Identification for control: can the optimal restricted complexity model always be identified ?[1]

F. De Bruyne, M. Gevers

CESAME, Université Catholique de Louvain, Bâtiment Euler, Avenue Georges Lemaître 4-6, B1348 Louvain-La-Neuve, Belgium.

Abstract: We consider model based certainty equivalence control design for a high order plant on the basis of a model that is identified within a restricted complexity model set. Each model in this model set yields a "restricted complexity" controller. This defines a controller set which contains a "best" controller for the high order plant. The following question is addressed: can the restricted complexity model (or models) that produces this "best" restricted complexity controller always be obtained by identification from plant data, given adequate experimental conditions? The answer is 'not always'.

Keywords: experiment design, input spectrum, open loop identification, closed loop identification, identifiability.

1 Problem formulation

Let P be some "true" linear time invariant but unknown system, which is typically of high order, and consider that the task is to design a feedback controller for P using a certainty equivalence design on the basis of a model of P identified from input-output data. It is assumed that after some initial analysis, the designer has selected a certain (typically low order) model set $\mathcal{M}$:

$$\mathcal{M} \triangleq \{\hat{P}(\theta), \theta \in D_\theta \subset \mathbf{R}^{\mathbf{d}}\}. \tag{1.1}$$

It is further assumed that the true plant is not contained in the (reduced order) model set, i.e. there exists no value of θ for which $\hat{P}(\theta) = P$ at almost all frequencies. The situation described above is typical of complex process control applications, in which the model used for control design is typically much simpler than the plant and is necessarily data-based.

It is assumed that the control design is based on the minimization of some control performance criterion, $J(P, C)$, that has been selected once and for all by the designer: an H_∞ design criterion, an LQG control criterion, etc. If the (high order) plant were known exactly, then the minimization over the class of all admissable full order controllers would result in the ideal optimal controller,

$$C_{\text{opt}}^{\text{ho}} = \arg\min_{C} J(P, C), \tag{1.2}$$

[1] **Acknowledgement:** This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility rests with its authors.

to which corresponds an optimal cost, denoted $J^{\text{ho}}_{\text{opt}} = J(P, C^{\text{ho}}_{\text{opt}})$[2]. This "optimal high order controller" depends on the true unknown plant P and is therefore not computable. The design of the controller is performed using the certainty equivalence principle on the basis of a nominal loop, in which the true plant is replaced by a model, $\hat{P}(\theta)$, identified in the reduced order model set $\mathcal{M}$. This allows us to define the "reduced order controller set"; it is the set of controllers that are optimal for all models in the reduced order model set:

$$\mathcal{C} = \{C(\hat{P}(\theta)) : C(\hat{P}(\theta)) = \arg\min_{C} J(\hat{P}(\theta), C), \forall \theta \in D_\theta\}. \tag{1.3}$$

When any one of the controllers $C(\hat{P}(\theta)) \in \mathcal{C}$ is applied to the true system, this produces an "achieved cost" $J(P, C(\hat{P}(\theta)))$. This cost can be infinite if the certainty equivalence controller $C(\hat{P}(\theta))$ does not stabilize the true plant P, but we shall assume that there are at least some controllers in the set $\mathcal{C}$ that produce a finite achieved cost. We can therefore define the best "low order controller" as the controller in the set $\mathcal{C}$ that minimizes the achieved cost on the real plant[3]:

$$C^{\text{lo}}_{\text{opt}} = \arg\min_{C \in \mathcal{C}} J(P, C). \tag{1.4}$$

The cost achieved by $C^{\text{lo}}_{\text{opt}}$ on the real plant is denoted $J^{\text{lo}}_{\text{opt}} = J(P, C^{\text{lo}}_{\text{opt}}) \geq J^{\text{ho}}_{\text{opt}}$. The optimal low order controller is again not computable since it also depends on the unknown true plant P, but the point is that such optimal low order controller can be defined.

The optimal low order controller $C^{\text{lo}}_{\text{opt}}$ is the one that one would like to select through our procedure of identification in a reduced order model set $\mathcal{M}$ followed by model based controller design. A question that arises immediately is the following: is there an identification setup, i.e. experimental conditions (open loop identification, closed loop identification, etc) and choices of the design parameters (input spectra, data filters, etc), such that

$$C^{\text{lo}}_{\text{opt}} = \arg\min_{C \in \mathcal{C}} J(P(\hat{\theta}), C) \text{ with } \hat{\theta} = \arg\min_{\theta \in D_\theta} V(\theta) \tag{1.5}$$

where $V(\theta)$ is some identification criterion. In other words, is it possible to identify, in the reduced order model set, the restricted complexity model which yields $C^{\text{lo}}_{\text{opt}}$ as corresponding optimal controller. Stated even more briefly, is it possible to "identify", in the reduced order controller set, the optimal low order controller $C^{\text{lo}}_{\text{opt}}$, from data collected on P ? The answer, perhaps somewhat surprisingly, is "not always". This is shown by an example.

The outline of the paper is as follows. In Section 2, we show what is actually meant by identification setup and we describe the "identification design tools" that are available to cope with the previously defined problem. In Section 3, we illustrate the problem with a Linear Quadratic Gaussian (LQG) control design for a third order (ARX) plant in which the control design is based on a first order ARX model to be identified using plant input output data. We conclude in the last section.

[2]The superscript 'ho' stands for high order.

[3]The set $\mathcal{C}$ might not be closed w.r.t. the operation $\min_{C \in \mathcal{C}} J(P, C)$, or there might be more than one optimum, but for the sake of simplicity we shall assume that there is a unique $C^{\text{lo}}_{\text{opt}}$ for which $C^{\text{lo}}_{\text{opt}} = \arg\min_{C \in \mathcal{C}} J(P, C)$.

2 What is meant by identification setup ?

We first briefly recall the procedure and properties of prediction error identification. The true plant is assumed to be representable as follows,

$$y_t = P(z)u_t + v_t \tag{2.1}$$

where $P(z)$ is a scalar strictly proper rational transfer function, u_t is the input, v_t is an unmeasurable disturbance acting on the output y_t. We consider that the model is described by

$$y_t(\theta) = \hat{P}(z,\theta)u_t + \hat{H}(z,\theta)e_t. \tag{2.2}$$

Here $\hat{P}(z,\theta)$ and $\hat{H}(z,\theta)$ are proper rational transfer functions parametrized by some real vector θ, $\hat{H}(z,\theta)$ is monic, and e_t is a zero mean white noise sequence of variance σ^2. From the model set (2.2) it is easy to write the one step ahead prediction for y_t [2]:

$$\hat{y}_{t|t-1}(\theta) = \hat{H}^{-1}(z,\theta)P(z,\theta)u_t + [1 - \hat{H}^{-1}(z,\theta)]y_t. \tag{2.3}$$

From (2.1) and (2.3), we find that the prediction error $y_t - \hat{y}_{t|t-1}(\theta)$ is given by

$$\epsilon_t(\theta) = y_t - \hat{y}_{t|t-1}(\theta) = \hat{H}^{-1}(z,\theta)\left[\left(P(z) - \hat{P}(z,\theta)\right)u_t + v_t\right]. \tag{2.4}$$

In Least-Squares prediction error identification, the estimation of the parameter vector θ on the basis of N input-output data is obtained by minimizing the sum of the squares of the prediction errors $\{\epsilon_t(\theta), t = 1,\ldots,N\}$. In most cases, the prediction error sequence is filtered through some stable transfer function $D(z)$. We denote by $\epsilon_t^f(\theta)$ the filtered errors:

$$\epsilon_t^f(\theta) = D(z)\epsilon_t(\theta). \tag{2.5}$$

Least-squares prediction error identification then amounts to estimate the optimal $\hat{\theta}_N$ that minimizes the identification criterion $V_N(\theta)$

$$\hat{\theta}_N = \arg\min_{\theta\in D_\theta} V_N(\theta) \quad \text{where } V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}[\epsilon_t^f(\theta)]^2. \tag{2.6}$$

Under reasonable conditions on the data and the model structure [2], $\hat{\theta}_N$ converges as $N \longrightarrow \infty$ to

$$\theta^\star = \arg\min_{\theta\in D_\theta} \overline{V}(\theta) \quad \text{where } \overline{V}(\theta) = \lim_{N\longrightarrow\infty} E\,V_N(\theta) = E\left[\epsilon_t^f(\theta)\right]^2, \tag{2.7}$$

this last equality being valid only if the data sequence is a realization of a stationary stochastic process. Using Parseval's relation

$$E\left[\epsilon_t^f(\theta)\right]^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{\epsilon^f}(\omega)\mathrm{d}\omega, \tag{2.8}$$

it is possible, by expressing the filtered prediction errors as a function of the "true system" and the model transfer functions, to obtain an expression for the frequency distribution of the asymptotic model error. This exercise makes it possible to express the variance of $\epsilon_t^f(\theta)$ in the

frequency domain both for the case of open loop identification and closed loop identification.

If we assume that the data have been collected while the process was operating in open loop, in which case u_t and e_s are uncorrelated for all t and s, then it follows from (2.4) that,

$$\overline{V}(\theta) = \int_{-\pi}^{\pi} \left(\left| P(z) - \hat{P}(z,\theta) \right|^2 \phi_u + \phi_v \right) \frac{|D(z)|^2}{|\hat{H}(z,\theta)|^2} d\omega. \tag{2.9}$$

This integral expression gives an implicit characterization of the model $\hat{P}(z,\theta^\star)$ to which $\hat{P}(z,\hat{\theta}_N)$ converges if the number of data go to infinity: see (2.7). In other words, it gives an implicit characterization of the asymptotic bias error. Expression (2.9) is useful because it shows that, in the situation where some restricted complexity model set has been chosen for $\hat{P}(z,\theta)$ and $\hat{H}(z,\theta)$, one can still manipulate the frequency distribution of the plant/model error to a certain extent by playing with the design variables $D(z)$ and ϕ_u, namely the data filter and the input spectrum.

We now consider that the data have been collected on the true processs when some one degree of freedom controller, $u_t = -C(z)y_t + r_t$, was operating, and we derive a similar expression for the frequency distribution of the plant/model error in this case of closed loop identification. First, we compute the expression of $\epsilon_t^f(\theta)$. Substituting the "closed loop" expression of u_t in (2.4) yields

$$\epsilon_t^f = \frac{D(z)}{\hat{H}(z,\theta)(1 + C(z)P(z))} \left[\left(P(z) - \hat{P}(z,\theta) \right) r_t + \left(1 + C(z)\hat{P}(z,\theta) \right) v_t \right]. \tag{2.10}$$

When identification is performed on closed loop data collected on the process operating under a one degree of freedom controller, the estimate $\hat{\theta}_N$ converges to the minimum of the following cost function (dropping the z dependence for simplicity of notation):

$$\overline{V}(\theta) = \int_{-\pi}^{\pi} \left(\left| P - \hat{P}(\theta) \right|^2 \phi_r + \left| 1 + C\hat{P}(\theta) \right|^2 \phi_v \right) \frac{|D|^2}{|\hat{H(\theta)}|^2 |1 + CP|^2} d\omega. \tag{2.11}$$

Expression (2.11) describes in an implicit way the asymptotic distribution of the error between the true system $P(e^{j\omega})$ and the estimated model $\hat{P}(e^{j\omega},\theta^\star)$ when the identification is performed on data collected in closed loop using a reduced order model set. The following remarks can be made [1].

- The model fit is definitely influenced by the controller: the controller $C(z)$ exerts its influence through the sensitivity function $\frac{1}{1+C(z)P(z)}$ of the actual loop. The weighting on both terms of the integrand will be large where the sensitivity is large, namely around the crossover frequency of the closed loop system.

- External excitation is clearly needed for closed loop identification. The model $\hat{P}(z,\theta)$ will approximate $P(z)$ only if the reference signal spectrum ϕ_r dominates the noise signal spectrum ϕ_v within the closed loop bandwith. Without external reference signal, the model will try to approximate the inverse of the controller, $C^{-1}(z)$.

- The data filter $D(z)$ can again be used to shape the fit globally.

Thus, by identification setup, we mean the choice of the experimental conditions: one can either do the identification in open loop or in closed loop. Also, tools are available to orient the frequency fit of the model to certain frequency ranges. By choosing a low pass input spectrum (ϕ_u or ϕ_r), for instance, the fit is enhanced at low frequencies. If a sinusoidal input signal is used (discrete input spectrum), the identified low order model is exact at the corresponding frequency. The data filter $D(z)$ has essentially the same role as the input spectrum, the difference being that it weighs both terms of the integrand in the same way. In the simplified case study that follows, the influence of the data filter is therefore not considered ($D(z) = 1$).

3 Analysis for a simplified case

In this section, we search a solution to the problem described in Section 1 in the case where the "true plant" is a third order continuous time ARMAX system, and where the controller is obtained by LQG design performed on a first order continuous time ARX model to be identified from data obtained on this third order plant. We will consider the case of prediction error identification and examine different experimental conditions. The identification will be performed by explicitly minimizing the integral expressions (2.9) and (2.11) over the parameter vector θ, i.e. the best asymptotic model is considered.
The "true" third order plant is a stable continuous time ARMAX process, with relative degree two and positive gain $\frac{b_0}{a_0}$, described by:

$$y = P(s)u + H(s)v \tag{3.1}$$

where

$$P(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \qquad H(s) = \frac{s + g_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \tag{3.2}$$

We consider a first order ARX model set[4] described as follows:

$$\hat{y}(\theta) = \hat{P}(s,\theta)u + \hat{H}(s,\theta)e \tag{3.3}$$

where

$$\hat{P}(s,\theta) = \frac{b}{s + a} \qquad \hat{H}(s,\theta) = \frac{1}{s + a}. \tag{3.4}$$

The control performance index is chosen to be the following LQG criterion:

$$J(P,C) = \lim_{T \longrightarrow \infty} E\left\{ \frac{1}{T} \int_0^T \{y_t^2 + \lambda u_t^2\} dt \right\}. \tag{3.5}$$

The optimal high order controller $C_{\text{opt}}^{\text{ho}}$ defined in Section 1 is the third order controller that results from the minimization of (3.5) over all admissable full order controllers.

The reduced order controller set corresponding (by minimization of (3.5)) to the reduced order model set of all first order ARX models consists of all proportional output controllers, $u_t =$

[4]This is an academic example. The reason for taking first order models is that they can be pictorially represented by their parameters (a, b) in a two-dimensional parameter space: see later.

$k\,y_t$. The optimal low order controller $C_{\text{opt}}^{\text{lo}}$ is then the solution of the following minimization problem:

$$k_{\text{opt}} = C_{\text{opt}}^{\text{lo}} = \arg\min J(P,k). \tag{3.6}$$

If we consider that the true system (3.1) is described by the following state-space equations,

$$\left\{ \begin{array}{rcl} \dot{x} & = & F\,x + G\,u + G'\,v \\ y & = & H\,x, \end{array} \right. \tag{3.7}$$

it is easily seen that the control design problem (3.5)-(3.6) can be solved by the minimization of the following criterion over all k:

$$J(P,k) = \int_{-\infty}^{\infty} \text{tr}\left((1+\lambda k^2) H\,[sI - (F+GHk)]^{-1}\,G'\,\sigma_v^2\,G'^T\,[sI-(F+GHk)]^{-*}\,H^T \right) \mathrm{d}\omega.$$

This minimization is equivalent to the following minimization problem:

$$k_{\text{opt}} = \arg\min \left(J(P,k) = \text{tr}\left\{ (1+\lambda k^2) H Q H^T \right\} \right) \tag{3.8}$$

under the constraint of the Lyapunov equation

$$O = (F+GHk)Q + Q(F+GHk)^T + G'G'^T\sigma_v^2. \tag{3.9}$$

Inserting the parameters of the true plant (3.2), this yields:

$$k_{\text{opt}} = \arg\min \left\{ (1+\lambda k^2) \left(\frac{\sigma_v^2 (g_1^2(a_0 - b_0 k) + a_2 g_0^2)}{2(a_0 - b_0 k)(a_1 a_2 - a_0 + (b_0 - a_2 b_1)k)} \right) \right\} \tag{3.10}$$

This last minimization problem is easily solved once the transfer function parameters are specified.

Since we now have a procedure to calculate k_{opt}, we can restate the question addressed in Section 1 as follows. Assuming that we perform an identification from data generated by the "plant" (3.1) using the model set (3.4), do there exist choices of the identification design parameters such that the identified first order model has a corresponding optimal controller which is equal to k_{opt} ? In other words, can the optimal reduced order controller be obtained by identification using an indirect scheme?

In the following, we characterize all first order models for which k_{opt} is optimal by their parameters (a,b) in the parameter space. By imposing that k_{opt} must be the stabilizing solution of the Riccati equation related to the low order control problem, we obtain a set of equations that define a half line $\mathcal{L}_1$ which characterizes, in the parameter space (a,b), all first order models for which k_{opt} is optimal [4]:

$$\mathcal{L}_1 : \left\{ \begin{array}{rcl} b(k_{opt}^2 - \frac{1}{\lambda}) - 2k_{opt}a & = & 0 \\ a - bk_{opt} & > & 0 \quad \text{(Stability of the nominal closed loop system).} \end{array} \right. \tag{3.11}$$

We now use a prediction error method to identify first order ARX models from data obtained on the third order plant, using different input designs. First, open loop identification is considered in which the input is generated with a first order low pass input spectrum of increasing bandwidth; the bandwidth parameter c is used as a design parameter:

$$\phi_u(\omega, c) = \frac{c^2}{c^2 + \omega^2}. \tag{3.12}$$

Substituting this $\phi_u(\omega, c)$, as well as the noise spectrum $\phi_v(\omega)$ corresponding to the "true" noise model H of (3.1) in (2.9), and minimizing with respect to a and b, we obtain, in the parameter space (a, b), the curve $\mathcal{L}_2$ of all ARX models that can be identified for all possible c.

Our identifiability question can then be rephrased as follows:

- Is there an intersection between the curve $\mathcal{L}_1$ of all first order models for which k_{opt} is optimal, and the curve $\mathcal{L}_2$ of all models that are identifiable ?
- Alternatively, does there exist a bandwidth c (that is an input spectrum $\phi_u = \frac{c^2}{c^2+\omega^2}$) for which the identified model is a point on the curve $\mathcal{L}_1$ of all first order models for which k_{opt} is optimal ?

We can make the following observations:

- When the model is identified in open loop and the bandwidth c of the low pass input spectrum is very small, the identified model will fit the static gain of the system. Thus, for $c \to 0$, the identified model will converge to a point on the line $b = \frac{b_0}{a_0} a$.
- When the bandwidth c of the input spectrum increases, the gain of the identified model will increase since for large c one attempts to fit a first order model of slope -20dB/decade to a third order model with relative degree two of slope -40dB/decade.
- When the bandwidth c of the input spectrum goes to infinity, the gain of the identified model also goes to infinity. The curve of all identified models $\mathcal{L}_2$ therefore becomes tangent to the b axis for very large c.

A typical curve $\mathcal{L}_2$ of open loop identified models is given by the full line in Figure 3.1.

The dashed line in this figure represents all (a, b) models that have the same static gain as the true system. By the observations made above, the curve $\mathcal{L}_2$ necessarily lies above that line. Therefore, the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ will intersect if and only if the slope of the half line $\mathcal{L}_1$ is higher than the system gain $\frac{b_0}{a_0}$. Using (3.10), we arrive at the following condition for the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\frac{2k_{\text{opt}}(\lambda)}{(k_{\text{opt}}(\lambda))^2 - \frac{1}{\lambda}} > \frac{b_0}{a_0}. \tag{3.13}$$

Figure 3.1: Typical curve of identified models $\mathcal{L}_2$ (—-).

By computing k_{opt} as a function of the system parameters and of λ, this condition can be rewritten as:

$$\frac{2a_0b_0 - a_1a_2b_0 - a_0a_2b_1}{a_0(a_0 - a_1a_2) - b_0\lambda^{-1}(b_0 - a_2b_1)} > \frac{b_0}{a_0}. \tag{3.14}$$

This condition implies that the existence of identification design conditions that allow one to obtain the optimal low order controller depends on the weighting of the input signal in the quadratic control criterion (3.5).

Now consider that closed loop identification is used, with a controller $u_t = r_t - ky_t$ (i.e. proportional output control), and with a reference signal that has a spectrum $\phi_r(\omega) = \frac{c^2}{c^2+\omega^2}$. The identified model is then influenced by the operating controller: see (2.11). When the bandwidth c of the reference spectrum becomes very small, the formula (2.11) tells us that the model will approximate the inverse of the controller: see also the comments made below that formula. Thus, as c tends to zero, a and b will tend to infinity so that the model $\frac{b}{s+a}$ tends

to the constant $-\frac{1}{k}$. This is exactly what happens, as we can see in the simulation to follow. The same qualitative conclusions apply as for open loop identification: there will be values of λ, and thus of k_{opt}, for which the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ do not intersect for any value of c.

The insight gained by this simple analysis allows us to construct an example in which the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ do not intersect, whatever the input design that is used during identification. Consider the same example as before with the following numerical values:

$$a_0 = 3000, \quad a_1 = 650, \quad a_2 = 45, \quad b_0 = 3000, \quad b_1 = 3000, \quad g_0 = 1, \quad g_1 = 0.$$

For these numerical values, we have represented in Figure 3.2 the half lines $\mathcal{L}_1$ of (a, b) values yielding k_{opt} for several values of λ.

Figure 3.2: Open and closed loop identification of a third order ARX plant by a first order ARX model using a low pass input spectrum (resp. reference spectrum) of increasing bandwidth c.

The left hand picture represents open loop identification, while the right hand picture represents closed loop identification with an operating controller $u_t = r_t - y_t$. The pictures also portray the set of models $\mathcal{L}_2$ that can be identified by applying the spectrum $\phi = \frac{c^2}{c^2+\omega^2}$ to the input signal (respectively the reference signal). The dashed line in the left hand picture represents the line $b = \frac{b_0}{a_0}a$, while the dashed line on the right hand side represents the line $b = -\frac{1}{k}a$, with $k = 1$ here.

It follows from the figure that for $\lambda < 5.03$, the curve $\mathcal{L}_1$ defined by (3.10) corresponds to negative values of a, that is unstable models: see Figure 3.2. If an output error model

structure is used for the identification of the first order model, then an unstable model cannot be obtained, whatever the choice of the input signal spectrum: thus, the curve $\mathcal{L}_2$ of identified models cannot be in the left hand quadrants of Figure 3.2. This is true both in the case of open loop or closed loop identification. Now, the half lines $\mathcal{L}_1$ of optimal (a, b) values for different values of λ are the same for ARX models and for output error models. Thus, we conclude that if an output error model structure is used, then there are values of the weighting λ in the LQG criterion for which the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ will not intersect.

4 Some concluding remarks

It is very often the case in practice that a certainty equivalence control design is based on a low order identified model, even though the plant to which this controller is applied has a more complex structure. To a given choice of restricted complexity model structure, there corresponds (by certainty equivalence design) a restricted complexity controller set, and hence a best controller within this set for the actual system. The question addressed in this paper has been whether the model (or set of models) that yields this best reduced order controller can always be obtained by identification, given that the choice of model structure is fixed but that the other identification design parameters are free. In other words: can the best reduced order controller always be obtained as a result of this indirect (i.e. model based) design scheme? We have shown, using a suitably designed example and a zest of analysis, that this is not always the case.

References

[1] Gevers M. (1993). "Towards a joint design of identification and control ?", (Plenary talk at the 2*nd* European Control Conference, Groningen, Holland), Birkhäuser, Boston, Basel, Berlin, pp. 111-151.

[2] Ljung L. (1987). *System Identification: Theory for the User.* Prentice-Hall, Englewood Cliffs, New Jersey.

[3] Maciejowski J. M. (1989). *Multivariable Feedback Control.* Addison-Wesley Publishing Company, Wokingham, England.

[4] Polderman J. W. (1985). "A note on the structure of two subsets of the parameter space in adaptive control problems", *Systems & Control letters*, Vol. 7, pp. 25-34.