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Computing LQG plant and controller perturbations.[1]

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Abstract: Using the dual Youla parametrizations of controller based coprime factor plant perturbations and plant based coprime factor controller perturbations, we provide a computational procedure for computing an optimal infinite horizon Linear Quadratic Gaussian (LQG) controller from any stabilizing controller. The method allows us to calculate a new optimal LQG controller from a previous one when the plant has slightly changed, and to quantify the change in the controller as a function of the change in the plant. In addition, we compute the degradation in the achieved LQG cost when the LQG controller is computed on the basis of a plant model that is "close to" the real plant, where the closeness is measured by some norm of the perturbation.

1 Introduction

Consider that you have some initial plant, P_0, and some controller, C_0, that stabilizes P_0. Using stable proper coprime factor descriptions of P_0 and C_0, and the Youla parametrizations, one can then characterize both the set $\mathcal{C}$ of all controllers stabilizing P_0, and the set $\mathcal{P}$ of all plants that are stabilized by C_0. The first set is parametrized in terms of normalized coprime factors of P_0 and C_0 and an arbitrary stable transfer function S, i.e. $\mathcal{C} = \{C(S)\}$. The transfer function S is often called the Youla parameter and its norm indicates the size of the perturbation away from C_0. The second (dual) set is parametrized in terms of normalized coprime factors of P_0 and C_0 and an arbitrary stable transfer function Q, i.e. $\mathcal{P} = \{P(Q)\}$. The transfer function Q is also called Youla parameter and its norm indicates the size of the perturbation away from P_0.
These parametrizations are explicitly described in the following Proposition, which contains a collection of results from [6, 8]. The results are expressed here for multivariable systems. They apply to both the discrete and continuous time case.

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Proposition 1.1 [6, 8] *Let P_0 and C_0 have the (left and right) fractional representations $P_0 = X_0^{-1}Y_0 = Z_0W_0^{-1}$ and $C_0 = L_0^{-1}M_0 = N_0D_0^{-1}$, where $X_0, Y_0, Z_0, W_0, L_0, M_0, N_0, D_0$ are matrices of appropriate dimensions whose elements are all in the ring of proper stable transfer functions. (We assume a negative feedback convention). Assume that the following "double Bezout" equations hold*

$$\begin{bmatrix} D_0 & Z_0 \\ N_0 & -W_0 \end{bmatrix} \begin{bmatrix} X_0 & Y_0 \\ M_0 & -L_0 \end{bmatrix} = \begin{bmatrix} X_0 & Y_0 \\ M_0 & -L_0 \end{bmatrix} \begin{bmatrix} D_0 & Z_0 \\ N_0 & -W_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{1.1}$$

These equations express both the fact that the factors are coprime and that the feedback loop formed by the plant P_0 and the controller C_0 is internally stable. In addition, they relate the left and right coprime factorizations. For any arbitrary stable (linear) operator S of compatible dimensions, define

$$\begin{array}{rclrcl} N & = & N_0 - W_0S, & D & = & D_0 + Z_0S \\ L & = & L_0 + SY_0, & M & = & M_0 - SX_0 \end{array} \tag{1.2}$$

1. *Then $C(S) = L^{-1}M = ND^{-1}$ is a stabilizing controller for $P_0 = X_0^{-1}Y_0 = Z_0W_0^{-1}$.*

2. *Furthermore,* any *controller C that stabilizes P_0 has fractional representations* (1.2).

The dual result can be stated in the following way. For any arbitrary stable (linear) operator Q of compatible dimensions, define

$$\begin{array}{rclrcl} X & = & X_0 + QM_0, & Y & = & Y_0 - QL_0 \\ Z & = & Z_0 - D_0Q, & W & = & W_0 + N_0Q \end{array} \tag{1.3}$$

1. *Then $P(Q) = X^{-1}Y = ZW^{-1}$ is stabilized by $C_0 = L_0^{-1}M_0 = N_0D_0^{-1}$.*

2. *Furthermore,* any *plant stabilized by C_0 has fractional representations* (1.3).

An important robust stabilization result is that $C(S)$ stabilizes $P(Q)$ if and only if S stabilizes Q (see [8]*).* ■

The previous Proposition provides powerful tools. It says that, once we know one stabilizing controller for a plant, we can easily generate the family of all stabilizing controllers, by means of fractional representations[2]. In this paper we use these parametrizations to solve a number of problems in the case where the control design criterion is a Linear Quadratic Gaussian (LQG) criterion. Our basic one degree of freedom control loop is that of Figure 1.1, where we assume that the reference r_t and the noise v_t are quasi-stationary stochastic processes, so that power spectral densities $\phi_r(\omega)$ and $\phi_v(\omega)$ can be defined. The control design criterion is the following LQG index (expressed here in discrete time).

$$J_{LQG} = \lim_{N\to\infty} \frac{1}{N} E\left\{ \sum_{t=1}^{N} \left\{ [y_t - r_t]^2 + \lambda u_t^2 \right\} \right\} \tag{1.4}$$

Using the Youla parametrizations and the LQG control design criterion (1.4), we solve the following problems.

[2]Similar statements can be made for the dual parametrization.

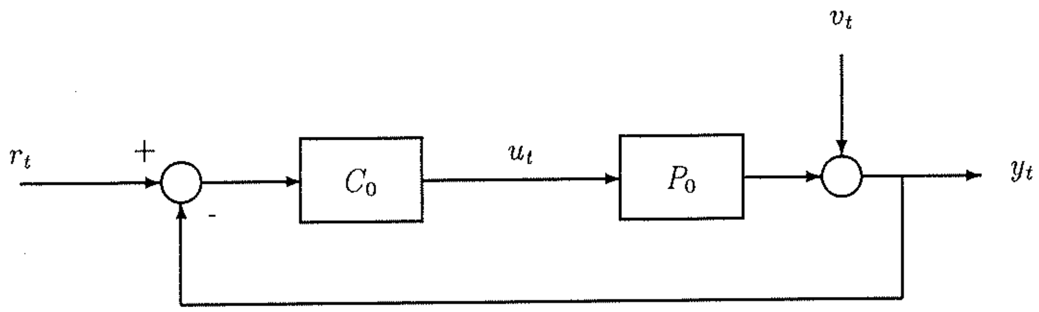

Figure 1.1: One degree of freedom control loop

1. For a given plant, P_0, we compute the optimal LQG controller as a function of an arbitrary stabilizing controller, C_0, and of the optimal Youla parameter, S_{opt}, without having to solve a Riccati equation.

2. Assume that the optimal LQG controller C_0 for a plant P_0 is known and consider a new plant P_1 that is stabilized by C_0 and that is obtained by a perturbation of size Q away from P_0. We then compute the optimal LQG controller C_1 for P_1 as a perturbation of size S away from C_0, where S is computed from P_0, C_0 and Q. This allows us to relate the size of a change in the plant to the size of the corresponding change in the optimal LQG controller.

3. Under the same assumptions as in 2 above, we compute the increase in the LQG cost (i.e. the performance degradation) that results from applying the controller C_1, optimal for P_1, to the initial plant P_0. This increase is expressed as a function of the size of the perturbation Q of P_1 away from P_0.

The main contribution (and our motivation for studying this problem) is in the framework of the currently emerging schemes for iterative identification and control design, in which models and model-based controllers are successively updated on the basis of new data collected on the real plant operating in feedback with the most recent controller: see [7], [5], [9] for a representative sample of these iterative design schemes and [4] for a tutorial presentation of the ideas. An implicit but unproven assumption underlying these schemes is that a small change in the plant model should result in a small change in the controller, and hence a small change in the actual closed loop system. This in turn should result in a slightly modified identified plant model. Our main contribution in this paper is to shed some light on this continuity question using the tools of coprime factor perturbations in the case of an LQG control criterion. Thus, in item 2 above, P_0 and P_1 could be seen as two successive plant models in an iterative design scheme, with C_0 and C_1 the corresponding optimal controllers. Alternatively, P_0 could also be the true plant, with P_1 a model that is close to it. We shall show that, under reasonable conditions, a small change in the plant yields a small change in the controller, with these changes being measured in either an H_2 or an H_∞ norm of the Youla parameter perturbation. The question addressed in item 3 is how much LQG cost increase is incurred by applying to the real plant P_0, say, an optimal controller C_1 computed on the basis of a plant model P_1 that is close to P_0. We shall give an explicit expression for this performance degradation.

In addition to these main results, our paper provides a number of new formulas that express various designed and achieved LQG costs in terms of coprime factor perturbations of an initial plant-controller pair. We believe that these formulas will prove to be useful in the solution of a number of related problems.

The outline of our paper is as follows. In Section 2 we present a solution[3] to the LQG controller design problem in the Youla parametrization framework starting from the plant model P_0 and any stabilizing controller C_0, using the set $\mathcal{C}(S)$ of all stabilizing controllers for P_0, i.e. we show how to compute S_{opt}. In Section 3 we apply our results to the case of a discrete time ARMAX process, together with an LQG regulation criterion and a minimum variance criterion. We compare these results with the classical solution using spectral factorization. In Section 4 we compute how much change is induced in a controller by a change in a plant model, while in Section 5 we express the degradation in the LQG cost that results from computing the LQG controller on the basis of a model that is a perturbed version of the actual plant. We conclude in Section 6.

2 Optimal LQG control in the Youla parametrization

Let $P_0 = X_0^{-1}Y_0 = Z_0W_0^{-1}$ and $C_0 = L_0^{-1}M_0 = N_0D_0^{-1}$ be left and right coprime factorizations of the plant P_0 and of an arbitrary stabilizing controller C_0 (see Figure 1.1), such that the "double Bezout" equations (1.1) hold. A rewriting of these equations gives

$$\left\{\begin{array}{l} D_0X_0 + Z_0M_0 = I \\ N_0Y_0 + W_0L_0 = I \\ D_0Y_0 - Z_0L_0 = 0 \\ N_0X_0 - W_0M_0 = 0 \end{array}\right. \qquad \left\{\begin{array}{l} X_0D_0 + Y_0N_0 = I \\ M_0Z_0 + L_0W_0 = I \\ X_0Z_0 - Y_0W_0 = 0 \\ M_0D_0 - L_0N_0 = 0 \end{array}\right. \tag{2.1}$$

It follows from Figure 1.1 that

$$\begin{aligned} y_t &= (I + P_0C_0)^{-1}P_0C_0\, r_t + (I + P_0C_0)^{-1}\, v_t \\ u_t &= (I + C_0P_0)^{-1}C_0\,(r_t - v_t) \end{aligned} \tag{2.2}$$

Using the "double Bezout" equations yields the following expressions for these transfer function matrices:

$$\begin{aligned} (I + P_0C_0)^{-1} &= D_0X_0 \\ (I + P_0C_0)^{-1}P_0C_0 &= I - D_0X_0 \\ (I + C_0P_0)^{-1}C_0 &= W_0M_0 \end{aligned} \tag{2.3}$$

According to the Proposition in Section 1, the set of all controllers stabilizing P_0 is given by

$$\mathcal{C} = (N_0 - W_0S)(D_0 + Z_0S)^{-1} = (L_0 + SY_0)^{-1}(M_0 - SX_0), \quad \text{where } S \text{ is stable.} \tag{2.4}$$

It is easy to show that, if (1.1) holds, the following double Bezout identities are verified.

$$\begin{aligned} &\begin{bmatrix} D_0 + Z_0S & Z_0 \\ N_0 - W_0S & -W_0 \end{bmatrix} \begin{bmatrix} X_0 & Y_0 \\ M_0 - SX_0 & -L_0 - SY_0 \end{bmatrix} \\ &= \begin{bmatrix} X_0 & Y_0 \\ M_0 - SX_0 & -L_0 - SY_0 \end{bmatrix} \begin{bmatrix} D_0 + Z_0S & Z_0 \\ N_0 - W_0S & -W_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{2.5}$$

[3]Our procedure is very similar to the one presented in [3].

Let C be any controller in the set $\mathcal{C}$ defined above. The transfer function matrices corresponding to (2.3) with C_0 replaced by C are now given by

$$\begin{aligned} (I+P_0C)^{-1} &= (D_0+Z_0S)X_0 \\ (I+P_0C)^{-1}P_0C &= I-(D_0+Z_0S)X_0 \\ (I+CP_0)^{-1}C &= W_0(M_0-SX_0) \end{aligned} \tag{2.6}$$

Consider now the LQG criterion (1.4). For simplicity, we shall assume that y_t, u_t and r_t are scalar; the extension to multivariable systems is straightforward.
Since $y_t - r_t = (1+P_0C)^{-1}(v_t - r_t)$ and $u_t = (1+CP_0)^{-1}C\,(v_t - r_t)$, the LQG index can be rewritten, using Parseval's theorem,

$$J_{LQG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left\{ \left|\frac{1}{1+P_0C}\right|^2 + \lambda\left|\frac{C}{1+P_0C}\right|^2 \right\} [\Phi_{rr}+\Phi_{vv}] \tag{2.7}$$

Using the expressions (2.6), specialized to this scalar case, we get an expression for the LQG index that is integrable in S:

$$J_{LQG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left\{ |D_0+Z_0S|^2 + \lambda|N_0 - W_0S|^2 \right\} |X_0|^2\Phi \tag{2.8}$$

where $\Phi = \Phi_{rr}+\Phi_{vv}$. If C_0 is the optimal LQG controller for P_0, then $S=0$ minimizes J_{LQG} over all stable S.

2.1 Computation of the optimal Youla parameter

We now consider that C_0 is an arbitrary stabilizing controller of P_0, and we compute the stable transfer function S that minimizes the previous LQG index.

$$\begin{aligned} \text{Integrand of } J_{LQG} &= \left\{ S^\star S[|Z_0|^2+\lambda|W_0|^2] + S[Z_0D_0^\star - \lambda W_0N_0^\star] \right. \\ &+ \left. S^\star[Z_0^\star D_0 - \lambda W_0^\star N_0] + [|D_0|^2+\lambda|N_0|^2] \right\} |X_0|^2\Phi \end{aligned}$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be defined as follows:

$$\mathcal{A}\mathcal{A}^\star = \left[|Z_0|^2+\lambda|W_0|^2\right]|X_0|^2\Phi \tag{2.9}$$

$$\mathcal{B} = [Z_0^\star D_0 - \lambda W_0^\star N_0]\,|X_0|^2\Phi \tag{2.10}$$

$$\mathcal{C} = \left[|D_0|^2+\lambda|N_0|^2\right]|X_0|^2\Phi \tag{2.11}$$

where $\mathcal{A}$ is minimum phase, stable and of relative degree zero.
Then the integrand of J_{LQG} is of the form

$$\begin{aligned} & S^\star S\mathcal{A}^\star\mathcal{A} + \mathcal{B}^\star S + S^\star\mathcal{B} + \mathcal{C} \\ = \; & [\mathcal{A}^\star S^\star + \mathcal{A}^{-1}\mathcal{B}^\star][\mathcal{A}S + \mathcal{A}^{-\star}\mathcal{B}] + \mathcal{C} - (\mathcal{A}^\star\mathcal{A})^{-1}\mathcal{B}^\star\mathcal{B} \end{aligned}$$

Let $T \triangleq \mathcal{A}S$. Minimizing J_{LQG} with respect to all stable S is equivalent to minimizing the following index with respect to all stable T:

$$\overline{J}_{LQG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega [T^\star + \mathcal{A}^{-1}\mathcal{B}^\star][T + \mathcal{A}^{-\star}\mathcal{B}] \tag{2.12}$$

The minimizing T is clearly given by $-[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{st}}$ where $[\ ]_{\mathrm{st}}$ denotes the stable part[4]. Then

$$S_{\mathrm{opt}} = -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{st}} \tag{2.13}$$

The optimal control cost is

$$\begin{aligned} J_{LQG}^{\mathrm{opt}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{unst}}|^2 + \mathcal{C} - (\mathcal{A}^\star\mathcal{A})^{-1}\mathcal{B}^\star\mathcal{B} \right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{unst}}|^2 + \left[(|D_0|^2 + \lambda|N_0|^2) - \frac{|Z_0^\star D_0 - \lambda W_0^\star N_0|^2}{|Z_0|^2 + \lambda|W_0|^2} \right] |X_0|^2\Phi \right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{unst}}|^2 + \frac{\lambda|D_0W_0 + N_0Z_0|^2}{|Z_0|^2 + \lambda|W_0|^2}|X_0|^2\Phi \right\} \qquad (2.14) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{unst}}|^2 + \frac{\lambda}{|Z_0|^2 + \lambda|W_0|^2}|X_0|^2\Phi \right\} \qquad (2.15) \end{aligned}$$

The last expression is only valid in the scalar case.

Remark: Suppose we restrict attention to optimizing over strictly proper (causal) controllers. We shall assume that the arbitrary stabilizing controller C_0 is strictly proper, so that in the fractional representation of the controller $N_0D_0^{-1}$, N_0 is strictly proper. Then the optimal controller, which has the structure $(N_0 - W_0S)(D_0 + Z_0S)^{-1}$ for some stable S, must be strictly proper. This holds if and only if:

$$\begin{gathered} N_0 - W_0S \text{ is strictly proper} \\ \Leftrightarrow \\ W_0S \text{ is strictly proper} \\ \Leftrightarrow \\ S \text{ is strictly proper} \end{gathered}$$

$S_{\mathrm{opt}} = -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{st}}$ will be strictly proper if and only if the constant term of $\mathcal{A}^{-\star}\mathcal{B}$ is assigned to $[\mathcal{A}^{-\star}\mathcal{B}]_{\mathrm{unst}}$.

2.2 Conditions for S_{opt} to be zero

We now consider the conditions on the coprime factors of P_0 and C_0 under which C_0 is optimal, i.e. the conditions under which $S_{\mathrm{opt}} = 0$ is optimal. First, we note the following result.

Lemma 2.1 *Let X be minimum phase and stable. Then $[X^{-\star}Y]_{\mathrm{st}} = 0$ if and only if $[Y]_{\mathrm{st}} = 0$.*

[4]Every finite rational operator $H(z)$ can be decomposed into the sum of its stable and unstable part, $H(z) = [H(z)]_{\mathrm{st}} + [H(z)]_{\mathrm{unst}}$.

Proof:

$$X^{-*}Y = [X^{-*}Y]_{\rm st} + [X^{-*}Y]_{\rm unst}$$

Therefore,

$$Y = X^{*}[X^{-*}Y]_{\rm st} + X^{*}[X^{-*}Y]_{\rm unst}$$

If $[X^{-*}Y]_{\rm st} = 0$, then Y is the product of two unstable functions; hence, $[Y]_{\rm st} = 0$.

On the other hand, $X^{-*}Y = X^{-*}[Y]_{\rm st} + X^{-*}[Y]_{\rm unst}$. If $[Y]_{\rm st} = 0$, then $X^{-*}Y$ is the product of two unstable functions, and hence $[X^{-*}Y]_{\rm st} = 0$. ∎

As a consequence, $S_{\rm opt} = 0$ if and only if

$$[\mathcal{B}]_{\rm st} = [(Z_0 D_0^{*} - \lambda W_0 N_0^{*}) X_0^{*} X_0 M_\Phi]_{\rm st} = 0, \tag{2.16}$$

where M_Φ is the minimum phase stable spectral factor of Φ.

If the plant P_0 is **stable**, X_0 is minimum phase and this condition simplifies to:

$$[\mathcal{B}]_{\rm st} = [(Z_0 D_0^{*} - \lambda W_0 N_0^{*}) X_0 M_\Phi]_{\rm st} = 0 \tag{2.17}$$

Finally, we compute the optimal LQG cost in the case where C_0 is optimal. In such case, $S_{\rm opt} = 0$, hence $[\mathcal{A}^{-*}\mathcal{B}]_{\rm st} = 0$, and therefore $[\mathcal{A}^{-*}\mathcal{B}]_{\rm unst} = \mathcal{A}^{-*}\mathcal{B}$. It follows from (2.8) that

$$J_{LQG}^{\rm opt} = \int_{-\pi}^{\pi} d\omega \left\{ |D_0|^2 + \lambda |N_0|^2 \right\} |X_0|^2 \Phi \tag{2.18}$$

3 Application in the case of an ARMAX plant model

In this section, we specialize our results to the case where the plant is described by an ARMAX model:

$$A(z) y_t = B(z) u_t + C(z) e_t \tag{3.1}$$

where e_t is white noise of zero mean and unit variance, $A(z)$, $B(z)$ and $C(z)$ are polynomials in z of degree n, $n-d$ and n respectively, with $A(z)$, $B(z)$ coprime, and $C(z)$ having all its zeros inside the unit circle (without loss of generality). Equation (3.1) is normalized so that the leading coefficients of the polynomials $A(z)$ and $C(z)$ are unity, i.e. $A(z)$ and $C(z)$ are monic. $B(z)$ can be factorized as $B(z) = B_-(z) B_+(z)$ where $B_+(z)$ has all its zeros strictly inside the unit circle and $B_-(z)$ has all its zeros on or outside the unit circle. We consider a disturbance rejection problem, and examine successively the case of the LQG criterion, $J_{LQG} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\{y_t^2 + \lambda u_t^2\}$, and the case of a minimum variance criterion, $J_{MV} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\{y_t^2\}$.

Our aim is first to substantiate our results of the previous section and connect them with the expressions obtained from the more classical spectral factorization method. In doing so, we also derive a number of new and interesting formulas for the LQG and minimum variance regulation problems.

3.1 The LQG regulation problem

The spectral factorization solution method for the infinite horizon LQG regulation problem consists in first computing the stable minimum phase spectral factor $G(z)$ of

$$G(z)G(z^{-1}) = \lambda A(z)A(z^{-1}) + B(z)B(z^{-1})$$

and then solving the following Diophantine equation for the polynomials $R(z)$ and $T(z)$:

$$G(z)C(z) = A(z)R(z) + B(z)T(z)$$

The resulting optimal controller is $u_t = -\frac{T(z)}{R(z)}y_t$ (See e.g. [2, 1]).
We will need an additional property of the polynomials constructed above.

Lemma 3.1 *There exists a polynomial $V(z)$ of degree at most n such that*

$$G(z^{-1})V(z) = R(z^{-1})B(z) - \lambda T(z^{-1})A(z) \tag{3.2}$$
$$A(z^{-1})V(z) = C(z^{-1})B(z) - T(z^{-1})G(z) \tag{3.3}$$

Proof:

$$\begin{aligned}
& A(z^{-1})\left[R(z^{-1})B(z) - \lambda T(z^{-1})A(z)\right] \\
&= A(z^{-1})R(z^{-1})B(z) - T(z^{-1})\left[G(z)G(z^{-1}) - B(z)B(z^{-1})\right] \\
&= B(z)\left[A(z^{-1})R(z^{-1}) + T(z^{-1})B(z^{-1})\right] - T(z^{-1})G(z)G(z^{-1}) \\
&= B(z)G(z^{-1})C(z^{-1}) - T(z^{-1})G(z)G(z^{-1}) \\
&= G(z^{-1})\left[B(z)C(z^{-1}) - G(z)T(z^{-1})\right] \qquad (3.4)
\end{aligned}$$

This gives the polynomial identity

$$\begin{aligned}
& A(z)\left\{R(z)[z^nB(z^{-1})] - \lambda T(z)[z^nA(z^{-1})]\right\} \\
&= G(z)\left\{[z^nB(z^{-1})]C(z) - T(z)[z^nG(z^{-1})]\right\}
\end{aligned}$$

Since $A(z)$ and $B(z)$ are relatively prime, so are $A(z)$ and $G(z)$. Hence there exists $\overline{V}(z)$, polynomial in z of degree n, such that

$$\begin{aligned}
A(z)\overline{V}(z) &= [z^nB(z^{-1})]C(z) - T(z)[z^nG(z^{-1})] \\
\text{or} \quad A(z)V(z^{-1}) &= B(z^{-1})C(z) - T(z)G(z^{-1})
\end{aligned}$$

where $V(z^{-1}) = z^{-n}\overline{V}(z)$. This proves (3.3).

Now (3.2) results from (3.4), which can be rewritten as

$$A(z^{-1})\left[R(z^{-1})B(z) - \lambda T(z^{-1})A(z)\right] = A(z^{-1})V(z)G(z^{-1})$$

■

Observe that a causal finite impulse response possesses a stable proper transfer function (i.e. polynomials in z^{-1} constitute stable transfer functions). Using earlier notations, we obtain:

$$\begin{aligned} Y_0 &= Z_0 = \frac{z^{-n}B(z)}{z^{-n}G(z)} & X_0 &= W_0 = \frac{z^{-n}A(z)}{z^{-n}G(z)} \\ N_0 &= M_0 = \frac{z^{-n}T(z)}{z^{-n}C(z)} & D_0 &= L_0 = \frac{z^{-n}R(z)}{z^{-n}C(z)} \end{aligned}$$

Note that these fractional representations fulfill the Bezout identities for general λ, and that each transfer function is stable and proper.

The computation of S_{opt} uses the two auxiliary transfer functions $\mathcal{A}$ and $\mathcal{B}$ defined in (2.9) and (2.10) respectively. We now compute these functions, dropping the arguments for convenience.

$$\mathcal{A}^\star\mathcal{A} = \left[|Z_0|^2 + \lambda|W_0|^2\right]|X_0|^2\Phi = \left[\left|\frac{B}{G}\right|^2 + \lambda\left|\frac{A}{G}\right|^2\right]\left|\frac{A}{G}\right|^2\left|\frac{C}{A}\right|^2 = \left|\frac{C}{G}\right|^2$$

It follows that the stable minimum phase spectral factor of relative degree zero is given by

$$\mathcal{A}(z) = \frac{C(z)}{G(z)}$$

Next

$$\begin{aligned} \mathcal{B}(z) &= [Z_0^\star D_0 - \lambda W_0^\star N_0]\,|X_0|^2\Phi \\ &= \left[\frac{B^\star}{G^\star}\frac{R}{C} - \lambda\frac{A^\star}{G^\star}\frac{T}{C}\right]\left|\frac{A}{G}\right|^2\left|\frac{C}{A}\right|^2 = [B^\star R - \lambda A^\star T]\frac{C^\star}{G^\star G G^\star} = \frac{V^\star C^\star}{G^\star G^\star} \end{aligned} \qquad (3.5)$$

where $V(z)$ is defined by (3.2).

Because all zeros of $G(z)$ are inside $|z| = 1$, $\mathcal{B}$ has no poles inside $|z| = 1$, except possibly at the origin. Observe that

$$\mathcal{B}(z) = \frac{z^n V(z^{-1}) z^n C(z^{-1})}{z^n G(z^{-1}) z^n G(z^{-1})}$$

Since $C(z)$ and $G(z)$ have degree exactly n, $z^n C(z^{-1})$ and $z^n G(z^{-1})$ have a finite non zero value at the origin. Since $V(z)$ has degree at most n, $z^n V(z^{-1})$ is finite at $z = 0$. Hence $\mathcal{B}$ has no pole at $z = 0$.

Since the constant term of $\mathcal{A}^{-\star}\mathcal{B}$ is part of the unstable part of $\mathcal{A}^{-\star}\mathcal{B}$, we have the expected result:

$$[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} = [\mathcal{B}]_{\text{st}} = 0 \text{ and hence } S_{\text{opt}} = 0.$$

The optimal cost is obtained from (2.14).

$$J_{LQG}^{\text{opt}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{\frac{|R|^2 + \lambda|S|^2}{|G|^2}\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega \left\{\frac{|V|^2 + \lambda|C|^2}{|G|^2}\right\}$$

3.2 The minimum variance regulation problem ($\lambda = 0$)

For $\lambda = 0$, the LQG regulation criterion reduces to the minimum variance criterion $J_{MV} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E\{y_t^2\}$.

The minimum variance disturbance rejection control [2] law is given by

$$u_t = -\frac{K(z)}{B_+(z)F(z)} y_t$$

where $K(z)$ and $F(z)$ are computed by the following equation:

$$z^{d-1}C(z)B_-^\star(z) = A(z)F(z) + B_-(z)K(z) \tag{3.6}$$

The plant and controller factorizations are defined by

$$\begin{aligned} Y_0 &= Z_0 = \frac{B(z)}{C(z)} & X_0 &= W_0 = \frac{A(z)}{C(z)} \\ N_0 &= M_0 = \frac{K(z)}{z^{d-1}B_+(z)B_-^\star(z)} & D_0 &= L_0 = \frac{F(z)}{z^{d-1}B_-^\star(z)} \end{aligned}$$

Note that these fractional representations fulfill the Bezout identity, and that each transfer function is stable and proper:

$$\begin{aligned} N_P N_{C_2} + D_P D_{C_2} &= \frac{B(z)K(z)}{C(z)B_+(z)z^{d-1}B_-^\star(z)} + \frac{A(z)F(z)}{C(z)z^{d-1}B_-^\star(z)} \\ &= \frac{B_-(z)K(z) + A(z)F(z)}{C(z)z^{d-1}B_-^\star(z)} = 1. \end{aligned}$$

We now have

$$\mathcal{A}^\star\mathcal{A} = \left[|N_P|^2 + \lambda|D_P|^2\right]|D_P|^2\Phi = \left|\frac{B}{C}\right|^2\left|\frac{A}{C}\right|^2\left|\frac{C}{A}\right|^2 = \left|\frac{B}{C}\right|^2$$

It follows that the stable minimum phase spectral factor of relative degree zero is given by

$$\mathcal{A}(z) = \frac{z^d B_+(z)B_-^\star(z)}{C(z)}$$

Also,

$$\mathcal{B}(z) = [N_P^\star D_C - \lambda D_P^\star N_{C_2}]\,|D_P|^2\Phi = \frac{B^\star(z)F(z)}{z^{d-1}C^\star(z)B_-^\star(z)} = \frac{B_+^\star(z)F(z)}{z^{d-1}C^\star(z)}$$

We have the expected result:

$$\mathcal{A}^{-\star}\mathcal{B} = \frac{z^d C^\star(z)}{B_+^\star(z)B_-(z)}\,\frac{B_+^\star(z)F(z)}{z^{d-1}C^\star(z)} = \frac{zF(z)}{B_-(z)}$$

$[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} = 0$ and $S_{\text{opt}} = 0$

The minimum variance control cost becomes

$$\begin{aligned} J_{MV}^{\text{opt}} &= J_{dr}^{\text{opt}} = \frac{1}{2\pi}\int d\omega |[\mathcal{A}^{-*}\mathcal{B}]_{\text{unst}}|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left\{ zz^{-1}\left|\frac{F(z)}{B_-(z)}\right|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left\{\left|\frac{F(z)}{B_-(z)}\right|^2\right\} \end{aligned}$$

Remark: In the case where the relative degree d is one and $B(z)$ has all its roots strictly inside the unit circle, i.e. $B_-(z) = 1$, $B_+(z) = B(z)$, $K(z) = C(z) - A(z)$ and $F(z) = 1$, we have the well known result:

$$u_t = -\frac{C(z) - A(z)}{B(z)} y_t \quad \text{and} \quad J_{MV}^{\text{opt}} = 1$$

4 Plant and corresponding controller perturbations

In this Section we examine the change that results in an optimal LQG controller when a plant model is changed from some initial model P_0 to a model P_1 that is expressed as a controller based perturbation of P_0. Consider first a plant model P_0 and its corresponding optimal (and hence stabilizing) controller C_0, both factorized as before. Let now P_1 be some plant that is stabilized by C_0. It can then be expressed as

$$P_1 = (X_0 + QM_0)^{-1}(Y_0 - QL_0) = (Z_0 - D_0Q)(W_0 + N_0Q)^{-1} \text{ for some stable } Q. \tag{4.1}$$

For any stable Q, the set of all controllers stabilizing P_1 is then given by

$$\overline{\mathcal{C}} = \left[N_0 - (W_0 + N_0Q)\overline{S}\right]\left[D_0 + (Z_0 - D_0Q)\overline{S}\right]^{-1} \tag{4.2}$$

$$= \left[L_0 + \overline{S}(Y_0 - QL_0)\right]^{-1}\left[M_0 - \overline{S}(X_0 + QM_0)\right] \tag{4.3}$$

where $\overline{S}$ is stable. We have called this parametrization $\overline{S}$ to distinguish it from S in (1.2) that parametrizes all controllers stabilizing P_0. Let C_1 be any controller in the set $\overline{\mathcal{C}}$.
Straightforward calculations show that, if (1.1) holds, the following double Bezout identities are also satisfied.

$$\begin{aligned} &\begin{bmatrix} D_0 + (Z_0 - D_0Q)\overline{S} & Z_0 \\ N_0 - (W_0 + N_0Q)\overline{S} & -W_0 \end{bmatrix} \begin{bmatrix} X_0 & Y_0 \\ M_0 - \overline{S}(X_0 + QM_0) & -L_0 - \overline{S}(Y_0 - QL_0) \end{bmatrix} \\ = &\begin{bmatrix} X_0 & Y_0 \\ M_0 - \overline{S}(X_0 + QM_0) & -L_0 - \overline{S}(Y_0 - QL_0) \end{bmatrix} \begin{bmatrix} D_0 + (Z_0 - D_0Q)\overline{S} & Z_0 \\ N_0 - (W_0 + N_0Q)\overline{S} & -W_0 \end{bmatrix} \\ = &\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{4.4}$$

The resulting LQG index is given by:

$$J_{LQG}(P_1, C_1) = \int_{-\infty}^{\infty} d\omega \left\{\left|D_0 + (Z_0 - D_0Q)\overline{S}\right|^2 + \lambda\left|N_0 - (W_0 + N_0Q)\overline{S}\right|^2\right\} |X_0 + QM_0|^2\Phi$$

It is integral in Q and S.

4.1 Computation of $\overline{S}_{\text{opt}}$ as a function of Q

In this Subsection, we characterize the optimal controller C_1^{opt}, i.e. we compute $\overline{S}_{\text{opt}}$ that minimizes J_{LQG} and express it as a function of Q and the coprime factorizations of the plant P_0 and its corresponding optimal controller C_0. Thus, $\overline{S}_{\text{opt}}$, which expresses C_1^{opt} as a perturbation of C_0^{opt}, will be defined as a function of Q, which expresses P_1 as a perturbation of P_0.

Recall that $\mathcal{A}$ and $\mathcal{B}$, related to the plant P_0 and its optimal controller C_0, are given by the following expressions:

$$\begin{aligned}\mathcal{A}\mathcal{A}^\star &= \left[|Z_0|^2 + \lambda |W_0|^2\right] |X_0|^2 \Phi \\ \mathcal{B} &= [Z_0^\star D_0 - \lambda W_0^\star N_0] |X_0|^2 \Phi.\end{aligned}$$

Two situations can occur when the system is perturbed: either the perturbation only influences the plant model, and the noise model remains unchanged (as happens in an OE model structure) or both the plant model and the noise model are influenced (as happens in an ARX or ARMAX model structure). We consider the case where Φ varies with Q in such a way that $|X_0 + QM_0|^2\Phi(Q)$ is independent of Q, i.e. $|X_0 + QM_0|^2\Phi(Q) = |X_0|^2\Phi(0)$. Other cases can be tackled in the same way and lead to similar conclusions.

We are now in a position to calculate the perturbed version of $\mathcal{B}$:

$$\begin{aligned}\overline{\mathcal{B}} &= [(Z_0^\star - Q^\star D_0^\star) D_0 - \lambda (W_0^\star + Q^\star N_0^\star) N_0] |X_0|^2 \Phi \\ &= \mathcal{B} - Q^\star \left[|D_0|^2 + \lambda |N_0|^2\right] |X_0|^2 \Phi.\end{aligned}$$

There will be a corresponding change from $\mathcal{A}$ to $\overline{\mathcal{A}}$.
The optimal $\overline{S}$ is given by $-\overline{\mathcal{A}}^{-1}[\overline{\mathcal{A}}^{-\star}\overline{\mathcal{B}}]_{\text{st}}$

$$\begin{aligned}[\overline{\mathcal{A}}^{-\star}\overline{\mathcal{B}}]_{\text{st}} &= [\overline{\mathcal{A}}^{-\star}\mathcal{B}]_{\text{st}} - \left[\overline{\mathcal{A}}^{-\star} Q^\star \left[|D_0|^2 + \lambda |N_0|^2\right] |X_0|^2 \Phi\right]_{\text{st}} \\ &= -\left[\overline{\mathcal{A}}^{-\star} Q^\star \left[|D_0|^2 + \lambda |N_0|^2\right] |X_0|^2 \Phi\right]_{\text{st}}\end{aligned}$$

where $\mathcal{B}$ is unstable by optimality and $\overline{\mathcal{A}}^{-\star}$ is by definition. Dropping high order terms in Q, we have:

$$\overline{\mathcal{A}} = \mathcal{A} + o(Q) \Rightarrow \overline{\mathcal{A}}^{-1} Q \simeq \mathcal{A}^{-1} Q + o(Q^2). \tag{4.5}$$

Therefore

$$\overline{S}_{\text{opt}} \simeq \mathcal{A}^{-1}\left[\mathcal{A}^{-\star} Q^\star \left[|D_0|^2 + \lambda |N_0|^2\right] |X_0|^2 \Phi\right]_{\text{st}} \tag{4.6}$$

It is possible to use a normalized coprime description of the controller by imposing that

$$|D_0|^2 + \lambda |N_0|^2 = 1. \tag{4.7}$$

The optimal value of $\overline{S}$ then reduces to

$$\overline{S}_{\text{opt}} \simeq \mathcal{A}^{-1}\left[\mathcal{A}^{-\star} Q^\star |X_0|^2 \Phi\right]_{\text{st}}. \tag{4.8}$$

4.2 A continuity question

The question we address is the following: Assume that the perturbation away from P_0 is small in some sense, will the optimal perturbation $\overline{S}_{\text{opt}}$ away from C_0 also be small ? To answer this question, we need the following lemma.

Lemma 4.1 *Let X and Z be transfer functions. Define $Y = [ZX]_{\text{st}}$ and let n be the degree of Y. Then the following results hold:*

$$\|Y\|_\infty = \|[ZX]_{\text{st}}\|_\infty \leq 2n\|ZX\|_\infty \leq 2n\|Z\|_\infty\|X\|_\infty \tag{4.9}$$

$$\|Y\|_2 = \|[ZX]_{\text{st}}\|_2 \leq \|ZX\|_2 \leq \|Z\|_\infty\|X\|_2 \tag{4.10}$$

Proof: $ZX = Y + U$ where U is the unstable part of ZX. Recall that the largest Hankel singular value $\sigma_1(Y)$ of Y can be characterized by

$$\sigma_1(Y) = \inf_{U_{\text{unstable}}}\|Y + U\|_\infty$$

It follows that

$$\sigma_1(Y) \leq \|ZX\|_\infty$$

The first result is proved from the following observation:

$$\|Y\|_\infty \leq 2(\sigma_1 + \sigma_2 + ... + \sigma_n) \leq 2n\sigma_1.$$

The second result follows from the fact that $\|X\|_2$ is finite. ■

By applying the previous lemma to (4.6) and (4.8), we observe that there is a risk, depending on Q, that $\|\overline{S}_{\text{opt}}\|_\infty$ could be large, even when $\|Q\|_\infty$ is small. However, if the degree of Q is limited and hence the degree of $\overline{S}_{\text{opt}}$, there is no serious problem.

In contrast to the ∞-norm case, there is no possibility for an explosion of $\|\overline{S}_{\text{opt}}\|_2$, provided $\|Q\|_2$ is small.

5 Plant and corresponding control cost perturbations

Let P_0 be the nominal plant and C_0 its optimal controller (i.e. condition (2.16) is satisfied). Let's assume that we have a model P_1 that is "Q apart" from the plant P_0 but still stabilized by C_0. It is obvious that P_1 is contained in

$$\mathcal{P}_1 = (X_0 + QM_0)^{-1}(Y_0 - QL_0) = (Z_0 - D_0Q)(W_0 + N_0Q)^{-1} \quad Q \text{ stable},$$

the set of all models stabilized by C_0. If C_1 is the optimal controller for P_1, one can try to find out how this controller performs on the real plant. One way to do that is to compare the optimal loop (P_0, C_0) and the achieved loop (P_0, C_1) by examining the respective costs.

The controller C_1 will be contained in

$$\begin{aligned}\overline{C_1} &= \left[N_0 - (W_0 + N_0 Q)\overline{S}\right]\left[D_0 + (Z_0 - D_0 Q)\overline{S}\right]^{-1} \\ &= \left[L_0 + \overline{S}(Y_0 - QL_0)\right]^{-1}\left[M_0 - \overline{S}(X_0 + QM_0)\right] \quad \overline{S} \text{ stable.}\end{aligned}$$

the set of all controllers stabilizing $P_1(Q)$.
The expression of the achieved control cost is now easily derived

$$J_{LQG}(P_0, C_1) = \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \frac{\left|D_0 + (Z_0 - D_0 Q)\overline{S}\right|^2 + \lambda\left|N_0 - (W_0 + N_0 Q)\overline{S}\right|^2}{\left|1 - Q\overline{S}\right|^2} \right\} |X_0|^2 \Phi$$

Since C_1 is optimal for P_1, $\overline{S}$ is equal to $\overline{S}_{\text{opt}}$. If we assume that $\|Q\|$ is small, then we have shown in the previous section that $\|\overline{S}_{\text{opt}}\|$ will be small and that therefore also $\|Q\overline{S}_{\text{opt}}\|$ will be small. Dropping second order terms, we obtain the following approximate expression for the control cost

$$J_{LQG}(P_0, C_1) \simeq \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \left|D_0 + (Z_0 - D_0 Q)\overline{S}_{\text{opt}}\right|^2 + \lambda\left|N_0 - (W_0 + N_0 Q)\overline{S}_{\text{opt}}\right|^2 \right\} |X_0|^2 \Phi$$

Expliciting the integrand and again dropping high order terms, the following approximate expression of the control cost is obtained

$$\begin{aligned}&J_{LQG}(P_0, C_1) \simeq J(P_0, C_0) \\ &\quad + \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ (D_0^\star Z_0 - \lambda N_0^\star W_0)\overline{S}_{\text{opt}} + (D_0 Z_0^\star - \lambda N_0 W_0^\star)\overline{S}^\star_{\text{opt}} \right\} |X_0|^2 \Phi\end{aligned}$$

$$J_{LQG}(P_0, C_1) \simeq J(P_0, C_0) + \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \mathcal{B}^\star \overline{S}_{\text{opt}}(Q) + \mathcal{B}\overline{S}^\star_{\text{opt}}(Q) \right\}$$

where $\mathcal{B}$ was defined earlier for the pair (P_0, C_0).

This shows that the increase in the control cost that results from applying the controller C_1, optimal for P_1, to the initial plant P_0 is small if the perturbation Q away from P_0 is small.

6 Conclusions

In this paper, we have presented a computational procedure to compute an infinite horizon LQG controller from a stabilizing controller using coprime factorizations. This procedure has allowed us to show that, under reasonable conditions, a small perturbation away from a given plant will produce a small perturbation away from the optimal controller corresponding to that plant. Also, the increase in the LQG cost that results from applying the "perturbed" controller to the real plant will be small as long as the plant/model perturbation is small.

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