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ESTIMATION OF TRANSFER FUNCTIONS : UNDERBIASED OR OVERBIASED ?

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ABSTRACT

Depending on whether the numerator or denominator of a transfer function model are chosen to be monic, the corresponding least squares estimate will be overbiased or underbiased, in a frequency weighted sense.

We consider the estimation of time-invariant transfer functions in a deterministic set up and under the typical situation where the true system has a transfer function $G_T(s)$ that is more complex than the parametrized model transfer function $G(s,\theta)$. Hence, there is no value of θ for which $G_T(s) = G(s,\theta)$ for all $s \in C$. We shall first consider that the model $G(s,\theta)$ is parametrized as follows

$$G(s,\theta) \stackrel{\Delta}{=} \frac{B(s,\theta)}{A(s,\theta)} \qquad (1)$$

where

$$A(s,\theta) \stackrel{\Delta}{=} s^n + a_1 s^{n-1} + \ldots + a_n \qquad (2)$$

$$B(s,\theta) \stackrel{\Delta}{=} b_0 s^m + b_1 s^{m-1} + \ldots + b_m \qquad (3)$$

The input-output representation of the true system is assumed to be noise free

$$y = G_T(s)u \qquad (3)$$

The exact input-output representation can also be written as

$$y = G(s,\theta)u + G_\Delta(s,\theta)u \qquad (4)$$

$$= \frac{B(s,\theta)}{A(s,\theta)}u + G_\Delta(s,\theta)u \qquad (5)$$

where $G_\Delta(s,\theta)$ represents the unmodelled dynamics. It is the set up and the model examined in [1].

In order to obtain a Least Squares estimate of θ, a standard procedure is to first transform (5) into regressor form by using an observer polynomial $E(s)$ of degree n, and to also filter the input and output data by a filter with transfer function $F(s)$ in order to focus the model fit into the desired frequency band. Thus (5) becomes

$$y_f = \frac{E(s) - A(s,\theta)}{E(s)} y_f + \frac{B(s,\theta)}{E(s)} u_f + \frac{A(s,\theta)}{E(s)} G_\Delta(s,\theta) u_f \qquad (6)$$

where

$$y_f = F(s)y, \quad u_f = F(s)u .$$

The input-output model is now rewritten in regressor form as

$$y_f(t) = \varphi^T(t)\theta + n(t) \qquad (7)$$

where

$$\varphi^T(t) = \left[\frac{s^{n-1} y_f(t)}{E(s)}, \ldots, \frac{y_f(t)}{E(s)}, \frac{s^m u_f(t)}{E(s)}, \ldots, \frac{u_f(t)}{E(s)} \right] \qquad (8)$$

$$\theta^T = [e_1 - a_1, \ldots, e_n - a_n, b_0, \ldots, b_m] \qquad (9)$$

$$n(t) = A(s,\theta) \frac{G_\Delta(s,\theta)}{E(s)} u_f(t) . \qquad (10)$$

If θ is estimated by Least Squares from (7), then it follows from Parseval's theorem (see [2]) that the optimal estimate $\hat{\theta}$ can be characterized in the frequency domain by

$$\hat{\theta} = \arg \min_\theta V(\theta) \qquad (11)$$

where

$$V(\theta) = \int_0^\infty |L(j\omega)|^2 \; |\hat{A}(j\omega)|^2 [G(j\omega,\theta) - G_T(j\omega)][G(j\omega,\theta) - G_T(j\omega)]^* d\omega \qquad (12)$$

and

$$L(j\omega) \stackrel{\Delta}{=} \frac{F(j\omega)}{E(j\omega)} U(j\omega) . \qquad (13)$$

We shall call the resulting optimal estimate $\hat{G}(j\omega)$:

$$\hat{G}(j\omega) \stackrel{\Delta}{=} G(j\omega,\hat{\theta}) . \qquad (14)$$

We note that the frequency weighting $L(j\omega)$ represents the combined effect of the identification filter $F(j\omega)$, the observer polynomial $E(j\omega)$ and the input $U(j\omega)$. It is thus a designer's choice.
Now in [1] the following remarkable result was proved :
Proposition 1 :

$$V(\hat{\theta}) = \int_0^\infty |L(j\omega)|^2 \; |\hat{A}(j\omega)|^2 \{ |G_T(j\omega)|^2 - |G(j\omega,\hat{\theta})|^2 \} d\omega \geq 0 \qquad (15)$$

The import of this result is that

$$\int_0^\infty |L(j\omega)|^2 \; |\hat{A}(j\omega)|^2 \; |G(j\omega,\hat{\theta})|^2 d\omega \leq \int_0^\infty |L(j\omega)|^2 \; |\hat{A}(j\omega)|^2 \; |G_T(j\omega)|^2 d\omega . \qquad (16)$$

Since this result holds whatever the designer's choice of the filter $L(j\omega)$, it means that, whatever the frequency band of interest, the gain of the model tends to be smaller than the gain of the true system. This came as a surprizing result, at least to the author of this note, but the proof in [1] is foolproof (as one would expect !), the result is confirmed by simulations, and other simulations in a paper of Ljung [3] on the deterministic estimation of transfer functions support the same result even though [3] does not focus on this problem.

We now show that the converse result holds (i.e. the gain of the estimated model overestimates the gain of the true system in a frequency weighted sense) if the model (1)-(2) is replaced by the equivalent model

$$G_1(s,\theta) \stackrel{\Delta}{=} \frac{C(s,\theta)}{D(s,\theta)} \qquad (17)$$

where

$$D(s,\theta) \stackrel{\Delta}{=} d_0 s^n + d_1 s^{n-1} + \ldots + d_n, \qquad (18a)$$

$$C(s,\theta) \stackrel{\Delta}{=} s^m + C_1 s^{m-1} + \ldots + C_m \qquad (18b)$$

Proposition 2 :
Let

$$V_1(\theta) = \int_0^{\infty} |L(j\omega)^2|\ \hat{D}(j\omega)|^2 [G_1(j\omega,\theta) - G_T(j\omega)]$$
$$[G_1(j\omega,\theta) - G_T(j\omega)]^* d\omega \quad (19)$$

and

$$\hat{\theta} = \arg\min_{\theta} V_1(\theta) \qquad (20)$$

Then

$$V_1(\hat{\theta}) = \int_0^{\infty} |L(j\omega)^2|\ \hat{D}(j\omega)|^2 \{ |G_1(j\omega,\hat{\theta})|^2 - |G_T(j\omega)|^2 \} d\omega \geq 0 \qquad (21)$$

Proof : The proof of Proposition 1 was obtained by equating to zero a clever linear combination of the derivatives of $V(\hat{\theta})$ w.r.t. the numerator coefficients $\hat{b}_i$ of $G(j\omega,\hat{\theta})$. Here we apply the same trick, but to the denominator coefficients $\hat{d}_i$ of $G_1(j\omega,\hat{\theta})$. We take $L(j\omega) \equiv 1$ for simplicity because it has no effect on the result. We then first rewrite

$$V_1(\hat{\theta}) = \int_0^{\infty} [\hat{C}(j\omega) - \hat{D}(j\omega)G_T(j\omega)][\hat{C}(j\omega) - \hat{D}(j\omega)G_T(j\omega)]^* d\omega \quad (22)$$

Differentiating w.r.t. $\hat{d}_1, \ldots, \hat{d}_n$ gives

$$\frac{\partial V_1(\hat{\theta})}{\partial \hat{d}_i} = 0 = \int_0^{\infty} [N(j\omega) + N^*(j)] d\omega \qquad (23)$$

where

$$N(j\omega) \stackrel{\Delta}{=} -(j\omega)^{n-i} [\hat{C}(j\omega) - \hat{D}(j\omega)G_T(j\omega)]^* G_T(j\omega) \qquad (24)$$

Therefore, dropping the explicit $j\omega$ dependence for simplicity of notation, we have

$$\sum_0^n \hat{d}_i \frac{\partial V_1(\hat{\theta})}{\partial \hat{d}_i} = 0 = \int_0^{\infty} [-\hat{D}G_T (\hat{C} - \hat{D}G_T)^* - \hat{D}^* G_T^* (\hat{C} - \hat{D}G_T)] d\omega$$

Therefore

$$V_1(\hat{\theta}) = \int_0^{\infty} (\hat{C} - \hat{D}G_T)(\hat{C} - \hat{D}G_T)^* d\omega$$
$$= \int_0^{\infty} [-\hat{D}G_T (\hat{C} - \hat{D}G_T)^* - \hat{D}^* G_T^* (\hat{C} - \hat{D}G_T) + \hat{C}(\hat{C} - \hat{D}G_T)^* + \hat{D}^* G_T^* (\hat{C} - \hat{D}G_T)] d\omega$$
$$= \int_0^{\infty} [\,|\hat{C}(j\omega)|^2 - |\hat{D}(j\omega)|^2\ |G_T(j\omega)|^2] d\omega$$
$$= \int_0^{\infty} |\hat{D}(j\omega)|^2 [\,|\hat{G}_1(j\omega,\hat{\theta})|^2 - |G_T(j\omega)|^2] d\omega \geq 0$$

It is obvious that the proof technique goes through unchanged if $L(j\omega) \neq 1$.

The consequence of Proposition 2 is that, if the parametrization (17) - (18) is used, then the gain of the estimated transfer function overestimates the true gain in the sense that

$$\int_0^{\infty} |L(j\omega)|^2 |\hat{D}(j\omega)|^2 |G_1(j\omega,\hat{\theta})|^2 d\omega \geq$$
$$\int_0^{\infty} |L(j\omega)|^2 |\hat{D}(j\omega)|^2 |G_T(j\omega)|^2 d\omega \quad (25)$$

This result holds true whatever the frequency weighting $L(j\omega)$ chosen by the designer.

We conclude that whether the gain of an estimated transfer function underestimates or overestimates the gain of the true system in a frequency weighted sense is intimately dependent on which parametrization, (1) - (2) or (17) - (18), is chosen. We notice that these two parametrizations represent the same class of transfer functions. For first order systems, this means that whether a person's favorite model is $\frac{b}{s+a}$ or $\frac{1}{d_0 s + d_1}$, she will either underestimate of overestimate the system's gain. The exact reason for this difference was unclear at the time of submitting this note and it was conjectured that some explanation might come from the concept of Total Least Squares.

EPILOGUE

The quest for an interpretation of the difference between the two models above, and the search for a prodecure to obtain an unbiased model have been successfully pursued with a Total Least Squares expert. A full paper is in preparation [4]. Without unveiling all the treasures of [4], we just mention that the clue to answering these two questions was obtained by noting that the estimation of the parameters in the two models above are special cases of the estimation of fully parametrized polynomials $A(s,\theta)$ and $B(s,\theta)$ with constraints, i.e. monicity of $A(s,\theta)$ or $B(s,\theta)$. By a different (and clever) choice of constraints, an unbiased model can be obtained.

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