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# Parametrization of all plants that have the same optimal LQG controller*


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## Abstract

Using the dual Youla parametrizations of controller-based coprime factor plant perturbations and plant-based coprime factor controller perturbations, we characterize the set of all plants that have the same optimal LQG or MV controller.


## 1 Introduction

Consider that you have some initial plant, $P_0$, and some controller, $C_0$, that stabilizes $P_0$. Using stable proper coprime factor descriptions of $P_0$ and $C_0$, and the Youla parametrizations, one can then characterize both the set $\mathcal{C}$ of all controllers stabilizing $P_0$, and the set $\mathcal{P}$ of all plants that are stabilized by $C_0$. The first set is parametrized in terms of coprime factors of $P_0$ and $C_0$ and an arbitrary proper stable transfer function $S$, i.e. $\mathcal{C} = \{C(S)\}$. The transfer function $S$ is often called the Youla parameter and its norm indicates the size of the perturbation away from $C_0$. The second (dual) set is parametrized in terms of coprime factors of $P_0$ and $C_0$ and an arbitrary proper stable transfer function $Q$, i.e. $\mathcal{P} = \{P(Q)\}$. The transfer function $Q$ is also called Youla parameter and its norm indicates the size of the perturbation away from $P_0$.

These parametrizations are explicitly described in the following proposition, which contains a collection of results from [7]. They are expressed here for scalar systems and apply to both the discrete and continuous time case; the extension to the multivariable case is straightforward.

**Proposition 1.1** [7] *Let $P_0$ and $C_0$ have fractional representations $P_0 = N_P D_P^{-1}$ and $C_0 = N_C D_C^{-1}$, where $N_P, D_P, N_C, D_C$ belong to $\mathbf{S}$, the ring of proper stable transfer functions. (We assume a negative feedback convention). Assume that the following Bezout equation holds*

$$N_C N_P + D_C D_P = 1. \qquad (1.1)$$

*This equation expresses both the fact that the factors are coprime and that the feedback loop formed by the plant $P_0$ and the controller $C_0$ is internally stable. For any $S \in \mathbf{S}$, define*

$$N_S = N_C - D_P S, \quad D_S = D_C + N_P S. \qquad (1.2)$$

1. *Then $C(S) = N_S D_S^{-1}$ is a stabilizing controller for $P_0 = N_P D_P^{-1}$.*
2. *Furthermore, any controller that stabilizes $P_0$ has a fractional representation $N_S D_S^{-1}$ with $N_S$, $D_S$ as in (1.2) for some $S \in \mathbf{S}$.*

*The dual result can be stated in the following way. For any $Q \in \mathbf{S}$, define*

$$N_Q = N_P - Q D_C, \quad D_Q = D_P + Q N_C. \qquad (1.3)$$

1. *Then $P(Q) = N_Q D_Q^{-1}$ is stabilized by $C_0 = N_C D_C^{-1}$.*
2. *Furthermore, any plant stabilized by $C_0$ has a fractional representation $N_Q D_Q^{-1}$ with $N_Q$, $D_Q$ as in (1.3) for some $Q \in \mathbf{S}$.* ■

In this paper we use these parametrizations to characterize all plants that have same optimal LQG or MV controller. Our basic one degree of freedom control loop is that of Figure 1.1, where $y_t$ is the plant output and $u_t$ is the control signal. The disturbance signal $v_t$ is assumed zero mean stationary with spectral density function $\phi_0$. Our control design criterion is the following regulation LQG index (expressed here in discrete time)

$$J_{\text{LQG}} = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{t=1}^{N} \{y_t^2 + \lambda u_t^2\}\right\}. \qquad (1.4)$$


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The solution of the minimization problem (1.4) using the Youla parametrization was first presented in [8]. [4] proposes a generalization to LQG control in a prescribed domain of stability. The analytic solution to the infinite horizon LQG control problem in the polynomial setting of [8] is recalled in Section 2. Here, we

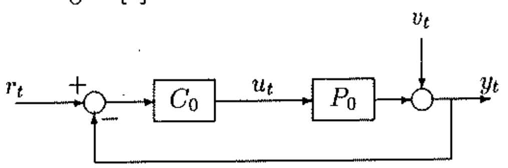


Figure 1.1: One degree of freedom control loop

characterize the set of all plants ($P_1$, $\phi_1$) that have the same optimal LQG controller, $C_0$, as the original plant ($P_0$, $\phi_0$).

We believe that this parametrization will prove to be useful in the analysis of iterative identification and control schemes that take an LQG control objective as a starting point. Indeed, using our parametrization, it is possible to characterize the common features of all models that admit the same optimal LQG controller as the "true" plant; features which are representative of a good identified model in the identification stage.

A similar parametrization has been introduced in [5] in the context of adaptive control and in a state-space framework. In that paper, the author characterizes the set of all models that are equivalent with the real system (which is supposed to be in the model set) in the sense that they lead to the same controlled behaviour as the desired behaviour of the real system. Notice that a "dual" problem was treated in [6], i.e. in that paper the authors parametrize the set of all solutions for the unconstrained $H_2$-optimal control problem in the state feedback case.

The outline of our paper is as follows. In Section 2 we present a solution to the LQG controller design problem in the Youla parametrization framework starting from the plant model $P_0$ and any stabilizing controller $C_0$, using the set $\mathcal{C}(S)$ of all stabilizing controllers for $P_0$, i.e. we show how to compute $S_{\text{opt}}$. In Section 3 we compute how much change is induced in a controller by a change in a plant model and we characterize the set of all plants that have the same optimal LQG controller. In Section 4, we particularize our results to the case of a discrete time ARMAX system. The validity of the theoretical results is checked on an LQG and MV example in Section 5. We conclude in Section 6.

## 2 Optimal LQG control in the Youla parametrization

Let $P_0 = N_P D_P^{-1}$ and $C_0 = N_C D_C^{-1}$ be coprime factorizations of the plant $P_0$ and of an arbitrary stabilizing controller $C_0$ such that the Bezout equation (1.1) holds. Let $C_0$ in Figure 1.1 be replaced by an arbitrary controller $C(S)$ defined in Proposition 1.1 and consider a disturbance rejection problem, i.e. $r_t = 0$. For this ($P_0$, $\phi_0$, $C(S)$) configuration we then have:

$$y_t = (D_C + N_P S) D_P\, v_t \text{ and } u_t = (N_C - S D_P) D_P\, v_t.$$

The LQG index (1.4) can then be rewritten, using Parseval's theorem, to obtain an expression[1] that is integrable in $S$:

$$\begin{aligned} J_{LQG}(P_0, \phi_0, C(S)) &= \frac{1}{2\pi}\int d\omega \,\{|D_C + N_P S|^2 \\ &\quad + \lambda |N_C - D_P S|^2\}\, |D_P|^2 \phi_0. \qquad (2.1) \end{aligned}$$

where $\phi_0$ is the spectral density function of $v_t$. If $C_0$ is the optimal LQG controller for $P_0$, then $S = 0$ minimizes $J_{\text{LQG}}$ over all $S \in \mathbf{S}$.

### Computation of the optimal Youla parameter

It is shown in [8] that a stable minimizing $S$ can be found analytically by means of spectral factorizations and projections, i.e. by taking stable parts. Indeed, it is straightforward to show that by completing the square, the LQG control criterion can be rewritten[2] as:

$$\begin{aligned} J_{\text{LQG}} &= \|\mathcal{A}S + \mathcal{A}^{-*}\mathcal{B}\|_2^2 \\ &\quad + \frac{1}{2\pi}\int d\omega \left\{\frac{\lambda}{|N_P|^2 + \lambda |D_P|^2} |D_P|^2\, \phi_0\right\} \qquad (2.2) \end{aligned}$$

$$\begin{aligned} \text{with} \quad \mathcal{A}\mathcal{A}^* &= [|N_P|^2 + \lambda |D_P|^2]\, |D_P|^2 \phi_0, \qquad (2.3) \\ \mathcal{B} &= [N_P^* D_C - \lambda D_P^* N_C]\, |D_P|^2\, \phi_0 \qquad (2.4) \end{aligned}$$

where $\mathcal{A}$ is minimum phase, stable and of relative degree zero[3]. The minimizing $S$ is clearly given by

$$S_{\text{opt}} = -\mathcal{A}^{-1} [\mathcal{A}^{-*}\mathcal{B}]_{\text{st}} \qquad (2.5)$$

where $[\;]_{\text{st}}$ denotes the stable part: see remark below. The optimal control cost is

$$\begin{aligned} J_{\text{LQG}}^{\text{opt}} &= \|[\mathcal{A}^{-*}\mathcal{B}]_{\text{unst}}\|_2^2 \qquad (2.6) \\ &\quad + \frac{1}{2\pi}\int d\omega \left\{\frac{\lambda}{|N_P|^2 + \lambda |D_P|^2} |D_P|^2\, \phi_0\right\}. \end{aligned}$$

[1] The integration bounds have been omitted to stress the fact that the expressions are valid in both the continuous ($\int_{-\infty}^{\infty}$) and discrete time case ($\int_{-\pi}^{\pi}$).

[2] If $S = \frac{B}{A}$ of relative degree $d$, with $A$ and $B$ polynomials, then $S^*$ is defined as $\frac{B(-s)}{A(-s)}$ in continuous time and as $\frac{z^d B^*(z)}{A^*(z)}$ in discrete time, which ensures that $(S^*)^* = S$, and that $S^*(e^{j\omega})$ is the complex conjugate of $S(e^{j\omega})$. $A^*(z)$ is defined here as the reciprocal polynomial of $A(z)$, i.e. if $A(z) = z^{n_a} + a_1 z^{n_a - 1} + \cdots + a_{n_a}$, then $A^*(z) = z^{n_a} A(z^{-1}) = 1 + a_1 z + \cdots + a_{n_a} z^{n_a}$.

[3] In the continuous time case, the relative degree zero constraint cannot always be imposed. In such cases, the infimum of $J_{\text{LQG}}$ is not attained for any $S \in \mathbf{S}$. However, one can still compute $\inf_{S \in \mathbf{S}} J_{\text{LQG}}(S)$ and construct a family $\{S_\epsilon \in \mathbf{S}\}$ such that $J(S_\epsilon)$ approaches the infimum as $\epsilon \longrightarrow 0$. See [7] for details.

**Remark:** Every finite rational transfer function $Z$ can be decomposed into the sum of its stable and unstable parts, $Z = [Z]_{st} + [Z]_{unst}$, as follows. Expand $Z$ into partial fractions (unique decomposition) and a polynomial; then $[Z]_{st}$ (respectively $[Z]_{unst}$) is the sum of the terms corresponding to poles in the open left half plane (respectively in the closed right half plane) in continuous time and inside (respectively on or outside) the unit circle in discrete time. The improper part of $Z$ is assigned to the unstable part. In the continuous time decomposition of $\mathcal{A}^{-*}\mathcal{B}$, it is necessary to take the unique solution with the constant part assigned to the stable part in order to make the cost (2.6) finite. In discrete time, one can partition the constant part between the stable and the unstable part: all these solutions lead to a finite cost. Since we optimize over all proper $S$, $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ has to reflect just that part of the associated impulse $\{h_k\}$ corresponding to $k < 0$, so that $[\mathcal{A}^{-*}\mathcal{B}]_{unst} = \sum_{k<0} h_k z^{-k}$. The constant term in the partial fraction expansion must be so partitioned between $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ and $[\mathcal{A}^{-*}\mathcal{B}]_{st}$ that $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ has $z = 0$ as a zero, i.e. there is a unique decomposition.

## 3 Parametrization of all plants that have the same LQG or MV controller

Consider a plant model $(P_0, \phi_0)$ and its corresponding optimal (and hence stabilizing) controller $C_0$, both factorized as before. Let now $P_1$ be some plant that is stabilized by $C_0$. It is obvious that $P_1$ is contained in the set

$$\mathcal{P}_1(Q) = (N_P - QD_C)(D_P + QN_C)^{-1} \quad \text{with } Q \in \mathbf{S},$$

of all models stabilized by $C_0$. The set of all controllers stabilizing $P_1(Q)$ is then given by

$$\begin{aligned}\overline{\mathcal{C}}_1(\overline{S}, Q) &= [N_{C_1}(\overline{S}, Q)]\,[D_{C_1}(\overline{S}, Q)]^{-1} \qquad (3.1)\\ &= [N_C - \overline{S}(D_P + QN_C)]\,[D_C + \overline{S}(N_P - QD_C)]^{-1}\end{aligned}$$

for some $\overline{S}, Q \in \mathbf{S}$. We have called this Youla parameter $\overline{S}$ to distinguish it from $S$ that parametrizes all controllers stabilizing $P_0$. Let $C_1$ be any controller in the set $\overline{\mathcal{C}}_1$. The LQG index (1.4) for $(P_1, \phi_1)$ with controller $C_1$, expressed in the frequency domain, is integral in $Q$ and $\overline{S}$:

$$\begin{aligned}J_{LQG}(P_1, \phi_1, C_1) &= \frac{1}{2\pi}\int d\omega \left\{ \left|D_{C_1}(\overline{S}, Q)\right|^2 \right.\\ &\left. + \lambda \left|N_{C_1}(\overline{S}, Q)\right|^2 \right\} |D_P + QN_C|^2 \phi_1. \qquad (3.2)\end{aligned}$$

Two situations can occur when a system is perturbed: either the perturbation $Q$ only influences the plant model, and the noise model remains unchanged (as happens in an OE model structure) or both the plant model and the noise model are influenced (as happens in an ARX or ARMAX model structure). We consider the second case, and we assume that $\phi_0$ varies with $Q$ in such a way that $|D_P + QN_C|^2 \phi_1$ is independent of $Q$, i.e.

$$|D_P + QN_C|^2 \phi_1 = |D_P|^2 \phi_0. \qquad (3.3)$$

This is typical of an ARX or ARMAX model structure. We note that the product of the last two terms in (3.2) is then independent of $Q$.

**Computation of $\overline{S}_{opt}$ as a function of $Q$**

In this subsection, we characterize the optimal controller $C_1^{opt}$, i.e. we compute $\overline{S}_{opt}$ that minimizes $J_{LQG}$ and express it as a function of $Q$ and the coprime factorizations of the plant $P_0$ and its corresponding optimal controller $C_0$. Thus, $\overline{S}_{opt}$, which expresses $C_1^{opt}$ as a perturbation of $C_0$, will be defined as a function of $Q$, which expresses $P_1$ as a perturbation of $P_0$ and $\phi_1$ as a function of $\phi_0$: see (3.3).

Recall that $\mathcal{A}$ and $\mathcal{B}$, related to the plant $P_0$ and its optimal controller $C_0$, are given by the following expressions:

$$\begin{aligned}\mathcal{A}\mathcal{A}^* &= [|N_P|^2 + \lambda|D_P|^2]\,|D_P|^2 \phi_0,\\ \mathcal{B} &= [N_P^* D_C - \lambda D_P^* N_C]|D_P|^2 \phi_0.\end{aligned}$$

We are now in a position to calculate the perturbed versions of $\mathcal{A}$ and $\mathcal{B}$. We start with

$$\begin{aligned}\overline{\mathcal{B}} &= [(N_P^* - Q^* D_C^*)D_C - \lambda(D_P^* + Q^* N_C^*)N_C]|D_P|^2\phi_0\\ &= \mathcal{B} - Q^*\,[|D_C|^2 + \lambda|N_C|^2]\,|D_P|^2 \phi_0. \qquad (3.4)\end{aligned}$$

There will be a corresponding change from $\mathcal{A}$ to $\overline{\mathcal{A}}$:

$$\overline{\mathcal{A}}\,\overline{\mathcal{A}}^* = [|N_P - QD_C|^2 + \lambda|D_P + QN_C|^2]\,|D_P|^2\phi_0. \qquad (3.5)$$

We obtain

$$\begin{aligned}\overline{S}_{opt} &= -\overline{\mathcal{A}}^{-1}\,[\overline{\mathcal{A}}^{-*}\,\overline{\mathcal{B}}]_{st} \qquad (3.6)\\ &= \overline{\mathcal{A}}^{-1}\left[\overline{\mathcal{A}}^{-*} Q^*\,[|D_C|^2 + \lambda|N_C|^2]\,|D_P|^2 \phi_0\right]_{st}\end{aligned}$$

because $\mathcal{B}$ is unstable by optimality of $C_0$ and $\overline{\mathcal{A}}^{-*}$ is unstable by definition.

We can now characterize the set of all plants $(P_1, \phi_1)$ that admit the same optimal controller, $C_0$, as $(P_0, \phi_0)$. This set is parametrized by all $Q \in \mathbf{S}$ for which $\overline{S}_{opt} = 0$, i.e. all $Q \in \mathbf{S}$ that solve

$$\left[\overline{\mathcal{A}}^{-*} Q^*\,[|D_C|^2 + \lambda|N_C|^2]\,|D_P|^2 \phi_0\right]_{st} = 0. \qquad (3.7)$$

Equation (3.7) is a very implicit characterization, not self-evidently allowing any nonzero solutions. In the next section, we shall display a nontrivial solution set more explicitly.

From (3.2) and by optimality of $C_0$, i.e. $Q \in \mathbf{S}$ solves (3.7) and $\overline{S}_{opt} = 0$, we observe that the optimal LQG cost that is associated to any triplet ($P_1(Q)$, $\phi_1(Q)$, $C_0$) satisfying (3.3) and for which $C_0$ is optimal, is now independent of $Q$:

$$\begin{aligned} J_{\mathrm{LQG}}(P_1, C_0) &= J_{\mathrm{LQG}}(P_0, C_0) \\ &= \frac{1}{2\pi}\int d\omega\{|D_C|^2 + \lambda|N_C|^2\}|D_P|^2\phi_0. \end{aligned}$$

## 4 Application in the case of an ARMAX plant model

Let us specialize our results to the case where the plant is described by a discrete time ARMAX model:

$$A(z)y_t = B(z)u_t + C(z)e_t \tag{4.1}$$

where $e_t$ is white noise of zero mean and unit variance, $A(z)$, $B(z)$ and $C(z)$ are polynomials in $z$ of degree $n$, $n-d$ and $n$, respectively, that have no common factor, with $A(z)$, $B(z)$ coprime, $d \geq 1$, and $C(z)$ having all its zeros inside the unit circle. Note that this system has a delay $d$. Equation (4.1) is normalized so that the leading coefficients of the polynomials $A(z)$ and $C(z)$ are unity, i.e. $A(z)$ and $C(z)$ are monic. $B(z)$ can be factorized as $B(z) = B_-(z)B_+(z)$ where $B_+(z)$ has all its zeros strictly inside the unit circle and $B_-(z)$ is monic and has all its zeros on or outside the unit circle. We examine successively the case of the LQG criterion, $J_{\mathrm{LQG}} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E\{y_t^2 + \lambda u_t^2\}$, and the case of a minimum variance criterion, $J_{\mathrm{MV}} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E\{y_t^2\}$.

### The LQG disturbance rejection problem

The spectral factorization solution method for the infinite horizon LQG regulation problem consists in first computing the stable polynomial spectral factor $G(z)$ of

$$G(z)G^\star(z) = \lambda A(z)A^\star(z) + z^d B(z)B^\star(z). \tag{4.2}$$

It can be shown that if $\deg A > \deg B$ then there always exists a unique polynomial $G(z)$ with $\deg G(z) = n$ and positive coefficient of the highest degree term [3]. The next step consists in solving the following Diophantine equation for the polynomials $X(z)$ and $Y(z)$:

$$G(z)C(z) = A(z)X(z) + B(z)Y(z). \tag{4.3}$$

The resulting optimal controller is

$$u_t = -\frac{Y(z)}{X(z)}\,y_t. \tag{4.4}$$

The complexity of the control law is determined by the polynomials $Y(z)$ and $X(z)$. Since $A(z)$ has degree $n$, it is obvious that we can choose $Y(z)$ such that $\deg Y(z) \leq n$. For any such $Y(z)$, it follows that $\deg X(z) \leq n$. Requiring this degree constraint does not uniquely specify the solution pair $X(z), Y(z)$. This will be done subsequently. For any such pair, define the polynomials

$$X^\star(z) = z^n X(z^{-1}) \text{ and } Y^\star(z) = z^n Y(z^{-1}). \tag{4.5}$$

Notice that if $X(z)$, say, happens to have degree less than $n$, this definition is non standard.

We then have the following lemma, which is essentially the same as the one in [3]. However the proof of [3] does not consider the possibility that $A$ and $B^\star$ could have a common zero. Accordingly, we present a complete proof.

**Lemma 4.1** *Let the polynomials $A(z)$, $B(z)$, $C(z)$ and $G(z)$ have degree $n$, $n-d$, $n$ and $n$ respectively, with $(A, B)$ coprime, and (4.2) holding for some $\lambda > 0$. Let polynomials $X(z)$, $Y(z)$ with degree at most $n$ be found satisfying (4.3). Then there exists a polynomial $V(z)$ of degree at most $n$ such that*

$$\begin{aligned} G^\star(z)V(z) &= X^\star(z)B(z) - \lambda Y^\star(z)A(z) & (4.6)\\ A^\star(z)V(z) &= C^\star(z)B(z) - Y^\star(z)G(z) & (4.7) \end{aligned}$$

**Proof:** Let $M_+M_-$ be the greatest common divisor of $A$ and $B^\star$, where $M_+$ has all zeros strictly inside the unit circle and $M_-$ all zeros on or outside the unit circle. Then for some $A_1$, $B_1$, with $A_1A_1^\star$ and $B_1B_1^\star$ coprime, we have

$$A = A_1M_+M_- \text{ and } B = B_1M_+^\star M_-^\star. \tag{4.8}$$

It follows that

$$G = G_1M_+M_-^\star \tag{4.9}$$

where $A_1A_1^\star$, $B_1B_1^\star$ and $G_1G_1^\star$ are mutually coprime with $G_1G_1^\star = \lambda A_1A_1^\star + z^d B_1B_1^\star$. Now observe that

$$\begin{aligned} A^\star[X^\star B - \lambda Y^\star A] &= A^\star X^\star B + z^d BB^\star Y^\star - GG^\star Y^\star \\ &= B\left[A^\star X^\star + z^d B^\star Y^\star\right] - GG^\star Y^\star \\ &= BG^\star C^\star - GG^\star Y^\star \\ &= G^\star[BC^\star - GY^\star]. & (4.10) \end{aligned}$$

The result of the lemma is immediate if we can conclude that $G^\star$ divides $X^\star B - \lambda Y^\star A$. If $A^\star$ and $G^\star$ are coprime, this would be immediate. However, $A^\star$ and $G^\star$ may not be coprime. Use (4.8) and (4.9) to rewrite (4.10) as

$$A_1^\star M_-^\star[X^\star B - \lambda Y^\star A] = G_1^\star M_-[BC^\star - GY^\star]. \tag{4.11}$$

Next $M_-^\star$ is a factor of $B$ and therefore is coprime with $A$; hence $M_-$ is coprime with $A^\star = A_1^\star M_+^\star M_-^\star$. From (4.11), we see then that $M_-$ divides $X^\star B - \lambda Y^\star A$ and therefore also $X^\star B$. Since $M_-$ divides $A$, it is coprime with $B$. Hence $M_-$ divides $X^\star$, i.e. $X^\star = X_1^\star M_-$. Also (4.3) yields

$$G_1M_+M_-^\star C = A_1M_+M_-X + BY \tag{4.12}$$

and it follows easily that $Y = M_+Y_1$. So (4.11) can be rewritten as

$$A_1^\star\left[X_1^\star B_1 M_-^\star - \lambda Y_1^\star A_1 M_+\right] = G_1^\star\left[B_1 C^\star - G_1 M_+ Y_1^\star\right]$$

Now $A_1^\star$ and $G_1^\star$ are coprime. Hence $A_1^\star$ divides $B_1C^\star - G_1M_+Y_1^\star$, i.e. there exists a polynomial $V(z)$ such that $A_1^\star V = B_1C^\star - G_1M_+Y_1^\star$. It follows that

$$\begin{aligned} A^\star V &= A_1^\star M_+^\star M_-^\star V \\ &= B_1 M_+^\star M_-^\star C^\star - (G_1 M_+ M_-^\star)(Y_1^\star M_+^\star) \\ &= BC^\star - GY^\star \end{aligned} \qquad (4.13)$$

as required. The identity (4.6) is then immediate from (4.13) and (4.10). The degree constraint on $V$ is immediate from those on $Y$, $G$, $A$, $B$ and $C$. ■

It remains to specify how to choose the particular $X$, $Y$ pair satisfying (4.3). A trivial calculation shows that if $X$, $Y$, $V$ is any triple of polynomials satisfying (4.3), (4.6) and (4.7), then these equations are also satisfied by

$$\overline{X} = X + kB,\ \overline{Y} = Y - kA \ \text{ and } \ \overline{V} = V + kG$$

for any constant $k$. If $V$ has degree $n$, choose $k$ such that $\overline{V}$ has degree less than $n$. Else choose $k = 0$. It follows that we can assume $\deg V < n$ in solving (4.3), (4.6) and (4.7).

Using earlier notations, we introduce the following plant and controller factorizations:

$$\begin{aligned} N_P &= \frac{B(z)}{G(z)}, & D_P &= \frac{A(z)}{G(z)}, \\ N_C &= \frac{Y(z)}{C(z)}, & D_C &= \frac{X(z)}{C(z)} \end{aligned} \qquad (4.14)$$

where $G$, $X$ and $Y$ are determined from $A$, $B$, $C$ via (4.2) and (4.3). Note that these fractional representations fulfill the Bezout identity (1.1) and that each transfer function is stable and proper.

The first term in (3.4) is given by

$$\begin{aligned} \mathcal{B}(z) &= \left[\frac{z^d B^\star X - A^\star Y}{G^\star C}\right]\left|\frac{C}{G}\right|^2, \\ &= \left[\frac{z^l G V^\star}{G^\star C}\right]\left|\frac{C}{G}\right|^2 = \frac{z^l V^\star C^\star}{G^\star G^\star} \end{aligned}$$

where $V(z)$ is defined by (4.6) and $l = n - \deg V \geq 1$. Because $\mathcal{B}|_{z=0} = 0$ and $G^\star G^\star$ is unstable, we have $[\mathcal{B}]_{\rm st} = 0$, as expected. The second term in (3.4) is

$$-Q^\star\left[|D_C|^2 + \lambda|N_C|^2\right]|D_P|^2\phi_0 = -Q^\star\frac{|X|^2 + \lambda|Y|^2}{GG^\star}.$$

If we take

$$Q = G^\star R \qquad (4.15)$$

where $R$ is any element of $\mathbf{S}$ that has relative degree $n+1$, then the second term in (3.4) is unstable and has a zero at $z = 0$. This means that $\overline{S}_{\rm opt} = -\overline{\mathcal{A}}^{-1}[\overline{\mathcal{A}}^{-\star}\overline{\mathcal{B}}]_{\rm st} = 0$ because $\overline{\mathcal{A}}^{-\star}$ is unstable by definition.

**Theorem 4.2** *Consider some ARMAX system ($A$, $B$, $C$) and let the system $(P_0, \phi_0) = (\frac{B}{A}, |\frac{C}{A}|^2)$ and its optimal LQG controller $C_0$ have fractional representations defined as in (4.14). Let*

$$P_1 = (N_P - QD_C)(D_P + QN_C)^{-1} \qquad (4.16)$$

*be a perturbation of $P_0$ and consider the following noise spectrum*

$$\phi_1 = \frac{|D_P|^2\phi_0}{|D_P + QN_C|^2} \qquad (4.17)$$

*which assures that assumption (3.3) is satisfied. Then, for fixed $\lambda$, any system $(P_1, \phi_1)$ with $Q$ defined as in (4.15) has the same optimal LQG controller, $C_0 = N_C D_C^{-1}$, as the unperturbed system $(P_0, \phi_0)$. Moreover, for fixed $\lambda$, any system $(P_1, \phi_1)$ that has optimal LQG controller $C_0 = N_C D_C^{-1}$ can be expressed as (4.16) and (4.17) for some $Q$ defined in (4.15).* ■

**The minimum variance disturbance rejection problem ($\lambda = 0$)**

For $\lambda = 0$, the LQG regulation criterion reduces to the minimum variance disturbance rejection criterion $J_{\rm MV} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}E\{y_t^2\}$. The minimum variance disturbance rejection control law is given by (see [3]):

$$u_t = -\frac{K(z)}{B_+(z)F(z)}y_t \qquad (4.18)$$

where $K(z)$ and $F(z)$ are polynomials that satisfy the Diophantine equation

$$z^{d-1}C(z)B_-^\star(z) = A(z)F(z) + B_-(z)K(z) \quad (4.19)$$

in which the polynomial $F(z)$ has degree $d + \deg B_- - 1$ and $\deg K < n$.

The plant and controller factorizations are defined by:

$$\begin{aligned} N_P &= \frac{B(z)}{C(z)}, & D_P &= \frac{A(z)}{C(z)}, \\ N_C &= \frac{K(z)}{z^{d-1}B_+(z)B_-^\star(z)}, & D_C &= \frac{F(z)}{z^{d-1}B_-^\star(z)}. \end{aligned} \qquad (4.20)$$

Note that these fractional representations fulfill the Bezout identity (1.1) and that each transfer function is stable and proper.

The first term in (3.4) is given by

$$\begin{aligned} \mathcal{B}(z) &= [N_P^\star D_C - \lambda D_P^\star N_C]|D_P|^2\phi_0 \\ &= \frac{z^d B^\star(z)}{C^\star(z)}\frac{F(z)}{z^{d-1}B_-^\star(z)} = \frac{zB_+^\star(z)F(z)}{C^\star(z)}. \end{aligned}$$

Because $\mathcal{B}|_{z=0} = 0$ and $C^\star$ is unstable, we have $[\mathcal{B}]_{\rm st} = 0$ as expected. The second term in (3.4) is

$$-Q^\star|D_C|^2|D_P|^2\phi_0 = -Q^\star\frac{|F|^2}{z^{d-1}B_-^\star B_-}.$$

If we take

$$Q = B_- R \qquad (4.21)$$

where $R$ is any element of **S** with relative degree $d + \deg B_-$ (i.e. $Q$ has relative degree $d$), then the second term in (3.4) is unstable and has a zero at $z = 0$. This means that $\overline{S}_{\rm opt} = -\overline{A}^{-1} [\overline{A}^{-\star}\overline{B}]_{\rm st} = 0$ because $\overline{A}^{-\star}$ is unstable by definition.

**Theorem 4.3** *Consider some ARMAX system (A, B, C) and let the system* $(P_0, \phi_0) = (\frac{B}{A}, |\frac{C}{A}|^2)$ *and its optimal MV controller* $C_0$ *have fractional representations defined as in* (4.20). *Let*

$$P_1 = (N_P - QD_C)(D_P + QN_C)^{-1} \qquad (4.22)$$

*be a perturbation of* $P_0$ *and consider the following noise spectrum*

$$\phi_1 = \frac{|D_P|^2 \phi_0}{|D_P + QN_C|^2} \qquad (4.23)$$

*which assures that assumption* (3.3) *is satisfied. Then any system* $(P_1,\phi_1)$ *with Q defined as in* (4.21) *has the same optimal MV controller,* $C_0 = D_C^{-1} N_C$*, as the unperturbed system* $(P_0,\phi_0)$*. Moreover, any system* $(P_1,\phi_1)$ *that has optimal MV controller* $C_0 = N_C D_C^{-1}$ *can be expressed as* (4.22) *and* (4.23) *with Q defined in* (4.21). ■

## 5 Numerical example

To illustrate the theoretical results proposed above, let us take an ARMAX system described by (4.1) with

$$A(z) = z,\ B(z) = b,\ C(z) = z + h.$$

The optimal LQG controller (4.4) is given by

$$u_t = -\frac{bhz}{(b^2+\lambda)z + \lambda h}\, y_t \qquad (5.24)$$

and the plant and controller factorizations are given by (4.14) with

$$\begin{aligned} G(z) &= (\sqrt{b^2+\lambda})\, z, \\ X(z) &= \frac{(b^2+\lambda)z + \lambda h}{\sqrt{b^2+\lambda}}, \\ Y(z) &= \frac{bhz}{\sqrt{b^2+\lambda}}. \end{aligned}$$

Take $Q = (\sqrt{b^2+\lambda})\, R$ where $Q \in$ **S** has relative degree 2. Then, for fixed $\lambda$, any ARMAX system $(P_1,\phi_1)$ that has optimal LQG controller (5.24) can be expressed as

$$\begin{aligned} P_1(Q) &= \frac{\frac{b}{z} - Q\,\frac{(b^2+\lambda)z+\lambda h}{z+h}}{1 + Q\,\frac{bhz}{z+h}}, \\ \phi_1(Q) &= \left| \frac{\frac{z+h}{z}}{1 + Q\,\frac{bhz}{z+h}} \right|^2. \end{aligned}$$

Let us take an ARMAX system (4.1) with

$$\begin{aligned} A(z) &= z^2 + a_1 z + a_2, \\ B(z) &= z + b_1, \\ C(z) &= z^2 + c_1 z + c_2 \end{aligned}$$

and $|b_1| < 1$. The optimal MV controller (4.18) is given by

$$u_t = -\frac{C(z) - A(z)}{B(z)}\, y_t \qquad (5.25)$$

and the plant and controller factorizations are given by (4.20) with

$$\begin{aligned} K(z) &= C(z) - A(z), F(z) = 1, \\ B_+(z) &= B(z) \text{ and } B_-(z) = 1. \end{aligned}$$

Any ARMAX system $(P_1,\phi_1)$ that has optimal MV controller (5.25) can be expressed as

$$\begin{aligned} P_1(Q) &= \frac{\frac{z+b_1}{z^2+c_1 z+c_2} - Q}{\frac{z^2+a_1 z+a_2}{z^2+c_1 z+c_2} + Q\,\frac{(c_1-a_1)z+(c_2-a_2)}{z+b_1}}, \\ \phi_1(Q) &= \left| \frac{1}{\frac{z^2+a_1 z+a_2}{z^2+c_1 z+c_2} + Q\,\frac{(c_1-a_1)z+(c_2-a_2)}{z+b_1}} \right|^2 \end{aligned}$$

where $Q \in$ **S** has relative degree 1.

## 6 Conclusions

In this paper, we have used the Youla parametrization to characterize the set of all plants that have the same optimal LQG or MV controller.

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