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Overbiased, underbiased and unbiased estimation of transfer functions [1]

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Abstract

The identification of an undermodeled transfer function from input-output data is stated as a constrained optimization problem. The constraints determine the identification procedure, the residual error and whether on the average the magnitude of the frequency response is overbiased, underbiased or unbiased, as measured by a certain weighted L_2-bias integral. The unbiased solutions are linear combinations of overbiased and underbiased solutions, which are precisely the classical least squares estimates. They can be obtained from the solution of certain eigenvalue problems. The results are illustrated with several numerical examples.

Automatica Key Words Index: Bias reduction, Constraint theory, Estimation theory, Frequency response, Identification, Least-squares estimation, Modeling, Model reduction, Parameter estimation.

1 Problem formulation

Consider a true linear system $G_T(s)$ with input-output representation $y(t) = G_T(s)u(t)$. The model $G(s,\theta)$ is parametrized as

$$G(s,\theta) = \frac{B(s,\theta)}{A(s,\theta)} = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \ldots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0}$$

in which all coefficients are real. The parameter vector θ is defined as $\theta^t = (a^t \;\; b^t)$ with $a^t = (\alpha_0 \; \alpha_1 \; \ldots \; \alpha_{n-1} \; \alpha_n)$ and $b^t = (\; \beta_0 \; \beta_1 \; \ldots \; \beta_{m-1} \; \beta_m \;)$. We can rewrite the true system equation as

$$y(t) = \frac{B(s,\theta)}{A(s,\theta)}u(t) + G_\Delta(s,\theta)u(t)$$

where $G_\Delta(s,\theta)$ represents the unmodelled dynamics. The input-output equation can also be written as:

$$A(s,\theta)y(t) - B(s,\theta)u(t) = A(s,\theta)G_\Delta(s,\theta)u(t) \quad (1)$$

Although the right hand side could be considered as an equation error, equation (1) as such can not be used for the purpose of identification because it contains transfer functions which are not proper, and hence require differentiation of signals. Therefore, we convert (1) into a system of proper transfer functions by introducing an observer polynomial $E(s)$ of degree $r \geq max(m,n)$. Moreover, we can improve the signal-to-noise ratio and avoid aliasing effects by first filtering the data with a filter with transfer function $F(s)$. This filter can also be used to focus the model fit into some desired frequency range. Thus (1) becomes

$$\begin{aligned}\frac{A(s)}{E(s)}F(s)y(t) - \frac{B(s)}{E(s)}F(s)u(t) &= \frac{F(s)G_\Delta(s)A(s)}{E(s)}u(t) \\ &= e(t) \qquad (say) \qquad (2)\end{aligned}$$

where $e(t)$ is to be considered as a residual or an equation error. We can rewrite this equation as

$$h^t(t)\begin{pmatrix} a \\ b \end{pmatrix} = e(t)$$

in which each element of $h(t)$ is a filtered version of the input or output signals of the form

$$\frac{F(s)s^i}{E(s)}y(t), i = 0,\ldots,n \quad \text{or} \quad -\frac{F(s)s^j}{E(s)}u(t), j = 0,\ldots,m$$

The object function $J(\theta)$ is defined as

$$J(\theta) = \int_0^T e^2(t)dt = (\; a^t \;\; b^t \;)[\int_0^T h(t)h^t(t)dt]\begin{pmatrix} a \\ b \end{pmatrix} \quad (3)$$

The 'information matrix' D is defined as

$$D = \int_0^T h(t)h^t(t)dt$$

It is positive definite if $h(t)$ spans $\Re^{m+n+2}$ over the interval $[0,T]$. This will be the case if the input $u(t)$ is '*sufficiently rich*' w.r.t. the dynamics $G_T(s)$ over the interval $[0,T]$ and if $G_T(s)$ cannot be modelled by a rational transfer function with polynomial degrees less than m and n. The problem of estimating the transfer function $G(s,\theta)$ can now be recast as a *constrained minimization problem*:

$$min_{over \; \theta \in \Re^{m+n+2}} \quad J(\theta) \qquad (4)$$

subject to constraints on $\theta = (a^t \;\; b^t)$. Without constraints on a and b a trivial and useless solution to the minimization problem would be $a = 0$ and $b = 0$.
Using Parseval's Theorem, the time domain criterion (3) (with $T = \infty$) can be rewritten as the following frequency domain least squares criterion (see e.g. [6]):

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$$J(\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |L(j\omega)|^2|A(j\omega,\theta)|^2 \dots$$
$$\times(|G_T(j\omega) - \frac{B(j\omega,\theta)}{A(j\omega,\theta)}|^2)d\omega \qquad (5)$$

where

$$L(j\omega) = \frac{F(j\omega)}{E(j\omega)}U(j\omega)$$

The minimization of (4) or equivalently, (5), subject to some specific constraints on a and b, yields a specific parameter vector $\hat{\theta}$ and a corresponding model $G(j\omega,\hat{\theta}) = B(j\omega,\hat{\theta})/A(j\omega,\hat{\theta})$. The fact that $\hat{\theta}$ can be described as the minimizing value of (5) shows how the fit between $G_T(j\omega)$ and the estimated model can be affected by specific choices of the filtered input spectrum $L(j\omega)$, i.e. how the bias can be shaped by appropriate frequency weighting.

2 Overbiased, underbiased and unbiased estimation

The constraints on a and b that are considered in this paper are of the form

$$v(a,b) = 0 \qquad (6)$$

where v is some linear or quadratic function of the coefficients in a and b. Using a Lagrange multiplier λ, the Lagrangean for the optimization problem is given by

$$\mathcal{L}(a,b,\lambda) = J(a,b) - \lambda v(a,b)$$

In order to minimize (5) subject to (6) one has to solve the following set of $m+n+3$ equations:

$$\frac{\partial\mathcal{L}}{\partial\alpha_i} = \frac{\partial J}{\partial\alpha_i} - \lambda\frac{\partial v}{\partial\alpha_i} = 0 \quad i = 0,\dots,n \qquad (7)$$
$$\frac{\partial\mathcal{L}}{\partial\beta_j} = \frac{\partial J}{\partial\beta_j} - \lambda\frac{\partial v}{\partial\beta_j} = 0 \quad j = 0,\dots,m \qquad (8)$$
$$v(a,b) = 0 \qquad (9)$$

It is straightforward to derive from (5) that

$$\frac{\partial J}{\partial\alpha_i} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |L|^2[(j\omega)^i A^* G_T G_T^* + (-j\omega)^i A G_T G_T^*$$
$$-(j\omega)^i G_T B^* - (-j\omega)^i B G_T^*]d\omega$$
$$\frac{\partial J}{\partial\beta_i} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |L|^2[(j\omega)^i B^* + (-j\omega)^i B \dots$$
$$- (-j\omega)^i A G_T - (j\omega)^i G_T^* A^*]d\omega$$

From this we find:

$$\sum_{i=0}^{n}\alpha_i\frac{\partial J}{\partial\alpha_i} - \sum_{i=0}^{m}\beta_i\frac{\partial J}{\partial\beta_i}$$
$$= \frac{2}{2\pi}\int_{-\infty}^{\infty} |L|^2|A|^2(|G_T|^2 - \frac{|B|^2}{|A|^2})d\omega$$
$$= 2V(\theta) \quad \text{say} \qquad (10)$$

The number $V(\theta)$ will be called the *L_2-bias integral.* The value of (10), for a specific model $\hat{\theta}$, reflects the bias, in a weighted square sense, between the magnitude of the true transfer function and that of the estimated model transfer function. Note that the frequency weighting is the same in (10) as in the identification criterion (5).

We shall call the estimated model *L_2-overbiased* if $V(\hat{\theta}) < 0$. In this case, the magnitude of the model's transfer function as a function of frequency, is on the average larger than the one of the true system. We call the model *L_2-underbiased* if $V(\hat{\theta}) > 0$. On the average, the magnitude of the model's transfer function is smaller than the one of the true system. The model is *L_2-unbiased* if $V(\hat{\theta}) = 0$. Intervals where the magnitude of the model's transfer function dominates the one of the true model, are compensated by regions where the true system's magnitude is larger than that of the model.

We can now combine equations (7)-(8)-(9) and equation (10) to find that

$$2V(\theta) = \lambda(\sum_{i=0}^{n}\alpha_i\frac{\partial v(a,b)}{\partial\alpha_i} - \sum_{j=0}^{m}\beta_j\frac{\partial v(a,b)}{\partial\beta_j}) \qquad (11)$$

The interpretation is the following: While we minimize the residual mean square error (5) subject to the constraint (6), we can at the same time obtain the numerical value of the *L_2-bias integral* (10) by substituting the optimal value of the Lagrange multiplier λ in (11).

In [7] it was observed that for a parametrization of the model transfer function $G(s)$ with $\alpha_n = 1$, the resulting least squares solution provides an *underbiased* model. In [4] it was observed that the constraint $\beta_m = 1$ results in an *overbiased* model. These observations and the conjecture formulated in [4] have stimulated the present research, in particular the quest for an *unbiased* identification scheme.

The main result of this paper is the observation that *the specific choice for a constraint on the vectors a and b determines the identification method on the one hand (least squares, eigenvalue decomposition, etc ...), the residual mean square error (3) and the bias integral (10) on the other hand.*

It will be shown how, by a careful choice of the constraints, one may construct identification schemes that are *unbiased*, and at the same time minimize $J(\hat{\theta})$ among all unbiased models.

Throughout we shall use the following notation: The information matrix D is partitioned as

$$D = \begin{matrix} & n+1 & m+1 \\ n+1 & D_{aa} & D_{ab} \\ m+1 & D_{ab}^t & D_{bb} \end{matrix}$$

The inverse of D is partitioned as:

$$D^{-1} = \begin{matrix} & n+1 & m+1 \\ n+1 & E & F \\ m+1 & F^t & G \end{matrix}$$

where $E = (D_{aa} - D_{ab}D_{bb}^{-1}D_{ab}^t)^{-1}$, $F = -D_{aa}^{-1}D_{ab}(D_{bb} - D_{ab}^t D_{aa}^{-1}D_{ab})^{-1}$ and $G = (D_{bb} - D_{ab}^t D_{aa}^{-1}D_{ab})^{-1}$. It is also a positive definite matrix and both E and G are square, symmetric, positive definite matrices. Estimates are denoted by a superscript $\hat{}$.

3 Linear constraints

Consider the minimization of (4) with a linear constraint on the k-th component of a:

$$\alpha_{k-1} = 1 \qquad 1 \le k \le n+1 \qquad (12)$$

We then find from (7)-(8)-(9) that

$$D\begin{pmatrix} a \\ b \end{pmatrix} = 1_k\lambda/2$$

The notation 1_k refers to the unit vector, which is zero everywhere, except for its k-th component, which is 1. Hence

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = D^{-1}1_k\lambda/2$$

which implies that the solution vector $(\hat{a}^t \ \ \hat{b}^t)^t$ is proportional to the k-th column of D^{-1}. The Lagrange multiplier λ can be determined from the constraint $\alpha_{k-1} = 1$:

$$J(\hat{\theta}) = \lambda/2 = 1/e_{kk} > 0$$

For the L_2-bias (10) we find from (11):

$$2V(\hat{\theta}) = \lambda$$

Hence, the bias integral is precisely equal to the k-th least squares residual! Its positivity implies that the linear constraint (12) leads to an *underbiased* model (which was observed in [7] for the constraint $\alpha_n = 1$).
Similarly, it can be derived from (7)-(8)-(9) and (11) that a constraint of the form

$$\beta_{k-1} = 1 \qquad 1 \leq k \leq m+1 \tag{13}$$

leads to an *overbiased* model:

$$V(\hat{\theta}) = -J(\hat{\theta}) = -\frac{1}{g_{kk}} < 0$$

This was already observed in [4] for the constraint $\beta_m = 1$. Here we find the more general result that the *overbiasedness* holds for *all* constraints (13). The solution vector $(a^t \ \ b^t)$ is now proportional to the $(k+n+1)-th$ column of D^{-1} and the Lagrange multiplier follows from the constraint (13).

Because all the solutions from the linearly constrained optimization problems of this section can be obtained via a 'classical' linear least squares scheme (see e.g. [1] [2] [3] [8]), we propose to call the columns of D^{-1} (normalized such that the constrained component is 1), the ***linear least squares solutions***. Hence there are $m+n+2$ linear least squares solutions, corresponding to the $m+n+2$ constraints (12) and (13).

4 Quadratic constraints

With a quadratic constraint of the form

$$a^t a = 1 \tag{14}$$

we find from (7)-(8)-(9) that

$$D_{aa}a + D_{ab}b = a\lambda \tag{15}$$
$$D^t_{ab}a + D_{bb}b = 0 \tag{16}$$

Since D is invertible and positive definite, it follows from Cauchy's eigenvalue interlacing property [5, p.269]) that the submatrix D_{bb} is also invertible. Hence $b = -D_{bb}^{-1}D^t_{ab}a$, so that

$$(\ D_{aa} - D_{ab}D_{bb}^{-1}D^t_{ab}\ )a = a\lambda \qquad a^t a = 1$$

Hence, we need to solve the eigenvalue problem for the Schur complement of the matrix D_{bb} in D (which is symmetric and positive definite) for its minimal eigenvalue and corresponding eigenvector. Observe that:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} E \\ F^t \end{pmatrix}\hat{a}\lambda$$

We find that the solution in this case is a linear combination of the first $n+1$ columns of D^{-1}, which are the $n+1$ least squares solutions of our identification problem, corresponding to the constraints $\alpha_0 = 1, \ldots \alpha_n = 1$.
We also observe from (15)-(16) that the optimal value $J(\hat{\theta})$ is given by:

$$J(\hat{\theta}) = (\ \hat{a}^t \ \ \hat{b}^t\ )D\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \lambda$$

Using (10) together with (15)-(16) we find:

$$V(\hat{\theta}) = J(\hat{\theta}) = \lambda > 0$$

The last two expressions show that this identification scheme provides an *underbiased* model.

Similarly, for the constraint

$$b^t b = 1 \tag{17}$$

the solution vector b is the eigenvector of the Schur complement of D_{aa} in D, corresponding to the smallest eigenvalue λ that satisfies:

$$(\ D_{bb} - D^t_{ab}D^{-1}_{aa}D_{ab}\ )b = b\lambda \qquad b^t b = 1 \tag{18}$$

The residual mean square error (3) is equal to the smallest eigenvalue λ of $D_{bb} - D^t_{ab}D^{-1}_{aa}D_{ab}$ (which is a positive definite matrix). The solution in this case is a linear combination of the last $m+1$ columns of D^{-1}, which are the $m+1$ least squares solution of our identification problem, corresponding to the constraints $\beta_0 = 1, \ldots, \beta_m = 1$. The value of the bias integral (10) is given by $-\lambda$ and is always negative. Hence, we have a systematic *overestimation* of the magnitude.
Observe that the identification method which follows from $b^t b = 1$ (i.e. the eigenvalue problem (18)) might be advantageous from the computational point of view if the numerator degree m is small.
Obviously, the constraint (14) can be viewed as a special case of constraints of the form

$$\sum_{i=0}^{r_1} \alpha_i^2 = 1 \tag{19}$$

with $r_1 \leq n$. This constraint will lead to an $(r_1+1) \times (r_1+1)$ symmetric positive definite eigenvalue problem. The solution will be a linear combination of the first r_1+1 columns of D^{-1}, which are the least squares solutions corresponding to $\alpha_0 = 1, \ldots, \alpha_{r_1} = 1$. The bias integral will be negative and hence we have an *underbiased* identification scheme.
It is interesting to note that, while all identifications with (19) are *underbiased*, the minimum of residual mean square error (3) decreases for increasing values of r_1. This is a direct consequence of the *eigenvalue interlacing theorem* [5, p.269] applied to the upper $(r_1+1) \times (r_1+1)$ blocks of the matrix D^{-1} for $r_1 = 0, \ldots, n$.
Similar conclusions hold of course for constraints on b

of the type $\sum_{j=0}^{r_2} \beta_i^2 = 1$ with $r_2 \leq m$. In this case, we have always an *overbiased* identification.

We can also combine the quadratic constraints (14) and (17) into one as

$$a^t a + b^t b = 1 \tag{20}$$

It now follows from (7)-(8)-(9) that

$$D \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \lambda$$

Obviously, the optimum value of λ is precisely the smallest eigenvalue λ_{min} of the matrix D. The solution for the vectors of polynomial coefficients a and b is given by

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = v_{min}$$

where v_{min} is the eigenvector of D corresponding to the smallest eigenvalue, normalized such that its norm equals 1. From (11) we find that

$$V(\hat{\theta}) = \lambda_{min}(\hat{a}^t\hat{a} - \hat{b}^t\hat{b})$$

Hence, the value of the bias integral depends not only on the smallest eigenvalue of D, but also on the difference of the norms of the vectors of polynomial coefficients (which are constrained by (20)). This identification scheme is *underbiased* if $\hat{a}^t\hat{a} - \hat{b}^t\hat{b} > 0$ and *overbiased* if $\hat{a}^t\hat{a} - \hat{b}^t\hat{b} < 0$.

The constraints (14), (17), (19) and (20) might be considered as a special case of a constraint of the form:

$$\sum_{i=0}^{r_1} \alpha_i^2 + \sum_{j=0}^{r_2} \beta_j^2 = 1 \tag{21}$$

with $0 \leq r_1 \leq n$ and $0 \leq r_2 \leq m$. This type of constraint leads to $(r_1 + r_2) \times (r_1 + r_2)$ eigenvalue problems for symmetric submatrices of D (which are necessarily positive definite because of the eigenvalue interlacing property). However, also because of the eigenvalue interlacing property, we know that only with the full quadratic constraint (20) (i.e. for $r_1 = n$ and $r_2 = m$), we get the minimal possible eigenvalue over all quadratic constraints, which is the minimal eigenvalue of D.

5 Multiplicative constraints

All identification schemes so far minimize the residual mean square error (3) but are *biased* with respect to the frequency criterion (10). The question remains whether there are certain types of constraints that give an *unbiased* model. Consider the minimization problem (4) with a constraint of the form:

$$\alpha_{k-1}\beta_{l-1} = \gamma \tag{22}$$

with $1 \leq k \leq (n+1)$, $1 \leq l \leq (m+1)$ and γ is a given real number. We'll show that this type of constraints leads to *unbiased* models. First observe that we are *not* free in the choice of the sign of γ. Indeed, assume that $k = l = 1$, then the real number β_0/α_0 is the *static gain* of the transfer function. Therefore, fixing the sign of γ corresponds to fixing the sign of the static gain, which implies a restriction on the model class. However, for the moment, we shall *assume* that we know the sign of γ. It will be shown below that we really do not need this information *a priori*. As a matter of fact, the identification scheme will always automatically allow both choices.

From (7)-(8)-(9) we find

$$D_{aa}a + D_{ab}b = 1_k\beta_{l-1}\lambda/2 \tag{23}$$

$$D_{ab}^t a + D_{bb}b = 1_l\alpha_{k-1}\lambda/2 \tag{24}$$

It follows from the invertibility of D that:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = D^{-1} \begin{pmatrix} 1_k\beta_{l-1} \\ 1_l\alpha_{k-1} \end{pmatrix} \lambda/2 \tag{25}$$

From this equation, we see that the optimal $\hat{\theta}$ can be obtained as a linear combination of two columns of D^{-1}, which are precisely 2 least squares solutions! The coefficients α_{k-1} and β_{l-1} can be calculated from the 2×2 eigenvalue problem:

$$\begin{pmatrix} f_{kl} & e_{kk} \\ g_{ll} & f_{kl} \end{pmatrix} \begin{pmatrix} \alpha_{k-1} \\ \beta_{l-1} \end{pmatrix} = \begin{pmatrix} \alpha_{k-1} \\ \beta_{l-1} \end{pmatrix} \kappa \tag{26}$$

where $\kappa = 2/\lambda$. The eigenvalues are given by

$$\kappa = f_{kl} \pm \sqrt{g_{ll}e_{kk}} \tag{27}$$

and are always real since the diagonal elements of E and G are positive. The 2×2 matrix in (26) is obtained by interchanging the columns of the 2×2 matrix

$$\begin{pmatrix} e_{kk} & f_{kl} \\ f_{kl} & g_{ll} \end{pmatrix} \tag{28}$$

But the 2×2 matrix in (28) is positive definite as a consequence of the eigenvalue interlacing property. In particular this implies that $e_{kk} > 0$, $g_{ll} > 0$ $e_{kk}g_{ll} > f_{kl}^2$. Hence there is always a positive and a negative eigenvalue in (27). Recall that we could not a priori fix the sign of γ in $\alpha_{k-1}\beta_{l-1} = \gamma$. But here we find precisely that such an a priori preference is not needed because we will always have the choice between a positive and a negative eigenvalue. In particular, when $k = 1$ and $l = 1$, the product $\alpha_0\beta_0$ is either positive or negative, corresponding to a static gain β_0/α_0 which is either positive or negative. We are interested in the eigenvalue κ with the largest absolute value (which corresponds to the λ with the least absolute value). Its sign will also determine the sign of γ. The eigenvectors have to be normalized such that $|\alpha_{k-1}\beta_{l-1}| = |\gamma|$. Having determined the coefficients $\alpha_{k-1}, \beta_{l-1}$ and λ, the remaining coefficients are determined from (25).

Premultiplying (23) with a^t and (24) with b^t and adding, we find

$$J(\hat{\theta}) = (\,\hat{a}^t \;\; \hat{b}^t\,) D \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \hat{\alpha}_{k-1}\hat{\beta}_{l-1}\lambda = \gamma\lambda$$

Obviously, from $J(\hat{\theta}) > 0$, we have that if $\gamma > 0$, then $\lambda > 0$ and if $\gamma < 0$, we must have $\lambda < 0$. For any specific choice of γ, we want to minimize $J(\theta)$, hence it suffices to look for the value of λ with least absolute value. From (10), it follows that

$$2V(\hat{\theta}) = (\hat{a}^t 1_k \hat{\beta}_{l-1} - \hat{b}^t 1_l \hat{\alpha}_{k-1})\lambda/2 = 0$$

Hence, this identification scheme is *unbiased*!

The results just derived have a very appealing interpretation: Recall that the columns of D^{-1} are precisely the *least squares solutions*. The first $n+1$ columns of D^{-1} are the solutions to the optimization problems (4) with one linear constraint (12) on a coefficient of $A(s)$, while the remaining $m+1$ columns are the solutions for a linear constraint (13) on a coefficient of $B(s)$. The former ones yield *underbiased* models while the latter ones yield *overbiased* models. For the constraint (22), we now see from (25) that the solution is described as a linear combination of the two least squares solutions obtained with a linear constraint on α_{k-1} and one on β_{l-1}. The respective weights attached to these two solutions follow from the 2×2 eigenvalue problem (26). The resulting solution is *unbiased*. Hence, we find that a certain linear combination of an *overbiased* and an *underbiased* solution, results in an *unbiased* one!

The multiplicative constraint (22) can be generalized to a constraint of the form

$$\sum_{i=1}^{r}\alpha_{i-1}\beta_{i-1}=\gamma \qquad 1\le r\le m+1 \tag{29}$$

where $\gamma\in\Re$ is given. For the same reason as before, the sign of γ is not fixed but will be determined furtheron. For the time being however, it is assumed that γ is a fixed given real number. Using the notation $a_r = (\ \alpha_0\ \ \ldots\ \ \alpha_{r-1}\ )^t$ and $b_r = (\ \beta_0\ \ \ldots\beta_{r-1}\ )^t$ we find from (7)-(8)-(9) that

$$D\begin{pmatrix} a\\ b\end{pmatrix}=\begin{pmatrix} b_r\\ 0\\ a_r\\ 0\end{pmatrix}\lambda/2$$

Observe that if $\hat\theta$ is a solution, then

$$J(\hat\theta)=(\ \hat a^t\ \ \hat b^t\ )D\begin{pmatrix}\hat a\\ \hat b\end{pmatrix}=\gamma\lambda$$

Since $J(\hat\theta)$ is always positive, it follows that λ and γ must have the same sign. We need to find the least absolute value of λ. It also follows from (10) that:

$$V(\hat\theta)=0$$

Hence, the resulting identification scheme is *unbiased*! From the nonsingularity of D, it follows that

$$\begin{pmatrix} a\\ b\end{pmatrix}=\begin{pmatrix} E & F\\ F^t & G\end{pmatrix}\begin{pmatrix} b_r\\ 0\\ a_r\\ 0\end{pmatrix}\lambda/2 \tag{30}$$

From this equation, we see that the solution will be a linear combination of the first $r+1$ columns of D^{-1} and its columns $(n+2)$ up to $(n+r+2)$. The coefficients $\alpha_i,\beta_i, i=0,1,\ldots,r-1$ can be determined from the $2r\times 2r$ eigenvalue problem:

$$\begin{pmatrix} F_r & E_r\\ G_r & F_r\end{pmatrix}\begin{pmatrix} a_r\\ b_r\end{pmatrix}=\begin{pmatrix} a_r\\ b_r\end{pmatrix}\kappa \tag{31}$$

where $\kappa=2/\lambda$. Here E_r, F_r and G_r are the $r\times r$ leading submatrices of E, F and G respectively. The matrix in (31) is obtained by interchanging the relevant *block columns* of the partitioned matrix D^{-1}. We are interested in the real eigenvalue κ of maximal absolute value. The corresponding eigenvector should be normalized so as to satisfy (29). The other coefficients can be determined from (30). It is interesting to note that all eigenvalues of (31) are real and that there are r positive and r negative ones. This is a direct consequence of the following lemma:

Lemma 1
Let Q be a $2q\times 2q$ real, symmetric, positive definite matrix with square $q\times q$ blocks: $Q=\begin{pmatrix} Q_{11} & Q_{12}\\ Q_{12}^t & Q_{22}\end{pmatrix}$
Then the eigenvalues of the matrix $\begin{pmatrix} Q_{12} & Q_{11}\\ Q_{22} & Q_{12}^t\end{pmatrix}$ *are real; q of them are positive and q negative.*

Proof: Define the block permutation matrix P as $P=\begin{pmatrix} 0 & I_q\\ I_q & 0\end{pmatrix}$. We are interested in the eigenvalues λ of QP, which are the roots of the characteristic equation $det(QP-\lambda I_{2q})=0$. Let the eigenvalue decomposition of Q be $Q=X\Lambda X^t$. Then:

$$det(P-\lambda Q^{-1})=0 \iff det(P-\lambda X\Lambda^{-1}X^t)=0$$

$$\iff det(X\Lambda^{-1/2}[\Lambda^{1/2}X^tPX\Lambda^{1/2}-\lambda I_{2q}]\Lambda^{-1/2}X^t)=0$$

$$\iff det(\Lambda^{1/2}X^tPX\Lambda^{1/2}-\lambda I_{2q})=0$$

So the eigenvalues of QP are the eigenvalues of $\Lambda^{1/2}X^tPX\Lambda^{1/2}$, which is symmetric, hence has real eigenvalues. Furthermore, $\Lambda^{1/2}X^tPX\Lambda^{1/2}$ is congruent to P, the eigenvalues of which are $+1$ (q times) and -1 (q times). Sylvester's Theorem [5, p.274] states that a congruence transformation preserves the inertia, which completes the proof. □

6 A numerical example

Consider a 'true' system $G_T(s)$ with a delay of 1 second:

$$G_T(s)=\frac{e^{-s}(s+1)}{s^2+2s+5}$$

This true system will be approximated by a transfer function

$$G(s)=\frac{\beta_0+\beta_1 s+\beta_2 s^2}{\alpha_0+\alpha_1 s+\alpha_2 s^2+\alpha_3 s^3}$$

The sampling time is 0.05 seconds and the simulation time is 40 sec. All simulations were done in *MATLAB*. As an input signal we apply white noise which is normally distributed with mean zero and variance 1. Both input and output are filtered through a 6-th order Butterworth filter with passband [1,3] rad/sec (see fig.1): $F(s)=B_f(s)/A_f(s)$ where $B_f(s)=8s^3$ and $A_f(s)=0.349s^6+1.983s^5+8.777s^4+19.904s^3+26.333s^2+17.855s+9.431$. The observer polynomial is $E(s)=(s+2)^3$.
In figure 2 we compare the identified transfer function with the one of the true system, for 4 different constraints. In figure 3, we show the L_2-bias integrand of the integral (10) as a function of frequency. The presence or absence of bias can clearly be seen from these figures and is conform with the theoretical results.

7 Conclusions

In this paper, we have shown how the estimation of undermodelled dynamics can be formulated as a constrained optimization technique. The constraints determine the identification method to be used (solving sets of linear equations or eigenvalue problems), the value of the residual mean square error and whether the magnitude of the estimated transfer function is overbiased, underbiased or unbiased as measured by a frequency

weighted integral. A survey of the results is given in the table at the end of this paper.
'Square root' versions of the algorithms in this paper are derived in [8]. These are algorithms where the explicit formation of the matrix D is avoided and the data matrix itself is used. The least squares solutions are obtained from QR-decompositions of the data matrix while the eigenvalue decompositions are replaced by singular value decompositions. In [8] we show that these square root versions are much more robust in certain modeling situations.

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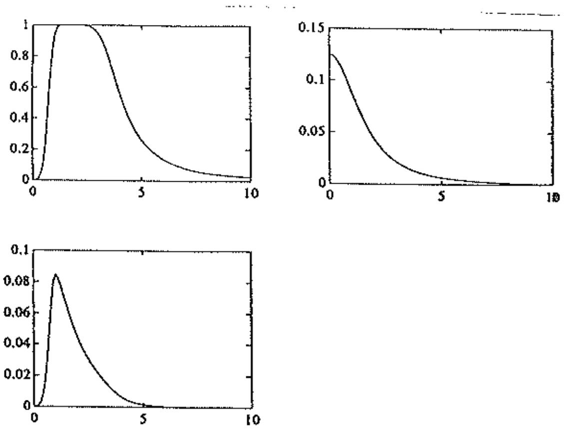

Figure 1: Magnitude of frequency responses of a/ the Butterworth filter. b/ the observer transfer function $1/E(s)$. c/ the filter $L(s) = F(s)/E(s)$.

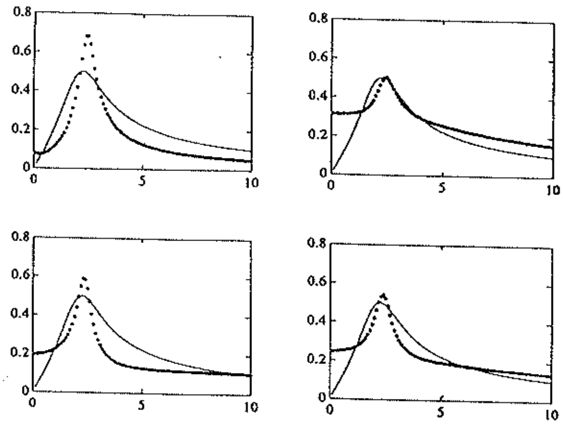

Figure 2: The magnitude of the real transfer function is plotted in full line. The lines with stars are the magnitude of the identified model for the constraints a/ $\alpha_1 = 1$, b/ $\beta_0 = 1$, c/ $a^t a + b^t b = 1$, d/ $\alpha_0\beta_0 = \pm 1$

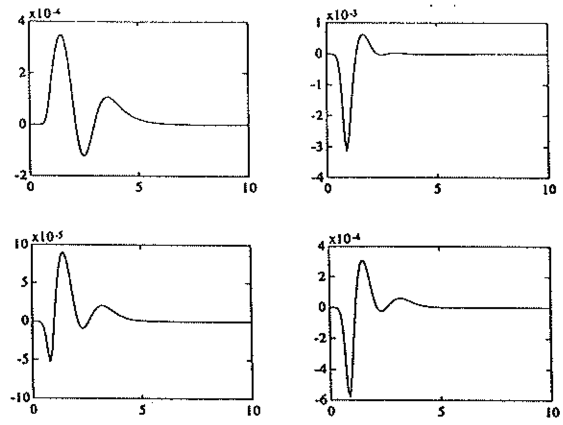

Figure 3: Integrand of the L_2-bias integral $V(\hat{\theta})$ as a function of frequency. We see that for the constraint a/ $\alpha_1 = 1$, the model is underbiased, for b/ $\beta_0 = 1$, overbiased, for c/ $a^t a + b^t b = 1$, underbiased ($a^t a - b^t b > 0$) and d/ $\alpha_0\beta_0 = \pm 1$, unbiased