Identifiability and Excitation of Polynomial Systems*

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Abstract— **This paper establishes identifiability and informativity conditions for a class of deterministic linearly parametrized polynomial systems. The class considered is polynomial in the states and in the inputs. The standard definitions of identifiability and informativity for linear systems are expanded to account for the situation where the identification is achieved either through the application of informative inputs or via the response to informative initial conditions. We provide necessary and sufficient conditions for identifiability from the initial state, respectively from the input, as well as necessary and sufficient conditions on the initial state, respectively on the input, to produce an informative experiment.**

I. INTRODUCTION

The question of identifiability of parametrized dynamical systems has occupied generations of system theorists, and the very definition of this concept has evolved over the years. For a long time, this concept embraced both the parametrization issue and the richness of the data set. Eventually, a clear separation was made between the *identifiability of the model structure*, which is a parametrization issue, and the *informativity of the data*, which is the issue of applying signals to the system that will produce different responses for different parameters.

The question of identifiability of the model structure can be succinctly summarized as follows: is the mapping from parameter vector θ to model $M(\theta)$ injective? The seminal paper [13] provided a broad answer to this question for large classes of linearly and nonlinearly parametrized systems using tools from differential algebra. It is important to observe that the identifiability is a property of the chosen model structure (i.e. the parametrization); it is totally independent of the true system and of the data.

The question of informativity of the data turns out to be harder to solve. For linear time invariant (LTI) systems, sufficient conditions on the input had been available for a very long time (see e.g. [12]), but necessary and sufficient conditions remained elusive until this question was solved in [9]. The relationship between identifiability, informativity and the uniqueness of the minimum in a Prediction Error Identification framework was established in [2] and required the introduction of the new concept of *local informativity*.

Informativity of the data - also known as input richness or transfer of excitation - is a topic that has attracted and continues to attract a wide attention. It is important not just in the context of identification, but also for the convergence of adaptive estimation and control schemes. Simply stated, the question can be summarized as follows: *what are the conditions on the excitation signal that will make the Gramian associated with a certain regression vector full rank?* The excitation signal is typically an input signal, but it can also be a properly chosen initial condition. The full rank condition on the Gramian is required for the estimation of the parameters.

Besides the results for the identification of LTI systems mentioned above, a wide range of questions related to informativity of the data and transfer of excitation have been addressed, dealing with different classes of systems, different types of input, and different convergence requirements.

In [1] informativity conditions on the input have been obtained for parameter convergence in linear discrete-time adaptive control schemes. Similar conditions for continuous-time linear adaptive control systems have been derived in [10]. In [11] nonlinear adaptive control schemes were studied, and it was shown that nonlinearities actually reduce the requirements on the richness of the external signal.

In the context of system identification, informativity conditions have been obtained for linear time-varying systems in [14], and more specific results have been obtained in [4] for Linear Parameter Varying systems with an ARX structure. In [6] informativity conditions on the input signal have been derived for a class of discrete-time linearly parametrized systems that are linear in the output and polynomial in the input. Bilinear systems are special members of this class. In [16] the question of which type of input signals (e.g. pulses, impulses, etc) are sufficient for the identification of bilinear systems has been studied.

In this paper we consider continuous time scalar linearly parametrized deterministic polynomial systems. The identifiability conditions for such systems have been established in [5]. *The contributions of this paper extend the results of [5] in several directions. Not only do we present necessary and sufficient conditions on the input to generate informative experiments, but we expand the traditional view by also presenting conditions for identifiability from the initial state, as well as necessary and sufficient conditions on the initial state to generate informative experiments.* Indeed, it is common

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in some application fields that the parameters are identified from data obtained as the response to some initial condition without any external input.

The remainder of the paper is organized as follows. In Section II we extend the definitions of local identifiability and local informativity from the linear stationary stochastic case to the case nonlinear deterministic systems; in addition, we distinguish between the classical definitions, where identifiability and informativity are secured from the input, to definitions that apply when these must be secured from the response to an initial condition. The specific model class of polynomial systems treated in this paper is introduced in Section III. Section IV recalls the results of [5] for the identifiability of linearly parametrized polynomial systems from the input. These serve as inspiration for the new results of Section V where we present necessary and sufficient conditions for identifiability, respectively informativity, from the initial state. The tools developed in Section V are then extended in Section VI where we present sufficient conditions for informativity from the input for the considered class of polynomial systems. Section VII concludes.

II. IDENTIFIABILITY AND INFORMATIVITY FOR NONLINEAR DETERMINISTIC SYSTEMS

We consider the following class of deterministic continuous-time nonlinear model structures:

$$\begin{aligned} \dot{x} &= f(x,\theta) + g(x,u) \\ y &= h(x,\theta) \end{aligned} \tag{1}$$

where x, u and y are scalar, and $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are given (i.e. known) analytic functions with $g(\cdot,0)=0$. The family of all models (1) generated by all $\theta \in \Re^d$ is called the model class $\mathcal{M}$. In addition, in the same spirit as [5] and [13], we shall make the following assumption on the input signal $u(.)$.

Assumption 1: The signal $u(t)$ is analytic and is such that the solution $x(t)$ of (1) is an analytic function.

The virtue of this assumption is that knowing all derivatives of an analytic signal at some time is equivalent to knowing that signal everywhere.

The choice of parameterization made in (1) should be such that the model class can describe exactly the real system $\mathcal{S}$; we shall throughout make the following assumption.

Assumption 2: There exists a parameter value θ_0 such that the real system is described by (1) with $\theta = \theta_0$.

We now present definitions that are the nonlinear deterministic counterpart of the classical definitions as can be found in [12], which are for linear time-invariant systems in a stochastic framework with quasi-stationary processes. These definitions clearly separate the concepts of identifiability, which is a property of the model structure, and of informativity, which is a property of the experimental data. In addition, we shall depart from the Linear Time Invariant (LTI) literature on identifiability and informativity by considering that the information content in the data, that allows estimation of the unknown parameters, can come either from the external input signal $u(.)$ or from the response to an initial condition x_0. Indeed in many engineering applications of nonlinear systems (e.g. in batch processes) the data used for identification are obtained by measuring the response to some initial condition; in particular, it is often the case that there are no external inputs to the system.

Consider the system (1) at some value θ_1 with initial condition x_0:

$$\begin{aligned} \dot{x} &= f(x,\theta_1) + g(x,u), \quad x(0) = x_0 \\ y &= h(x,\theta_1) \end{aligned} \tag{2}$$

and the same system at θ with initial condition $\hat{x}_0$:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},\theta) + g(\hat{x},u), \quad \hat{x}(0) = \hat{x}_0 \\ y &= h(\hat{x},\theta) \end{aligned} \tag{3}$$

Definition 1: **(Identifiability at θ_1)** The model (2) is *locally identifiable at* θ_1 if there exists a $\delta > 0$ and an experiment $z(.) \triangleq \{u(.), x_0\}$ such that, for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the models (2) and (3), driven by the same $u(.)$ and with the same initial condition $x_0 = \hat{x}_0$ are identical (i.e. $y(t,\theta) = y(t,\theta_1) \quad \forall t \geq 0$) only if $\theta = \theta_1$. The model (2) is *globally identifiable* at θ_1 if the same holds for all $\delta > 0$.

This definition relies on the possible existence of an excitation data set $z(.)$ which allows to differentiate between different values of θ by measuring the output. This data set may consist of an appropriate input sequence, an initial state, or a combination of both. Such a data set, when it exists, will be called informative. In [2], the new concept of *local informativity* was introduced for stationary stochastic LTI systems; the next definition is its deterministic counterpart.

Definition 2: **(Informativity at θ_1)** The excitation data set $z(.) \triangleq \{u(.), x_0\}$ is locally informative at θ_1 for the model set (1) if there exists a $\delta > 0$ such that, for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the models (2) and (3), driven by the same data set $z(.)$, are identical (i.e. $y(t,\theta) = \hat{y}(t,\theta_1) \quad \forall t \geq 0$) only if $\theta = \theta_1$.

These definitions exhibit the two ingredients that are necessary for a meaningful identification: informativity, which is a property of the applied data (input signal, initial state, or both), and identifiability, which refers to the possible existence of an informative data set given a particular model structure, and thus is a property of the model structure.

We shall from now on consider the following two situations separately:

1) the input signal $u(.)$ must provide informative data whatever the initial condition x_0, i.e. one must assure that "adversarial" initial conditions do not kill the excitation coming from $u(.)$. If the model structure is identifiable through $u(.)$, then the informativity question amounts to finding a sufficiently rich $u(.)$.
2) the system has no external input, i.e. $u(t) \equiv 0$, and identifiability must be secured through the initial condition $x(0)$. If the model structure is identifiable through $x(0)$, then the informativity question amounts to finding a x_0 that delivers informative data.

This first situation is the most commonly treated and probably the most commonly found in practice, but the second one is also found in a variety of applications, particularly in (bio-)chemical batch process [15], [3], [7].

A. Identifiability and informativity from the input

Definition 3: **(Identifiability at θ_1 from u)** The model structure (2) is *locally identifiable at θ_1 from the input u* if there exists a $\delta > 0$ and an input $u(.)$ such that, for all initial conditions $x_0 = \hat{x}_0$ and for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the systems (2) and (3) are identical (i.e. $y(t) = \hat{y}(t) \quad \forall t \geq 0$) only if $\theta = \theta_1$. The system (2) is **globally identifiable** at θ_1 from u if the same condition holds for all $\delta > 0$.

Definition 4: **(Global identifiability from u)** The model structure (2) is called *globally identifiable from u* if it is globally identifiable at almost all θ from u.

This last definition is consistent with the definition of global identifiability used in [12] and adopted in [2] for LTI systems.

The concept of *local informativity* was introduced for the first time in [2] in the context of LTI systems in a stochastic framework. For the nonlinear deterministic systems of this paper, we introduce the following definition.

Definition 5: **(Informativity of the input at θ_1)** The input signal $u(.)$ is locally informative at θ_1 for the system (2) if there exists a $\delta > 0$ such that, for all initial conditions $x_0 = \hat{x}_0$ and for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the models (2) and (3) with this input $u(.)$ are identical (i.e. $y(t) = \hat{y}(t) \quad \forall t \geq 0$) only if $\theta = \theta_1$.

B. Identifiability and informativity from the state

We now consider the case where no input is applied to the system, that is, $u(t) \equiv 0$, which corresponds to the model structures (2) and (3) with $g(x,u) \equiv 0$:

$$\begin{aligned} \dot{x} &= f(x,\theta_1), \quad y = h(x,\theta_1), \quad x(0) = x_0 && (4) \\ \dot{\hat{x}} &= f(\hat{x},\theta), \quad y = h(\hat{x},\theta), \quad \hat{x}(0) = \hat{x}_0 && (5) \end{aligned}$$

Definition 6: **(Identifiability at θ_1 from $x(0)$)** The model structure (4) is *locally identifiable at θ_1 from the initial condition $x(0)$* if there exists an initial condition $x_0 = \hat{x}_0$ and a $\delta > 0$ such that, for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the models (4) and (5) are identical (i.e. $y(t) = \hat{y}(t) \quad \forall t \geq 0$) only if $\theta = \theta_1$. The model structure (4) is *globally identifiable* at θ_1 from x_0 if the same holds for all $\delta > 0$.

Definition 7: **(Informativity of the initial condition at θ_1)** The initial condition x_0 is locally informative at θ_1 for the model structure (4) if there exists a $\delta > 0$ such that for all $\theta \in ||\theta - \theta_1|| \leq \delta$, the outputs of the systems (4) and (5) with initial condition $x_0 = \hat{x}_0$ are identical (i.e. $y(t) = \hat{y}(t) \quad \forall t \geq 0$) only if $\theta = \theta_1$.

III. THE MODEL CLASS OF POLYNOMIAL SYSTEMS

We now specialize to the class of scalar polynomial systems studied in [5]. The contribution of [5] for this class of systems was to establish local identifiability conditions from the input, which we shall recall in Section IV. Our new contribution, for this class of systems, will be to establish identifiability and informativity conditions from the initial state in Section V, and informativity conditions from the input in Section VI.

Thus, we consider the model structure (1) with the following form for $f(x,\theta)$ and $g(x,u)$:

$$\begin{aligned} f(x,\theta) &= \theta^T \phi(x) + m(x) && (6) \\ g(x,u) &= \sum_{i=1}^{l} g_i(x) u^i = G(x)U && (7) \end{aligned}$$

where $\phi(x) \in \Re^d$ is a known polynomial vector in the scalar x, $m(x)$ is a known polynomial in x, $\theta \in \Re^d$ is an unknown vector, $g_i(x) \in \Re$ are known polynomials in x, and where $G(x)$ and U are defined as follows:

$$G(x) = [g_1(x) \;\; g_2(x) \;\; \dots \;\; g_l(x)], \quad U = [u \;\; u^2 \;\; \dots \;\; u^l\;]^T \tag{8}$$

We denote by q the polynomial degree of $\phi(x)$, i.e. the degree of the highest degree polynomial in $\phi(x)$. The model structure (1) can then be rewritten as

$$\dot{x} = \theta^T \phi(x) + m(x) + G(x)U, \quad x(0) = x_0 \tag{9}$$

IV. IDENTIFIABILITY OF POLYNOMIAL SYSTEMS FROM THE INPUT

In this section we recall the main result of [5] on identifiability from the input. The key observation is as follows. If the model structure (9) is not identifiable at some θ_1 it means, from Definition 3, that for every input signal $u(.)$ there exists an initial condition x_0 and a $\theta \neq \theta_1$ such that the model (9) with $x(0) = x_0$ and the model

$$\dot{\hat{x}} = \theta_1^T \phi(\hat{x}) + m(\hat{x}) + G(\hat{x})U, \quad \hat{x}(0) = x_0 \tag{10}$$

have identical solutions: $\hat{x}(t) \equiv x(t) \quad \forall t \geq 0$. This means that for every analytic input $u(.)$ there is a nonzero vector $\beta = \theta - \theta_1$ such that

$$\beta^T \phi(x) \equiv 0 \quad \forall t \geq 0 \tag{11}$$

or, equivalently:

$$\beta^T R_\infty(\theta, x) = \mathbf{0} \tag{12}$$

at any particular value, for example at the initial condition $x(0)$, where

$$R_\infty(\theta, x) = \Big(\phi(x) \;\; \dot{\phi}(x) \;\; \ddot{\phi}(x) \;\; \phi^{(3)}(x) \;\; \dots \Big). \tag{13}$$

Define the $d \times (q+1)$ matrix:

$$J_q(x) \triangleq \Big(\; \phi(x) \quad \frac{\partial \phi(x)}{\partial x} \quad \frac{\partial^2 \phi(x)}{\partial x^2} \quad \dots \quad \frac{\partial^q \phi(x)}{\partial x^q} \; \Big) \tag{14}$$

It then follows from $\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x}$, $\ddot{\phi} = \frac{\partial^2 \phi}{\partial x^2}\dot{x}^2 + \frac{\partial \phi}{\partial x}\ddot{x}$, etc, and the fact that $\frac{\partial^{(q+1)} \phi}{\partial x^{(q+1)}} = \mathbf{0}$ that $R_\infty(\theta, x)$ can be written as

$$R_\infty(\theta, x) = J_q(x) Q_\infty(\theta, x, \dot{x}, \ddot{x}, \dots, u, \dot{u}, \ddot{u}, \dots) \tag{15}$$

where the left $(q+1) \times (q+1)$ submatrix of Q_∞ is upper-triangular with $(1, \dot{x}, \dot{x}^2, \dots, \dot{x}^q)$ on the diagonal, while all non-zero elements of Q_∞ are functions of $\dot{x}$, u, and higher order derivatives of x and u.

It is shown in [5] that a necessary condition for local identifiability at any θ is that the matrix $J_q(x)$ has full row rank $\forall x \in \Re$. Note that this requires $q \geq d-1$, where q is the degree of $\phi(x)$ and d is the number of parameters in θ. This condition is necessary both for identifiability from the input and for identifiability from the initial state, as defined in Section II.

The main identifiability result of [5] is stated in the following theorem; it relates to identifiability from the input.

Theorem 4.1: Consider the model structure (9) with $deg(\phi(x)) = q$ and let $d > 1$. This system is identifiable at θ_1 from the input u if and only if the following two conditions hold simultaneously:
(i) $J_q(x)$ has full row rank for all $x \in \Re$;
(ii) the polynomials $\theta_1^T\phi(x)+m(x)$ and $\{g_i(x), i = 1, \dots, l\}$ have no common real root w.r.t. x.

The condition $d > 1$ is there for the sake of making the statement necessary and sufficient. When $d = 1$, conditions (i) and (ii) are still sufficient, but condition (ii) is not necessary [5].

V. IDENTIFIABILITY AND INFORMATIVITY FROM THE INITIAL STATE

Inspired by the analysis of [5], and for pedagogical reasons, we now address the problem of identifiability and informativity from the initial state. The latter problem appears considerably simpler than the informativity from the input, which we return to in Section VI. To give some intuition, we start with an example.

A. An example for motivation

Example 1
Consider the model structure

$$\dot{x} = \theta_1 x^2 + \theta_2 x = \theta^T\phi(x),\ \ \theta^T = [\theta_1\ \ \theta_2],\ \ \phi(x) = [x^2\ \ x]^T. \tag{16}$$

We first observe that $q = 2$, that

$$J_q(x) = \begin{pmatrix} x^2 & 2x & 2 \\ x & 1 & 0 \end{pmatrix}$$

and hence the necessary condition for identifiability is satisfied. Now we write down the first $q+1$ equations (12), expressed at the initial time $t = 0$, and we introduce the notations: $x_1 = x(0),\ x_2 = \dot{x}(0),\ x_3 = \ddot{x}(0)$. This yields:

$$\beta_1 x_1^2 + \beta_2 x_1 = 0 \tag{17}$$
$$2\beta_1 x_1 x_2 + \beta_2 x_2 = 0 \tag{18}$$
$$2\beta_1 x_2^2 + 2\beta_1 x_1 x_3 + \beta_2 x_3 = 0 \tag{19}$$

Expressing the system equation (16) and its derivative at the same $t = 0$, we get, with the same notation:

$$x_2 = \theta_1 x_1^2 + \theta_2 x_1 \tag{20}$$
$$x_3 = 2\theta_1 x_1 x_2 + \theta_2 x_2 \tag{21}$$

Substituting x_2 and x_3 from (20)-(21) into (17)-(19) yields the following equivalent system of equations which now involves only (θ_1, θ_2) and x_1:

$$(\beta_1\ \ \beta_2)\, R_2(\theta_1, \theta_2, x_1) = \mathbf{0} \tag{22}$$

where

$$R_2(\theta, x_1) = \tag{23}$$
$$\begin{bmatrix} x_1^2 & 2x_1^2(\theta_1 x_1 + \theta_2) & 2x_1^2(3\theta_1^2 x_1^2 + 5\theta_1\theta_2 x_1 + 2\theta_2^2) \\ x_1 & (\theta_1 x_1 + \theta_2)x_1 & x_1(2\theta_1^2 x_1^2 + 3\theta_1\theta_2 x_1 + \theta_2^2) \end{bmatrix}$$

Following the analysis of the previous section, the model will be unidentifiable at θ from the initial condition if for all x_1 there exists a solution (β_1, β_2) with $[\beta_1\ \beta_2] \neq \mathbf{0}$ that solves (22). The two rows of R_2 are linearly independent except when either $x_1 = 0$ or $\theta_1 x_1 + \theta_2 = 0$. This shows that the model structure (16) is locally identifiable at every θ from the initial condition, for all initial conditions except $x_0 = 0$ or $x_0 = -\frac{\theta_2}{\theta_1}$. This a set of measure zero. Hence the model structure is not only locally, but also globally identifiable. As for the informativity, any initial condition other than $x_0 = 0$ or $x_0 = -\frac{\theta_2}{\theta_1}$ is informative for the estimation of the parameters θ_1 and θ_2 of the model structure (16).

B. The general case

We now provide necessary and sufficient conditions for the identifiability and the informativity at a given θ from the initial state, for general polynomial model structures of the form

$$\dot{x} = \theta^T\phi(x) + m(x), \quad x(0) = x_0. \tag{24}$$

where $\phi(x) \in \Re^d$ is a known polynomial vector in the scalar x, $m(x)$ is a known polynomial in x, $\theta \in \Re^d$ is an unknown vector to be estimated from the response of this system to some initial condition x_0. We assume again that $deg(\phi(x)) = q$. We first derive local identifiability conditions at some θ.

It follows from Definition 6 that the model structure (24) is not identifiable from $x(0)$ at θ if there exists $\theta_1 \neq \theta$ such that the model (24) and the model $\dot{\hat{x}} = \theta_1^T\phi(\hat{x}) + m(\hat{x}),\ \ \hat{x}(0) = x_0$ have identical responses: $\hat{x}(t) = x(t)\ \ \ \forall t \geq 0$. This implies that there exists a nonzero vector $\beta \in \Re^d$ such that (11) holds, or equivalently that (12) holds at any particular value, for example at the initial condition $x(0)$.

It can be shown, after lengthy calculations that exceed the limit of this conference paper, that

$$\beta^T R_\infty(\theta, x) = \beta^T J_q(x) Q_\infty(\theta, x, \dot{x}, \ddot{x}, \dots) = \mathbf{0} \tag{25}$$

if and only if

$$\beta^T R_q(\theta, x) = \beta^T J_q(x) Q_q(\theta, x, \dot{x}, \ddot{x}, \dots, x^{(q)}) = \mathbf{0} \tag{26}$$

where R_q and Q_q are the left $(q+1) \times (q+1)$ submatrices of R_∞ and Q_∞, respectively [8]. In particular

$$R_q(\theta, x) = \begin{pmatrix} \phi(x) & \dot{\phi}(x) & \ddot{\phi}(x) & \dots & \phi^{(q)}(x) \end{pmatrix}. \tag{27}$$

By computing the higher order derivatives of x from (24) and by successive substitutions for $\dot{x}, \dots, x^{(q)}$, etc in $Q_\infty(\theta, x, \dot{x}, \dots, x^{(q)})$, we can express this matrix as a function of $x_1 \triangleq x(0)$ and θ. We denote the resulting matrix $W(\theta, x_1)$ and we can thus write the equations (12) at the initial condition $x_1 = x(0)$ in the following equivalent form:

$$\beta^T J_q(x_1) W(\theta, x_1) = \mathbf{0} \quad (28)$$

Recalling Definition 7, we have thus proved the following main result.

Theorem 5.1: The model structure (24) is locally identifiable at θ from the initial condition if and only if there exists an initial condition x_1 such that the rows of $R_q(\theta, x_1)$ are linearly independent.

Assume that the model structure (24) is locally identifiable at θ from the initial condition x_1, so that in particular $q+1 \geq d$. Since the elements of $R_q(\theta, x_1)$ are polynomial functions of x_1 and θ, the drop of row rank of $R_q(\theta, x_1)$ occurs at solutions of polynomial equations in x_1. If $q+1 = d$, the rank drops at the solutions x_1 of one polynomial equation. If $q+1 > d$ the rank drops at solutions x_1 that are the common roots of several polynomial equations; the set of such x_1 is generically empty. This leads to the following result.

Theorem 5.2: Let the model structure (24) be locally identifiable at a given θ. The set of initial conditions which do not result in an informative experiment at θ is either empty or it forms a thin set in $\Re$.

In fact we can characterize this thin set completely. To this end we first present a Lemma.

Lemma 5.1: Consider any nontrivial interval $\mathcal{I} \subset \Re$, and assume that $J_q(x)$ has full rank for all $x \in \Re$. Then there does not exist a nonzero $\eta \in \Re^d$ such that

$$\eta^T \phi(x) = 0 \quad \forall x \in \mathcal{I}. \quad (29)$$

Proof: Consider any $\bar{x} \in \mathcal{I}$ and define

$$e(x) = \left[1, (x-\bar{x}), \frac{(x-\bar{x})^2}{2!}, \ldots, \frac{(x-\bar{x})^q}{q!}\right]^T.$$

Then the polynomial nature of $\phi(x)$ ensures that for all $x \in \mathcal{I}$:

$$\phi(x) = \phi(\bar{x}) + J_q(\bar{x}) e(x).$$

Under (29), there holds for all $x \in \mathcal{I}$:

$$\eta^T J_q(\bar{x}) e(x) = 0.$$

Thus as $\mathcal{I}$ is a nontrivial interval one has

$$\eta^T J_q(\bar{x}) = 0.$$

As $J_q(\bar{x})$ has full rank this must imply that $\eta = 0$. ■

It is worth noting that in fact this Lemma holds as long as J_q has full rank at *any* as opposed to *all* points in $\mathcal{I}$. We now prove that the thin set in question comprises the roots of $\phi^T(x)\theta + m(x)$, a fact that accords with the example presented above.

Theorem 5.3: Let the model structure (24) be locally identifiable at a given θ and $d > 1$. Then an initial condition $x_0 \in \Re$ yields an informative experiment at θ if and only if x_0 is not a root of the polynomial equation

$$\phi^T(x_0)\theta + m(x_0) = 0. \quad (30)$$

Proof: Local identifiability ensures that $J_q(x)$ has full rank for all $x \in \Re$. To establish a contradiction suppose (30) is violated but x_0 is not informative. Then there exists a nonzero $\beta \in \Re^d$, such that along the trajectory $x(t)$ starting from $x(0) = x_0$,

$$\beta^T \phi(x(t)) = 0, \quad \forall t \geq 0. \quad (31)$$

However, as x_0 is not a stationary point of (24), there is a nontrivial interval containing x_0, such that for every point in that interval there exists a $t \geq 0$ at which $x(t)$ equals this point. Then from Lemma 5.1 $\beta = 0$, establishing a contradiction. Thus (30) is a necessary condition for x_0 to be uninformative.

Now consider any x_0 satisfying (30). Such an x_0 is a stationary point of (24). Thus for all $t \geq 0$, $\phi(x(t)) = \phi(x_0)$. As $d > 1$, there exists a nonzero $\beta \in \Re^d$ such that for all $t \geq 0$ (31) holds. Thus x_0 cannot be informative. ■

VI. INFORMATIVITY FROM THE INPUT

Consider the polynomial model structure of the form (9) with $deg(\phi(x)) = q$, and assume that it obeys the conditions of Theorem 4.1. The models (9) and (10) have identical solutions if and only if there exists a nonzero vector $\beta \in \Re^d$ such that (12) holds at any particular value, for example at the initial condition $x(0) = x_1$. Remember now that $R_\infty(\theta, x)$ can be written as in (15).

By the same procedure used in Section V, we compute the higher order derivatives of x from the model equation (9), we introduce the notations $x_1 = x(0)$ and $u_1 = u(0), u_2 = \dot{u}(0), u_3 = u^{(3)}(0), \ldots$, and we make successive substitutions in the matrix Q_∞; we denote the resulting matrix $W_\infty(\theta, x_1, u_1, u_2, \ldots)$. The equation $\beta^T R_\infty(\theta, x) = \mathbf{0}$ can then be rewritten as

$$\begin{aligned} &\beta^T R_\infty(\theta, x_1, u_1, u_2, \ldots) \\ &= \beta^T J_q(x_1) W_\infty(\theta, x_1, u_1, u_2, \ldots) = \mathbf{0} \end{aligned} \quad (32)$$

$\beta = \mathbf{0}$ is the only solution of (32) for a given θ if and only if $u_1, u_2, \ldots$ are such that $R_\infty(\theta, x_1, u_1, u_2, \ldots)$ has full column rank for all x_1. Thus we have proven the following result.

Theorem 6.1: An analytic input signal $u(.)$ is informative at θ for the model structure (9) if and only if at any given time the signal and its derivatives, $u_1, u_2, \ldots$, are such that the matrix $R_\infty(\theta, x_1, u_1, u_2, \ldots)$ in (32) has rank d for all x_1.

We have the following immediate but useful consequence.

Corollary 6.1: Let the model structure (9) be locally informative at θ. A sufficient condition for an input u to be locally informative at θ is that the matrix $R_q(\theta, x_1, u_1, \ldots, u_q)$ formed from the first $q+1$ columns of $R_\infty(\theta, x_1, u_1, u_2, \ldots)$ has full row rank for all x_1.

We illustrate the use of Corollary 6.1 with the following example, which is Example 1 with an added input.

Example 2

Consider the system

$$\dot{x} = \theta_1 x^2 + \theta_2 x + u = \theta^T \phi(x) + u, \quad (33)$$

with θ and $\phi(x)$ as in (16). Following the same derivation as in Example 1, we get the same UE equations (17)-(19). The two SE equations now become

$$x_2 = \theta_1 x_1^2 + \theta_2 x_1 + u_1 \quad (34)$$

$$x_3 = 2\theta_1 x_1 x_2 + \theta_2 x_2 + u_2 \quad (35)$$

Following the procedure described above we arrive at the following expression for the matrix $R_2(\theta, x_1, u_1, u_2)$:

$$R_2(\theta, x_1, u_1, u_2) = \begin{bmatrix} x_1^2 & 2x_1(\theta_1 x_1^2 + \theta_2 x_1 + u_1) & 6\theta_1^2 x_1^4 + 10\theta_1\theta_2 x_1^3 + (4\theta_2^2 + 8\theta_1 u_1)x_1^2 + (6\theta_2 u_1 + 2u_2)x_1 + 2u_1^2 \\ x_1 & \theta_1 x_1^2 + \theta_2 x_1 + u_1 & 2\theta_1^2 x_1^3 + 3\theta_1\theta_2 x_1^2 + (2\theta_1 u_1 + \theta_2^2)x_1 + \theta_2 u_1 + u_2 \end{bmatrix} \quad (36)$$

Clearly this matrix $R_2(\theta, x_1, u_1, \ldots, u_q, \theta)$ has rank two if the left 2×2 submatrix has full rank; call it $\bar{R}$. A sufficient condition for informativity of $u(.)$ at θ is therefore a pair u_1, u_2 such that $det(\bar{R}) \neq 0$ for all x_1. Since $det(\bar{R}) = -x_1^2(\theta_1 x_1^2 + \theta_2 x_1 + u_1)$, we see that $\bar{R}$ is singular only if $x_1 = 0$ or if $\theta_1 x_1^2 + \theta_2 x_1 + u_1 = 0$. We consider these two cases separately.

Case 1: $x_1 = 0$

The condition $\beta^T R_2(\theta, x_1, u_1, u_2) = \mathbf{0}$ becomes, after substitution in (36):

$$[\beta_1 \quad \beta_2] \begin{bmatrix} 0 & 0 & 2u_1^2 \\ 0 & u_1 & \theta_2 u_1 + u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (37)$$

We conclude from Theorem 6.1 that any input $u(.)$ that is such that $u(0) \neq 0$ will ensure that $\beta = \mathbf{0}$ and hence such input is informative at all θ even when the initial condition is $x(0) = 0$.

Case 2: $\theta_1 x_1^2 + \theta_2 x_1 + u_1 = 0$.

This implies by (34)-(35) that $x_2 = 0$ and hence $x_3 = u_2$. The condition now becomes

$$[\beta_1 \quad \beta_2] \begin{bmatrix} x_1^2 & 0 & 2x_1 u_2 \\ x_1 & 0 & u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (38)$$

We observe that R_2 has full rank if and only if $x_1 u_2 \neq 0$. This is achieved for any input signal $u(.)$ such that $u_1 \neq 0$ and $u_2 \neq 0$. Indeed, $u_1 \neq 0$ means that $x_1 \neq 0$ since $\theta_1 x_1^2 + \theta_2 x_1 + u_1 = 0$. We conclude that a choice of input such that, at any given time t_0, $u(t_0) \neq 0$ and $\dot{u}(t_0) \neq 0$ makes that input globally informative.

VII. CONCLUSIONS

In [5] the question of identifiability for the class of linearly parametrized polynomial systems was solved, in the sense that necessary and sufficient conditions on the model structure for identifiability from the input were provided. The related question of informativity, i.e. which input signals will allow the estimation of the parameters, was left open. In the meantime, progress has been accomplished on the concept of informative experiments with the introduction of the concept of *local informativity* in [2].

In this paper we have provided sufficient conditions for the generation of informative experiments from the input for the same class of polynomial systems. But in addition, we have provided necessary and sufficient conditions for identifiability from the initial state, as well as necessary and sufficient conditions for informativity from the initial state.

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