H'_{\theta^l}(\theta_0)]$. Here $G'_{\theta^l},\ H'_{\theta^l}$ denote the gradients of $G(z;\theta), H(z;\theta)$ with respect to the l-th entry of the parameter vector θ. If the model structure is rational, then (8) is affine in a finite number of matrices $m_0, m_1, \dots, m_n$ for a suitably chosen polynomial $d(z)$. In addition, most experiment design criteria are formulated as scalar functions of $\bar{M}$. Therefore, Assumption 1 covers a wide variety of problem formulations in closed-loop optimal experiment design, see also [19],[12]. In particular, all classical designs (D-optimal, A-optimal, L-optimal etc.) fall within the framework of Assumption 1.

III. CENTRAL EXTENSIONS

In this section we introduce the concept of central extensions. Before we focus on the generalized moments (7) of the structured power spectrum (6), we will first consider the case of moment sequences of general power spectra.

Let $\Phi(\omega)$ be an integrable 2π-periodic complex-Hermitian matrix-valued positive semi-definite function of size $l \times l$, possibly containing a singular part consisting of Dirac δ-functions. Define the moments of Φ by

$$m_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Phi(\omega) e^{jk\omega}\, d\omega. \qquad (9)$$

Note that $m_{-k} = m_k^*$. Then the block-Toeplitz matrices

$$T_k = \begin{pmatrix} m_0 & m_1^* & \ddots & m_{k-1}^* & m_k^* \\ m_1 & m_0 & \ddots & m_{k-2}^* & m_{k-1}^* \\ \ddots & \ddots & \ddots & & \\ m_k & m_{k-1} & \ddots & m_1 & m_0 \end{pmatrix} \qquad (10)$$

are positive semi-definite for all k. On the other hand, given an infinite sequence of matrices m_k, $k \in \mathbb{Z}$, satisfying $m_{-k} = m_k^*$ and such that all block-Toeplitz matrices T_k are positive semi-definite, there exists a unique positive semi-definite function $\Phi(\omega)$ producing the matrices m_k as in (9) [4, Theorem 1]. Note that if $\Phi(-\omega) = \Phi(\omega)^T$, then all moments m_k are real, and the complex conjugate transpose in (10) becomes the ordinary transpose. For the sake of simplicity, we will consider only the case when the matrices T_k are positive definite. However, with the methods in [5, Appendix A] it is also possible to treat the singular case, and the degeneracy of T_k is not a fundamental obstacle.

Following [4], define the $l \times (k+1)l$ matrix functions

$$U_k(z) = \begin{pmatrix} z^k I_l & z^{k-1} I_l & \cdots & I_l \end{pmatrix},$$

the matrix-valued polynomials

$$A_k(z) = U_k(z) T_k^{-1} U_k^T(0),$$

and the rational matrix-valued functions

$$\Phi_k(\omega) = A_k(e^{j\omega})^{-*} A_k(0) A_k(e^{j\omega})^{-1}.$$

Note that Φ_k depends only on the moments $m_0, \dots, m_k$. By [4, Theorem 6] the polynomials $A_k(z)$ have no zeros in the closed unit disk, by [4, Theorem 3] the functions Φ_k are positive definite, and by [4, Theorem 9] $(m_0, \dots, m_k)$ is the truncated moment sequence of Φ_k up to order k. The function Φ_k hence shares the first $k+1$ moments with Φ. It is called the k-th order *rational approximation* of Φ.

Given a finite moment sequence $(m_0, m_1, \dots, m_n)$ such that the block-Toeplitz matrix T_n constructed from this sequence is positive definite, we can define the function

$$\Phi(\omega) = A_n(e^{j\omega})^{-*} A_n(0) A_n(e^{j\omega})^{-1}. \tag{11}$$

By the above, Φ is the n-th order rational approximation of itself and $(m_0, m_1, \dots, m_n)$ are its first $n+1$ moments. This Φ uniquely defines higher order moments $m_{n+1}, m_{n+2}, \dots$ via (9). Moreover, Φ also coincides with its rational approximation $\Phi_{n'}$ and $A_{n'}(z) = A_n(z)$ for all $n' \geq n$ [4, Theorem 9]. Let A_n^k be the matrix coefficient of the polynomial $A_n(z)$ at z^k. By definition, A_n^k is the $(n+1-k, n+1)$-th $l \times l$ block of the inverse T_n^{-1}. Note also that $A_n(0) = A_n^0$.

An *extension* of a truncated moment sequence $(m_0, \dots, m_n)$ is a, possibly infinite, sequence $(m_0, \dots, m_n, m_{n+1}, \dots)$ obtained from some positive semi-definite function Φ by formula (9). By the Carathéodory-Fejer theorem (see, e.g., [15, Chapter VI, Theorem 4.1]) an extension exists if and only if $T_n \succeq 0$. If Φ is given by (11), then $(m_0, \dots, m_n, m_{n+1}, \dots)$ is called the *central extension*. If the m_k are real, then the coefficients A_n^k are also real, and $\Phi(-\omega) = \Phi(\omega)^T$. Hence all moments of the central extension will be real.

IV. MAIN RESULT

In this section we return to our optimal closed-loop experiment design problem described in Assumption 1. First we describe the linear relations between the optimal moments $m_0, \dots, m_n$. With the help of these relations we then use the central extension of the truncated moment sequence $(m_0, \dots, m_n)$ to recover the joint power spectrum (6) which realizes the sequence according to formula (7).

A. Structure of the moments

In this subsection we briefly recall the results from [10],[9] on the linear relations between the moments $m_0 = m_0^T, m_1, \dots, m_n$. Set $m_{-k} = m_k^T$ and partition the $l \times l$ matrix moments m_k into 4 blocks $m_{k,11}, m_{k,12}, m_{k,21}, m_{k,22}$, according to the partition of $\mathbb{R}^l$ into a sum $\mathbb{R}^{l_1} \oplus \mathbb{R}^{l_2}$. The moment matrices m_k, defined by formula (7), depend on the spectra Φ_u, Φ_{ue}, which in turn determine the experimental conditions. However, not all pairs (Φ_u, Φ_{ue}), and hence not all sequences $(m_0, \dots, m_n)$, correspond to valid experiment designs.

From (7) it follows that

$$m_{k,22} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_{l_2}}{|d(e^{j\omega})|^2} e^{jk\omega} \, d\omega \tag{12}$$

for all $k = -n, \dots, n$. The positivity of the joint power spectrum Φ_{χ_0} implies by the Carathéodory-Fejer theorem that the block-Toeplitz matrix

$$T_n = \begin{pmatrix} m_0 & m_1^T & \ddots & m_{n-1}^T & m_n^T \\ m_1 & m_0 & \ddots & m_{n-2}^T & m_{n-1}^T \\ \ddots & \ddots & \ddots & & \\ m_n & m_{n-1} & \ddots & m_1 & m_0 \end{pmatrix} \tag{13}$$

is positive semi-definite. Further, the transfer functions from the signals r, e to the signals u, y are stable. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. Then the function $f_{ue} : \mathbb{T} \to \mathbb{C}^{l_1 \times l_2}$, defined by the cross spectrum Φ_{ue} by means of $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$, can be extended to a holomorphic function outside of the unit disc, including the point at infinity (compare also [12]). From

$$m_{k,12} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1}{d(e^{j\omega})} \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})} e^{jk\omega} \, d\omega$$

it follows that

$$\sum_{i=0}^{m} d_i m_{k+i,12} = \frac{1}{2\pi j} \int_{\mathbb{T}} \frac{f_{ue}(z)}{d(z^{-1})} z^{k-1} \, dz.$$

Since all zeros of $d(z^{-1})$ are in the open unit disc, the ratio $f_{ue}(z)/d(z^{-1})$ is also holomorphic outside of the unit disc. It follows that $\sum_{i=0}^{m} d_i m_{k+i,12} = 0$ for all $k < 0$, and hence

$$\sum_{i=0}^{m} d_i m_{k-i,21} = 0, \qquad k = 1, \dots, n. \tag{14}$$

Similarly it follows that the matrices (12) satisfy

$$\sum_{i=0}^{m} d_i m_{k-i,22} = 0, \qquad k = 1, \dots, n. \tag{15}$$

In [9, Theorem 1] we have shown for the single-input single-output case that the conditions (12), (14), and $T_n \succeq 0$ are sufficient to guarantee the existence of a positive semi-definite joint power spectrum (6), satisfying $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$, such that Φ_{ue} represents a stable transfer function, which reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula (7). The proof extends also to the MIMO case considered here. The result [9, Theorem 1] is, however, non-constructive, because it does not yield an explicit power spectrum Φ_{χ_0}, but merely proves its existence.

In the next section we give a constructive proof by showing that the explicit power spectrum obtained by the central extension is feasible. We will require the non-degeneracy condition $T_n \succ 0$, but, as mentioned above, including the singular case is a technical but not a fundamental difficulty.

B. Feasibility of the central extension

In this subsection we shall prove the following result.

Theorem 1: Let $(m_0, \dots, m_n)$ be a $(n+1)$-tuple of real $l \times l$ matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k = 1, \dots, n$. Suppose that these matrices satisfy conditions (12), (14), and $T_n \succ 0$, where T_n is given by (13). Then the rational power spectrum $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot \Phi(\omega)$, where $\Phi(\omega)$ is given by (11) as an explicit function of $m_0, \dots, m_n$, satisfies the following properties. It is of the form (6), positive definite, satisfies $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$, its upper right block Φ_{ue} represents a stable transfer function, and it reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula (7).

Proof: That Φ_{χ_0} is positive definite and reproduces the truncated moment sequence $(m_0, \dots, m_n)$ is a consequence of Theorems 3 and 9 in [4], respectively, and the absence of roots of $d(z)$ on the unit circle. The relation $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$ follows from the realness of the coefficients A_n^k, which in turn is a consequence of the realness of the matrices m_k.

By (14), (15) the last l_2 rows of the $l \times (n+1)l$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & d_m I_l & d_{m-1} I_l & \cdots & d_0 I_l \end{pmatrix} T_n$$

are given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \sum_{i=0}^m d_i m_{-i,21} & \sum_{i=0}^m d_i m_{-i,22} \end{pmatrix}.$$

Recall that the last l rows of the inverse T_n^{-1} are given by $((A_n^n)^T \; (A_n^{n-1})^T \; \dots \; A_n^0)$. It follows that

$$\begin{pmatrix} 0 & d_k I_{l_2} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^m d_i m_{-i,21} & \sum_{i=0}^m d_i m_{-i,22} \end{pmatrix} (A_n^k)^T,$$

where we put $d_k = 0$ for $k > m$ by convention. Multiplying by z^k and summing over k, we obtain after transposition

$$\begin{pmatrix} 0 \\ d(z) I_{l_2} \end{pmatrix} = A_n(z) \begin{pmatrix} \sum_{i=0}^m d_i m_{i,12} \\ \sum_{i=0}^m d_i m_{i,22} \end{pmatrix}. \tag{16}$$

The lower right $l_2 \times l_2$ block of $\Phi_{\chi_0}(\omega)$ equals

$$\begin{aligned} & \begin{pmatrix} 0 \\ d(e^{j\omega}) I_{l_2} \end{pmatrix}^* \Phi(\omega) \begin{pmatrix} 0 \\ d(e^{j\omega}) I_{l_2} \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=0}^m d_i m_{i,12} \\ \sum_{i=0}^m d_i m_{i,22} \end{pmatrix}^T A_n^0 \begin{pmatrix} \sum_{i=0}^m d_i m_{i,12} \\ \sum_{i=0}^m d_i m_{i,22} \end{pmatrix} \\ = & d_0 \sum_{i=0}^m d_i m_{i,22} = d_0 \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_{l_2}}{d(e^{-j\omega})} d\omega \\ = & d_0 \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_{l_2}}{d(e^{j\omega})} d\omega = \lambda_0 I_{l_2}. \end{aligned}$$

Here we used (11), (16) for the first relation, the constant term in the identity (16) for the second one, and (12) for the third one. The fourth relation is obtained by the change of variables $\omega \mapsto -\omega$, and the last relation comes from the fact that the function $\frac{1}{d(z)}$ is holomorphic in the closed unit disc and assumes the value $\frac{1}{d_0}$ at $z = 0$. Thus the lower $l_2 \times l_2$ block of $\Phi_{\chi_0}(\omega)$ equals Φ_e.

The upper right $l_1 \times l_2$ block Φ_{ue} of $\Phi_{\chi_0}(\omega)$ equals

$$\begin{aligned} & \begin{pmatrix} d(e^{j\omega}) I_{l_1} \\ 0 \end{pmatrix}^* \Phi(\omega) \begin{pmatrix} 0 \\ d(e^{j\omega}) I_{l_2} \end{pmatrix} \\ = & \begin{pmatrix} d(e^{j\omega}) I_{l_1} \\ 0 \end{pmatrix}^* A_n(e^{j\omega})^{-*} A_n^0 \begin{pmatrix} \sum_{i=0}^m d_i m_{i,12} \\ \sum_{i=0}^m d_i m_{i,22} \end{pmatrix} \\ = & d(e^{-j\omega}) \begin{pmatrix} I_{l_1} \\ 0 \end{pmatrix}^T A_n(e^{-j\omega})^{-T} \begin{pmatrix} 0 \\ d_0 I_{l_2} \end{pmatrix}. \end{aligned}$$

Here we used (11), (16) for the first relation and the constant term in (16) for the second one. Thus the function $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$ can be extended from the unit circle to the function $d(z^{-1}) \begin{pmatrix} I_{l_1} \\ 0 \end{pmatrix}^T A_n(z^{-1})^{-T} \begin{pmatrix} 0 \\ d_0 I_{l_2} \end{pmatrix}$, which is holomorphic outside of the closed unit disc, including the point at infinity. This completes the proof. ∎

V. SOLUTION ALGORITHM

We now outline a general scheme for the solution of problems satisfying Assumption 1, in two steps. First we find the optimal truncated moment sequence by solving a semi-definite program, and then we recover the experimental conditions, i.e., the power spectrum Φ_r of the external input and the controller K from this moment sequence.

Apart from the constraints following from the formulation of the particular problem instance under consideration, the moment sequence $(m_0, \dots, m_n)$ has to satisfy conditions (12), (14), and $T_n \succeq 0$. Condition (12) determines the blocks $m_{k,22}$ explicitly. Condition (14) yields linear relations on the blocks $m_{k,21}$, while the last condition amounts to a linear matrix inequality. The optimal experiment design problem defined in Assumption 1 is thus turned into the following semi-definite program.

$$\min \left(\sum_{k=0}^n \langle C_k, m_k \rangle + \sum_{k=1}^N c_k x_k \right) \tag{17}$$

with respect to the constraints

$$\mathcal{A}(m_0, m_1, \dots, m_n, x_1, x_2, \dots, x_N) \succeq 0,$$

$$m_{k,22} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_{l_2}}{|d(e^{j\omega})|^2} e^{jk\omega} d\omega, \quad k = -n, \dots, n,$$

$$\sum_{i=0}^m d_i m_{k-i,21} = 0, \quad k = 1, \dots, n,$$

$$T_n = \begin{pmatrix} m_0 & m_1^T & \ddots & m_n^T \\ m_1 & m_0 & \ddots & m_{n-1}^T \\ \ddots & \ddots & \ddots & \ddots \\ m_n & m_{n-1} & \ddots & m_0 \end{pmatrix} \succeq 0,$$

where $m_{-k} = m_k^T$. By solving this semi-definite program, the user obtains the optimal truncated moment sequence $(m_0, \dots, m_n)$ and the optimal value of the cost function.

If the matrix T_n corresponding to the solution happens to be positive definite, then Theorem 1 allows to explicitly recover the joint power spectrum (6) by the explicit formula

$$\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot A(e^{j\omega})^{-*} A(0) A(e^{j\omega})^{-1},$$

where $A(z) = U(z) T_n^{-1} U^T(0)$ and $U(z) = \begin{pmatrix} z^n I_l & z^{n-1} I_l & \cdots & I_l \end{pmatrix}$. The power spectrum Φ_r and the controller K may then be recovered from the upper left $l_1 \times l_1$ block Φ_u and the upper right $l_1 \times l_2$ block Φ_{ue} of Φ_{χ_0} by formulas (5).

The methods presented above can also be extended to the case when the matrix T_n happens to be singular. The technicalities are more involved, however, and will be treated in a future paper.

VI. EXAMPLE

Let us consider an ARX model structure $G = \frac{\theta z^{-1}}{1+\vartheta z^{-1}}$, $H = \frac{1}{1+\vartheta z^{-1}}$ with true parameters θ_0, ϑ_0, where $|\vartheta_0| < 1$. We wish to identify the system in closed-loop under a constraint on the output power, $\overline{E}y^2 \le c$, where $c > \lambda_0$, such that the determinant of the information matrix is maximized (D-optimality). Set $d(z) = 1 + \vartheta_0 z$, then the elements of the information matrix and the output power can be expressed by the generalized moments as $\overline{M}_{11} = \lambda_0^{-1}((1+\vartheta_0^2)m_{0,11} + 2\vartheta_0 m_{1,11})$, $\overline{M}_{12} = \lambda_0^{-1}(-\theta_0 m_{1,11} - (1-\vartheta_0^2)m_{0,12} - \theta_0\vartheta_0 m_{0,11})$, $\overline{M}_{22} = \lambda_0^{-1}(-2\theta_0\vartheta_0 m_{0,12} + \frac{\lambda_0}{1-\vartheta_0^2} + \theta_0^2 m_{0,11})$, $\overline{E}y^2 = -2\theta_0\vartheta_0 m_{0,12} + \frac{\lambda_0}{1-\vartheta_0^2} + \theta_0^2 m_{0,11}$. Here we used that for the given choice of d we have $m_{0,22} = \frac{\lambda_0}{1-\vartheta_0^2}$, $m_{1,22} = -\frac{\lambda_0\vartheta_0}{1-\vartheta_0^2}$, $m_{1,21} = -\vartheta_0 m_{0,12}$. The constraint and the cost function involve only the moments m_0, m_1, and we can set $n = 1$. The resulting maxdet problem on the remaining unknown moments has the explicit solution $m_{0,11} = \frac{(c(1-\vartheta_0^2)+\lambda_0\vartheta_0^2)(c\vartheta_0^2+c-\lambda_0)}{\theta_0^2(1-\vartheta_0^2)(c+(c-\lambda_0)\vartheta_0^2)}$, $m_{0,12} = \frac{\lambda_0\vartheta_0(2c-\lambda_0)}{\theta_0(1-\vartheta_0^2)(c+(c-\lambda_0)\vartheta_0^2)}$, $m_{1,11} = -\frac{\lambda_0\vartheta_0(c\vartheta_0^2+c-\lambda_0)}{\theta_0^2(1-\vartheta_0^2)(c+(c-\lambda_0)\vartheta_0^2)}$, $m_{1,12} = -\frac{m_{0,12}\vartheta_0(\Delta+(c-\lambda_0)\lambda_0(1-\vartheta_0^2)^2)}{\Delta}$, where we denoted $\Delta = c^2(1+\vartheta_0^2)^2 - c\lambda_0(2\vartheta_0^4+\vartheta_0^2+1) + \lambda_0^2\vartheta_0^4$. This solution gives rise to a positive definite block-Toeplitz matrix T_1. The controller K and power spectrum Φ_r resulting from the central extension of T_1 are given by

$$\begin{aligned} K &= -\frac{\vartheta_0(2c-\lambda_0)(c\vartheta_0^2+c-\lambda_0)(1+\vartheta_0 z^{-1})}{\theta_0(\Delta+\vartheta_0(2c(c-\lambda_0)(1+\vartheta_0^2)+\lambda_0^2\vartheta_0^2)z^{-1})}, \\ \Phi_r &= \frac{(c-\lambda_0)(c\vartheta_0^2+c-\lambda_0)(c+(c-\lambda_0)\vartheta_0^2)\Delta|e^{j\omega}+\vartheta_0|^2}{\theta_0^2|\Delta e^{j\omega}+\vartheta_0(2c(c-\lambda_0)(1+\vartheta_0^2)+\lambda_0^2\vartheta_0^2)|^2}. \end{aligned}$$

For the values $\lambda_0 = 1$, $c = 1.4$, $\theta_0 = 0.5$, $\vartheta_0 = 0.4$ we first identify the system with an open-loop experiment using white noise with variance $\sigma^2 = 1$ as input. From the identified parameters two experimental configurations are computed, namely the optimal open-loop input, and the optimal closed-loop input-controller pair. An optimal open-loop and an optimal closed-loop experiment are then performed and the parameter vector identified. The data length in each of the experiments is $N = 1000$. The empirical covariance matrices of the 500 identified parameter vectors have determinant $0.49736N^{-2}$ and $0.38796N^{-2}$ for the open-loop and the closed-loop experiments, respectively. We see that the empirical covariance matrix has a 28% smaller determinant for the closed-loop experiments.

VII. CONCLUSIONS

We have addressed the problem of optimal experiment design for the identification of linear time-invariant systems operating in closed loop, where the optimization is performed jointly over the controller and the spectrum of the external excitation. Our first contribution has been to extend the results of [9] to multiple-input multiple-output systems. But the most important contribution is to have produced a solution procedure that is significantly simpler and more transparent than that of [9]. The key observation was that a particular extension of the finite set of optimal moments, called the central extension, automatically satisfies the constraints of the optimal experiment design problem. An important side effect, which is the third contribution of this paper, is that the generation of the optimal "controller – external input" pair from this central extension becomes straightforward.

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