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Avoidance and intersection in the complex plane, a tool for simultaneous stabilization

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Abstract

In this paper we study the following problem: "under what condition(s) is it possible to find a single controller which stabilizes k siso linear time invariant plants $p_i(s)$ $(i = 1, ..., k)$?". We show that the problem admits a solution if and only if an avoidance condition in the complex plane is satisfied and we use this last result to derive a sufficient condition for k plants to be simultaneously stabilizable.

Keywords: Simultaneous stabilization, avoidance, intersection.

1 Introduction

Simple questions can not always be simply answered. In this paper we give a very partial answer to a simple question in control theory which, for being open for ten years, does not seem to have a simple answer. The question is known under the name of *simultaneous stabilization problem* and is the following: "Under what condition(s) is it possible to find a single controller $c(s)$ which stabilizes k siso linear time invariant systems $p_i(s)$ $(i = 1, ..., k)$?". This question has already been solved for one or two plants: when $k = 1$, it is always possible to find a stabilizing controller, and when $k = 2$ a tractable necessary and sufficient condition, known as the *parity interlacing property*, exists (see [11], [17], [14]). The problem becomes harder when $k \geq 3$ (in fact no satisfactory answer exists even when $k = 3$) and most papers on the simultaneous stabilization problem deal either with necessary or with sufficient conditions ([1], [3], [8], [9], [15]).

The connection between interpolation in the complex plane and the simultaneous stabilization problem was pointed out by various authors (Ghosh [4],[5],[6], Helton [7]). In the same spirit we present the problem in this paper as an avoidance problem between complex valued functions. Roughly speaking, a set of k siso linear time invariant plants $\{p_1(s), ..., p_k(s)\}$ will be shown to be simultaneously stabilizable iff there exists a $k+1^{th}$ plant $p_{k+1}(s)$ which avoids, in a sense that we will define, the plants $p_1(s), ..., p_k(s)$ for all s in the extended closed right half plane. With this view of the problem we will prove a new sufficient condition under which k plants are simultaneously stabilizable.

2 Notations-Definitions

$\mathbf{R}(s)$ is the set of real rational functions. $\mathbf{C}_\infty$ is the extended complex plane $\mathbf{C} \cup \{\infty\}$ adequately topologized. Ω is any subset of $\mathbf{C}_\infty$. We shall suppose troughout this note that Ω is symmetric with respect to the real axis (if $s \in \Omega$ then $\bar{s} \in \Omega$), that it is closed, simply connected and that its complement in $\mathbf{C}_\infty$ contains at least one value of $\mathbf{R} \cup \{\infty\}$. Ω is to be thought of as the complement in $\mathbf{C}_\infty$ of a region of stability. Classical examples of regions Ω are the closed unit disc and the extended closed right half plane which correspond respectively to the complement in $\mathbf{C}_\infty$ of the discrete and continuous time stability regions. A real rational function $f(s) \in \mathbf{R}(s)$ is *Ω-stable* if it has no poles in Ω. $S(\Omega)$ is the set of all Ω-stable functions. A real rational function $f(s) \in \mathbf{R}(s)$ belongs to $M(\Omega)$ if it has no zeros in Ω. Finally, we define $U(\Omega) := M(\Omega) \cap S(\Omega)$.

3 Avoidance and intersection

Immediate checking shows that, whatever Ω, $S(\Omega)$ is a commutative ring. It is also known that under our hypothesis on Ω, the field of fractions of $S(\Omega)$ is $\mathbf{R}(s)$ (see for example [12], p.50). This means that if $p(s) \in \mathbf{R}(s)$ then there exist $n(s), d(s) \in S(\Omega)$ such that $p(s) = \frac{n(s)}{d(s)}$ where $n(s)$ and $d(s)$ have no common zeros in Ω. Such a fractional decomposition of $p(s)$ is called an *Ω-coprime decomposition.* We may now define what we mean by the intersections of two functions $p_1(s), p_2(s) \in \mathbf{R}(s)$ in Ω.

Definition. *Let $p_1(s), p_2(s) \in \mathbf{R}(s)$ and let $n_i(s), d_i(s) \in S(\Omega)$ be fractional Ω-coprime decompositions of $p_i(s)$ $i = 1, 2$. The intersections of $p_1(s)$ and $p_2(s)$ in Ω are the zeros of $n_1(s)d_2(s) - d_1(s)n_2(s) \in S(\Omega)$ in Ω. If $n_1(s)d_2(s) - d_1(s)n_2(s) \in U(\Omega)$ then $p_1(s)$ and $p_2(s)$ have no intersections in Ω and we say that they avoid each other in Ω.*

This definition may look somewhat mysterious. In fact it is very natural and the procedure to compute the intersections between plants is very simple. If Ω does not contain the point at infinity then the decompositions $p_1(s) = \frac{n_1(s)}{d_1(s)}$ and $p_2(s) = \frac{n_2(s)}{d_2(s)}$, where $n_i(s), d_i(s)$ are polynomials with no common zeros ($i = 1, 2$), are Ω-coprime decompositions and hence the intersections of $p_1(s)$ and $p_2(s)$ in Ω are simply the zeros in Ω of the polynomial $n_1(s)d_2(s) - d_1(s)n_2(s)$. If Ω contains the point at infinity then the possible additional intersections at infinity may be checked by inspection of the relative degree and gain of the functions. For example, the rational functions $p_1(s) = \frac{2s}{(s+1)(s-1)}$ and $p_2(s) = \frac{1}{s-3}$ have their intersection at the zeros of $2s(s-3) - (s+1)(s-1) = s^2 - 6s + 1$ and at the point at infinity since $p_1(\infty) = p_2(\infty) = 0$.

From our assumptions on Ω it follows that the module of a function in $S(\Omega)$ has an upper bound on Ω and that of a function in $M(\Omega)$ has a lower bound on Ω. This allows us to prove the next lemma.

Lemma 1. *Let $p_1(s) \in S(\Omega)$ and $p_2(s) \in M(\Omega)$. Then there exists $L > 0$ such that $lp_2(s)$ avoids $p_1(s)$ in Ω $\forall l > L$.*

Proof. $p_1(s) \in S(\Omega)$ and $p_2(s) \in M(\Omega)$ and hence there exist trivial fractional Ω-coprime decompositions $p_1(s) = \frac{n_1(s)}{1}$ and $p_2(s) = \frac{1}{d_2(s)}$ where $n_1(s), d_2(s) \in S(\Omega)$. Define $L = \sup_{s \in \Omega} \mid n_1(s)d_2(s) \mid > 0$. L is finite because

Ω is closed in the extended complex plane and $n_1(s)d_2(s)$ has no poles in Ω. It is clear that for every $l > L$, $n_1(s)d_2(s) - l$ is an element of $S(\Omega)$ which never takes the value zero when $s \in \Omega$. That is, $n_1(s)d_2(s) - l \in U(\Omega)\ \forall l > L$. In other words, $lp_2(s)$ avoids $p_1(s)$ in Ω for every $l > L$.

4 Simultaneous stabilization

In the usual sense a controller $c(s) \in \mathbf{R}(s)$ is said to be a stabilizing controller for a plant $p(s)$ if $p(s)c(s)(1+p(s)c(s))^{-1}$ is proper and has no poles with positive real part. In other words $c(s)$ stabilizes $p(s)$ if $p(s)c(s)(1+p(s)c(s))^{-1} \in S(\mathbf{C}_{+\infty})$ where $\mathbf{C}_{+\infty}$ is the extended complex right half plane. It was shown in [12] that this is an ill-stated definition of stability and that it is necessary for practical purposes to ask internal as well as external stability. A controller $c(s)$ is an *internal stabilizer* of a plant $p(s)$ if all the transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}, c(s)(1+p(s)c(s))^{-1}$ and $p(s)(1+p(s)c(s))^{-1}$ are in $S(\mathbf{C}_{+\infty})$. Since we want to treat stabilization problems in a general framework, encompassing continuous as well as discrete time stability, we will say that a controller $c(s) \in \mathbf{R}(s)$ *internally Ω-stabilizes* (or is an *internal Ω-stabilizer* of) $p(s) \in \mathbf{R}(s)$ if all the transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}, c(s)(1+p(s)c(s))^{-1}$ and $p(s)(1+p(s)c(s))^{-1}$ are in $S(\Omega)$. This notion of internal Ω-stabilization is strongly connected to that of avoidance in Ω.

Lemma 2. *Let $p(s), c(s) \in \mathbf{R}(s)$. Then the controller $c(s)$ internally Ω-stabilizes $p(s)$ if and only if $-c^{-1}(s)$ avoids $p(s)$ in Ω.*

Proof. Let $p(s) = \frac{n_p(s)}{d_p(s)}$ and $c(s) = \frac{n_c(s)}{d_c(s)}$ be Ω-coprime decompositions of $p(s)$ and $c(s)$. It is well known that $c(s)$ internally Ω-stabilize $p(s)$ iff $n_p(s)n_c(s) + d_p(s)d_c(s) \in U(\Omega)$ (see [12]). This last condition is satisfied if and only if $-c(s)^{-1}$ avoids $p(s)$ in Ω.
With this result we may formulate the simultaneous stabilization problem under the form of an avoidance problem.

Corollary 1. *Let $p_i(s) \in \mathbf{R}(s)$ $(i = 1, ..., k)$. The plants $p_i(s)$ are simultaneously internally Ω-stabilizable if and only if there exists a $c(s) \in \mathbf{R}(s)$ such that $-c^{-1}(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k)$.*

By using this last result and Lemma 1 it is straightforward to prove the next theorem.

Theorem 1. *Let $p_i(s) \in M(\Omega)$ $(i = 1, ..., k)$ and consider any $c(s) \in M(\Omega)$. Then there exists $\lambda \in \mathbf{R}$ such that $\lambda c(s)$ internally Ω-stabilizes $p_i(s)$ $(i = 1, ..., k)$.*

Proof. By using Lemma 1, for each $p_i(s)$ there exists $L_i > 0$ such that $-c^{-1}(s)$ avoids $lp_i(s)$ in Ω for every $l > L_i$. Define $L_{max} = max_{i=1,...,k}L_i$. Then clearly $-c^{-1}(s)$ avoids $lp_i(s)$ $\forall l > L_{max}$ $(i = 1, ..., k)$. Choose an $\lambda > L_{max}$ then $-c^{-1}(s)$ avoids $\lambda p_i(s)$ $(i = 1, ..., k)$ or, equivalentely, $-(\lambda c(s))^{-1}$ avoids $p_i(s)$ $(i = 1, ..., k)$ and by Corollary 1 the theorem is proved.

This last theorem is in fact a well known result in simultaneous stabilization when Ω is the extended right half plane (see for example [15], [16]). If k plants are minimum phase and proper but not strictly proper then there exists a controller, with arbitrarily specified poles and arbitrarily specified stable zeros, that internally stabilizes $p_i(s)$ $(i = 1, ..., k)$.

The condition of the theorem is a particular situation under which the simultaneous stabilization problem of k plants admits a solution. In the next theorem we prove a new result in the same vein; we give a sufficient condition under which k plants are simultaneously internally Ω-stabilizable. The underlying idea is the following: a finite set of plants is simultaneously stabilizable iff there exists an additional "plant" which avoids all of them (see Corollary 1). Suppose now that in a set of k plants $\{p_1, ..., p_k\}$ one of the plants (say p_1) avoids all the others. Then by Corollary 1 the plants $p_2, p_3, ..., p_k$ are simultaneously stabilizable by $-p_1^{-1}$. In fact it is then possible to do more than that: it is then possible to find a stabilizing controller for the whole set $\{p_1, ..., p_k\}$. This is essentially what is contained in our next theorem which is the central result of this paper.

Theorem 2. *Let $p_i(s) \in \mathbf{R}(s)$ $(i = 1, ..., k)$ and suppose that there exist a j $(1 \leq j \leq k)$ such that $p_j(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k$ and $i \neq j)$. Then the plants $p_i(s)$ $(i = 1, ..., k)$ are simultaneously internally Ω-stabilizable.*

Proof. Suppose without loss of generality that $j = 1$. Find an Ω-coprime fractional decomposition of $p_1(s)$, $p_1(s) = \frac{n_1(s)}{d_1(s)}$ with $n_1(s), d_1(s) \in S(\Omega)$. We know that under our assumptions on Ω (Ω is symmetric, simply connected and its complement contains at least one value in $\mathbf{R} \cup \{\infty\}$), $S(\Omega)$ is an Euclidean ring (see [12] for more details). Hence there exist $x(s), y(s) \in S(\Omega)$ such that $n_1(s)x(s)+d_1(s)y(s) = 1$. Since $p_1(s)$ avoids $p_i(s)$ in Ω $(i = 2, ..., k)$ we have that $n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, ..., k)$ and we define $u_i(s) = n_i(s)d_1(s) - d_i(s)n_1(s) \in U(\Omega)$ $(i = 2, ..., k)$. Finally we define $\delta = min_{i=2,...,k} \frac{inf_{s\in\Omega}|u_i(s)|}{sup_{s\in\Omega}|x(s)n_i(s)+y(s)d_i(s)|} > 0$ and choose ϵ with $0 < \epsilon < \delta$. We claim that $q(s) := \frac{n_1(s)-\epsilon y(s)}{d_1(s)+\epsilon x(s)} \in \mathbf{R}(s)$ avoids $p_i(s)$ in Ω $(i = 1, ..., k)$. Indeed, if $i = 1$ then $n_1(s)(d_1(s) + \epsilon x(s)) - d_1(s)(n_1(s) - \epsilon y(s)) = \epsilon(n_1(s)x(s) + d_1(s)y(s)) = \epsilon \in U(\Omega)$. Whereas for $i \geq 2$ we have $n_i(s)(d_1(s) + \epsilon x(s)) - d_i(s)(n_1(s) - \epsilon y(s)) = n_i(s)d_1(s) - d_i(s)n_1(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) = u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s))$. By construction of ϵ it is clear that $u_i(s) + \epsilon(x(s)n_i(s) + y(s)d_i(s)) \neq 0$ for every $s \in \Omega$ $(i = 2, ...k)$. This shows that $u_i(s)+\epsilon(x(s)n_i(s)+y(s)d_i(s)) \in U(\Omega)$ $(i = 2, ..., k)$ and thus $q(s) = \frac{n_1(s)-\epsilon y(s)}{d_1(s)+\epsilon x(s)}$ avoids $p_i(s)$ in Ω $(i = 2, ...k)$. Finally, $q(s)$ avoids $p_i(s)$ $(i = 1, ..., k)$ and, by applying Corollary 1, $-q^{-1}(s)$ is a simultaneous stabilizer for all $p_i(s)$ $(i = 1, ..., k)$.

5 Example

Let $p_1(s) = \frac{1}{s-1}, p_2(s) = \frac{-s}{3s+1}, p_3(s) = -\frac{s-2}{5s-1}$ and $p_4(s) = -\frac{s^2-3s+1}{7s^2-s+2}$. It is easy to see that $p_1(s)$ does not intersect any of the $p_i(s)$ in $\mathbf{C}_{+\infty}$ $(i = 2, 3, 4)$ and hence, by Theorem 2, the plants p_1, p_2, p_3 and p_4 are simultaneously internally $\mathbf{C}_{+\infty}$-stabilizable. It is even possible to say more. $p_1(s)$ intersect $p_i(s)$ $(i = 2, 3, 4)$ at the unique point $1 \in \mathbf{C}$ and hence the plants p_1, p_2, p_3 and p_4 are simultaneously internally Ω-stabilizable for any region Ω that does not contains $\{1\}$.

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