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BENEFITS OF FEEDBACK IN EXPERIMENT DESIGN (York, 1985)

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Abstract

We investigate the design of identification experiments using some recent asymptotic expressions for the variance of the estimated transfer function. In particular we highlight a number of applications for which it is beneficial to let the experiment be carried out in closed loop.

1 INTRODUCTION

The identification of transfer functions from experimental data is a fundamental problem in applications of control theory. A typical approach to such identification can be outlined as follows.

Suppose that the true system can be described as

$$y(t)=G_0(q)u(t)v_0(t)=[\sum_{k=1}^{\infty} g_0(k)q^{-k}]u(t)+v_0(t)=$$

$$= \sum_{k=1}^{\infty} g_0(k)u(t-k)v_0(t) \qquad (1.1)$$

Here $y(t)$ and $u(t)$ are the scalar output and input, respectively, q^{-1} is the backward shift operator, $G_0(q)$ is the transfer function operator, and $g_0(k)$ are the impulse response coefficients. Moreover, $\{v_0(t)\}$ is assumed to be a zero-mean stationary stochastic process with spectral density $\Phi_v(\omega)$. We shall frequently work with the following representation of $v_0(t)$:

$$v_0(t) = H_0(q)e(t) = 1 + \sum_{k=1}^{\infty} h_0(k)e(t-k) \qquad (1.2)$$

where $\{e(t)\}$ is a sequence of independent random variables with zero mean values and variances σ^2. Then

$$\Phi_v(\omega) = \sigma^2|H_0(e^{i\omega})|^2 \qquad (1.3)$$

The function $G_0(e^{i\omega})$ is the transfer function of the system (1.1). In order to estimate G_0 and H_0 from observed data $Z^n=(u(1),y(1),....,u(N),y(N))$, one often proceeds as follows.

Postulate a set of candidate models

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t); \quad \theta\in D_M \qquad (1.4)$$

Let $\hat{y}(t|\theta)$ denote the one-step ahead prediction according to the model corresponding to the value θ:

$$\hat{y}(t|\theta)=H^{-1}(q,\theta)[G(q,\theta)]u(t)-[1-H^{-1}(q,\theta)]y(t), \qquad (1.5)$$

and let $\varepsilon(t,\theta)$ be the prediction error $\varepsilon(t,\theta)=y(t)-\hat{y}(t|\theta)$. Then let the estimate be

$$\hat{\theta}_N = \arg\min_{\theta\in D_M} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta) \qquad (1.6)$$

In this way we obtain the estimates

$$\hat{G}_N(e^{i\omega})=G(e^{i\omega},\hat{\theta}_N), \qquad \hat{H}_N(e^{i\omega})=H(e^{i\omega},\hat{\theta}_N) \qquad (1.7)$$

The quality of the estimates $\hat{G}_N$ and $\hat{H}_N$ can be evaluated in terms of the difference

$$\Delta T_N(e^{i\omega}) \triangleq \hat{T}_N(w^{i\omega}) - T_0(e^{i\omega}) \qquad (1.8)$$

where

$$\hat{T}_N(e^{i\omega}) \triangleq \begin{pmatrix} \hat{G}_N(e^{i\omega}) \\ \hat{H}_N(e^{i\omega}) \end{pmatrix} ; \quad T_0(e^{i\omega}) \triangleq \begin{pmatrix} G_0(e^{i\omega}) \\ H_0(e^{i\omega}) \end{pmatrix} \qquad (1.9)$$

It is natural to split up the error $\Delta T_N(e^{i\omega})$ into a random part and a bias part:

$$\Delta T_N(e^{i\omega}) = \hat{T}_N(e^{i\omega}) - E\hat{T}_N(e^{i\omega}) + E\hat{T}_N(e^{i\omega}) - T_0(e^{i\omega}) \qquad (1.10)$$

A scalar criterion can be formed as

$$J_N = \int_{-\pi}^{\pi} \mathrm{tr}[Q(\omega)\Pi_N(\omega)]d\omega \qquad (1.11)$$

where

$$\Pi_N(\omega) = E\, \Delta T_N(e^{i\omega})\Delta T_N^{\,T}(e^{-i\omega}) \qquad (1.12)$$

is the mean square error of the estimates (a 2×2 real valued matrix function) formed by expectation w.r.t the random vector

$\hat{\theta}_N$.

$$Q(\omega) = \begin{pmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{pmatrix} \quad (1.13)$$

is a weighting matrix that reflects the intended use of the model $\hat{T}_N$. In Ljung (1984a) several examples of Q corresponding to typical model applications such as simulation, prediction and control are given. Specific examples will be given in Section 4 below.

Our point now is that the value of the criterion (1.11) will depend on a number of design variables that are at the user's disposal. His objective is thus to choose these so as to minimize (1.11) subject to whatever constraints are imposed. A comprehensive treatment of this problem is given in Ljung (1984b). Here we shall concentrate on input design issues.

We then suppose that the user has at his disposal to choose the input spectrum and possibly also some feedback mechanism. The input may thus be generated as

$$u(t) = -F(q)y(t) + w(t) \quad (1.14)$$

where F is the feedback regulator and w(t) is some extra input signal. We suppose that F and the spectrum of w, $\Phi_w(\omega)$ is at the designer's disposal, subject to constraints we are going to specify later. We would thus like to minimize the criterion (1.11) with respect to these design variables. This is the problem discussed in the present paper.

This problem relates closely to questions discussed in some other papers. Asymptotic variance expressions for the random term in (1.10) were derived in Ljung (1984a), while expressions for the bias term are discussed in Wahlberg and Ljung (1984) and Ljung (1984b). Open loop design is studied in Yuan and Ljung (1984a,b).

The paper is organised as follows. The basic expressions for the analysis are given in Section 2. The connection between performance degradation and input desing is established in Section 3. Minimisation of the variance contribution for typical applications discussed in Section 4, while Section 5 deals with the special case where the objective of the identification is to design a minimum variance regulator.

2 ASYMPTOTIC RESULTS FOR TRANSFER FUNCTION ESTIMATES

Basic convergence results for the estimate $\hat{\theta}_N$, given by (1.6) can be quoted from Ljung (1978) and Ljung and Caines (1979) as follows

$$\hat{\theta}_N \to \theta^* \text{ w.p.1} \quad \text{as } N \to \infty \tag{2.1}$$

$$\theta^* = \arg \min_{\theta \in D} E\, \varepsilon^2(t,\theta) \tag{2.2}$$

$$\sqrt{N}\,(\hat{\theta}_N - \theta^*) \underset{N\to\infty}{\overset{d}{\to}} N(0,P) \quad \text{(convergence in distribution)} \tag{2.3}$$

An expression for P can be given, but we do not deetail with that here.

The asymptotic normality result (23) can be translated to a result on the asymptotic distribution of $\hat{T}_N(e^{i\omega})-T^*(e^{i\omega})$ $[T^*(e^{i\omega}) \triangleq T(e^{i\omega},\theta^*)]$, but this gives, in general, quite a complex expression for the corresponding covariance matrix. For a wide class of model sets, however, a simple asymptotic expression can be derived. Suppose that

The model set it subject to the following shift property

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \quad \dim \theta_k = s \tag{2.4a}$$

$$\frac{\partial}{\partial \theta_k} T(q,\theta) = q^{-k+1} \frac{\partial}{\partial \theta_1} T(q,\theta) \tag{2.4b}$$

In Ljung (1984a) it is shown that (2.4) is the typical structure for an n:th order linear, black-box model, involving s different polynomials in the delay operator. Then the following result holds:

$$\sqrt{N}\left(\hat{T}_N(e^{i\omega})-T^*(e^{i\omega})\right) \underset{N\to\infty}{\overset{d}{\to}} N\left(0,P_n(\omega)\right) \tag{2..5a}$$

$$\lim_{n\to\infty} \frac{1}{n} P_n(\omega) = \Phi_v(\omega)\cdot\left[\bar{\Phi}(\omega)\right]^{-1} \triangleq \bar{P}(\omega) \tag{2.5b}$$

where $\bar{\Phi}(\omega)$ is given by:

$$\bar{\Phi}(\omega) = \begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{eu}(\omega) & \sigma^2 \end{bmatrix} \tag{2.6}$$

with $\Phi_u(\omega)$ being the input spectrum and $\Phi_{ue}(\omega)$ the cross spectrum between the input u and the white noise sequence e, in the system description (1.1)-(1.2). Clearly, for an experiment performed in open loop, $\Phi_{ue}(\omega)\equiv 0$.

Heuristically we could rewrite (2.5) as

$$\operatorname{cov}\left(\hat{T}_N(e^{i\omega})\right) \simeq \frac{n}{N}\,\bar{P}(\omega) \tag{2.7}$$

We shall now introduce the approximation

$$E\,\hat{T}_N(e^{i\omega}) \simeq T^*(e^{i\omega}) \tag{2.8}$$

which is reasonable, since we typically have

$$E\,\hat{T}_N(e^{i\omega}) = T^*(e^{i\omega}) + \sigma(1/\sqrt{N}) \tag{2.9}$$

In addition, in this Conference paper, we shall assume that the contribution of the bias term $E\hat{T}_N(e^{i\omega})-T_0(e^{i\omega})$ in (1.12) is negligable w.r.t. the contribution of the random term (see (1.10)), so that the mean square error $\Pi_N(\omega)$ can be approximated by

$$\Pi_N(\omega) \simeq \frac{n}{N} \bar{P}(\omega) \tag{2.10}$$

This amounts to assuming that the model set is large enough. The influence of the bias term on experiment design will be analyzed in a full version of this paper. With these assumptions, we can rewrite (1.11) as

$$J_N \simeq \int_{-\pi}^{\pi} \mathrm{tr}\left[Q(\omega)\bar{\Phi}^{-1}(\omega)\right]\Phi_v(\omega)d\omega \triangleq J(\bar{\Phi}) \tag{2.11}$$

We have appended the argement $\bar{\Phi}$ to the criterion to stress that our objective is to minimize $J(\bar{\Phi})$ w.r.t. Φ_u and Φ_{ue}.

3. PERFORMANCE DEGRADATION AND INPUT DESIGN

Errors on the transfer function estimates will of course degrade the performance whenever the model is used. In this section we derive some basic formulas for performance degradation due tot he random error $\hat{T}_N(e^{i\omega})-T^*(e^{i\omega})$, and we shall show how this degradation can be kept small by proper input design.

Let s(t) be a signal derived from a model application. It could, e.g. be the output of the system when a minimum variance regulator, conputed using the model, is applied. Specific examples will be given in Section 4. Conceptually we could write

$$s(t) = f\big(T(q)\big)w(t) \tag{3.1}$$

to denote that the transfer functions T, as well as some additional signal (reference signals and/or noise) are used to determine s(t).

If the true transfer function $T_0(q)$ is used in (3.1), we obtain the "true" or "best" result

$$s_0(t) = f\big(T_0(q)\big)w(t) \tag{3.2}$$

When, instead, $\hat{T}_N$ is used we get the result

$$\hat{s}_N(t) = f\big(\hat{T}_N(q)\big)w(t) \tag{3.3}$$

Similarly the expected transfer function estimate T* gives

$$s^*(t) = f\big(T^*(q)\big)w(t) \tag{3.4}$$

It is now of interest to evaluate the performance degradation due to the variance of the estimates. Let

$$\tilde{s}_N(t) \triangleq \hat{s}_N(t) - s^*(t) \tag{3.5}$$

When the error $\tilde{T}_N(q)=\hat{T}_N(q)-T^*(q)$ is small, we can use Taylor's expansion to derive

$$\tilde{s}_N(t) = \tilde{T}_N^T(q)F(q)w(t) \tag{3.6}$$

where

$$F(q) = \frac{\partial}{\partial T} f(T)\Big|_{T=T_0(q)} \qquad (2\times 2 \text{ matrix; } r=\dim w) \tag{3.7}$$

The spectrum of $\tilde{s}_N(t)$ is,

$$\Phi_{\tilde{s}_N}(\omega)=\tilde{T}_N^T(e^{i\omega})F(e^{i\omega})\Phi_w(\omega)F^T(e^{-i\omega})\tilde{T}_N(e^{-i\omega})=\mathrm{tr}[\tilde{S}_N(\omega)Q(\omega)] \tag{3.8}$$

where

$$\tilde{S}_N(\omega) = \tilde{T}_N(e^{i\omega})\tilde{T}_N^T(e^{-i\omega}) \qquad (2\times 2 \text{ matrix}) \tag{3.9}$$

and

$$Q(\omega) = F(e^{i\omega})\Phi_w(\omega)F^T(e^{-i\omega}) \qquad (2\times 2 \text{ matrix}) \tag{3.10}$$

The expected value of $\Phi_{\tilde{s}_N}(\omega)$ is

$$\phi_s(\omega) = E\,\Phi_{\tilde{s}_N}(\omega) = \mathrm{tr}[E\tilde{S}_N(\omega)\;Q(\omega)] \tag{3.11}$$

with

$$E\,\tilde{S}_N(\omega) \simeq \frac{n}{N}\,\Phi_v(\omega)\bar{\Phi}(\omega)^{-1} \tag{3.12}$$

Denoting

$$Q(\omega) = \begin{pmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{pmatrix} \tag{3.13}$$

We have the alternative expression

$$\frac{N}{n}\,\phi_s(\omega) \simeq \frac{[\sigma^2 Q_{11}(\omega) - 2\mathrm{Re}(Q_{12}(\omega)\Phi_{ue}(-\omega)) + Q_{22}(\omega)\Phi_u(\omega)]}{\sigma^2\Phi_v(\omega) - |\Phi_{ue}(\omega)|^2}\,\Phi_v(\omega) \tag{3.14}$$

If the identification has been performed in open loop, then $\Phi_{ue} \equiv 0$. If it has been performed under the feedback law (1.14), where $\{w(t)\}$ is an additive stationary stochastic process with spectrum $\Phi_w(\omega)$, independent of $\{e(t)\}$, then we have (deleting arguments):

$$\Phi_{ue} = -\frac{FH_0}{1 + FG_0}\,\sigma^2 \tag{3.15}$$

$$\Phi_u = \left|\frac{FH_0}{1 + FG_0}\right|^2 \sigma^2 + \left|\frac{1}{1 + FG_0}\right|^2 \Phi_w(\omega) \tag{3.16}$$

The variance of $\hat{s}_N$ is

$$\mathrm{Var}\ \tilde{s}_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_s(\omega)d\omega \tag{3.17}$$

Our experiment design task therefore is to minimise Var $\tilde{s}_N$ with respect to Φ_u and Φ_{ue} (or equivalently F and Φ_w) for different applications under either input variance constraint or output variance constraint

$$\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \leq C \tag{3.18}$$

$$\int_{-\pi}^{\pi} \Phi_y(\omega)d\omega \leq C \tag{3.19}$$

4. REDUCTION OF VARIANCE BY FEEDBACK

In this section, we solve the minimisation problem first described for a variety of applications (simulation, prediction, minimum variance control,...). Each application will lead to a particular set of weights $Q_{ij}(\omega)$, i,j=1,2 in (3.13). We shall in particular highlight some applications in which the use of feedback during identification has a beneficial effect.

For simplicity we shall in the sequel assume that $\sigma^2=1$. This can always be achieved by absorbing σ into H_0. We then denote (see (3.14)):

$$\frac{N}{n}\ \phi_s(\omega)\simeq J(\omega)\triangleq \frac{Q_{11}(\omega)-2\mathrm{Re}\left(Q_{12}(\omega)\Phi_{ue}(-\omega)\right)+Q_{22}(\omega)\Phi_u(\omega)}{\Phi_u(\omega)-\left|\Phi_{ue}(\omega)\right|^2}\ \Phi_v(\omega) \tag{4.1}$$

Before going into specific applications, it is worth noting the following.

<u>Preliminary result</u>:

If $Q_{12}=0$ and **the input variance is constrained** , then open loop operation is optimal, i.e. $\Phi_{ue}(\omega)=0$. This follows directly from (3.14). It also follows that the input spectrum is

$$\Phi_u^{opt}(\omega) = \mu \sqrt{Q_{11}(\omega)\Phi_v(\omega)} \tag{4.2}$$

where μ is adjusted so that equality holds in (3.18).

4.1 Estimation of the I/O dynamics $G(e^{i\omega})$ only

Suppose our objective is to minimize $\mathrm{Var}(\tilde{G}_N)$ only. It is easy to show that if the input power is constrained, the optimal strategy is to maximise the input power in open loop operation. Consider now the case where the output power is constrained.

Suppose there is a delay d in the plant, i.e.

$$G_0(q) = \sum_{k=d}^{\infty} g_0(k)q^{-k} \tag{4.3}$$

Then we decompose $H_0(q)$ as

$$H_0(q) = H_0^*(q) + \tilde{H}_0(q) \tag{4.4a}$$

where

$$H^*{}_0(q)=1+h_1q^{-1}+\ldots+h_{d-1}q^{-d+1};\ \tilde{H}_0(q)=\sum_{k=d}^{\infty} h_kq^{-k} \tag{4.4b}$$

Then (3.19) is equivalent with

$$\Phi_y(\omega)=|G_0|^2\Phi_u+|H_0^*|^2+|\tilde{H}_0|^2+2\mathrm{Re}(G_0\overline{\tilde{H}}_0\Phi_{ue}) \leq D \tag{4.5}$$

where $\overline{\tilde{H}}_0$ is the complex conjugate of $\tilde{H}_0$ with D possibly a function of ω. (We have dropped the argument ω for convenience). Our task therefore is to minimise $\mathrm{Var}(\hat{G}_N(e^{i\omega}))$ subject to (4.5). By (3.9), (3.12) and (2.6), this is equivalent with minimising

$$J(\omega) = \frac{\Phi_v}{\Phi_u - |\Phi_{ue}|^2} \tag{4.6}$$

subject to (4.5). The optimal solution is

$$\Phi_{ue}^{opt}(\omega) = -\frac{\tilde{H}_0(e^{i\omega})}{G_0(e^{i\omega})}; \quad \Phi_u^{opt}(\omega) = |\Phi_{ue}^{opt}(\omega)|^2 + \sqrt{\frac{\Phi_v(\omega)}{\mu |G_0(e^{i\omega})|^2}} \tag{4.7}$$

where μ is a Lagrange multiplier. This corresponds to the feedback law

$$u(t) = -\frac{\tilde{H}_0(q)}{G_0(q)} y(t) + w(t) \tag{4.8}$$

(4.8) is the minimum variance control law, which is of course applicable only if $G_0(q)$ has minimum phase. If the power of the external signal $w(t)$ is adjusted so as to satisfy the constraint (4.5) with equality, then the corresponding optimal variance of $\tilde{G}_N(e^{i\omega})$ can be approximated by (see (3.12)):

$$\mathrm{Var}(\tilde{G}_N(e^{i\omega})) \simeq \frac{n}{N} \frac{\Phi_v}{\Phi_u - |\Phi_{ue}|^2} = \frac{n}{N} \Phi_v \frac{|G_0|^2}{D - |H_0^*|^2} \tag{4.9}$$

<u>Comment</u>: It is interesting to compare (4.9) with the best achievable error variance using open loop identification. With $\Phi_{ue}=0$ and Φ_u chosen such that $\Phi_y=D$, the error variance becomes

$$\mathrm{Var}(\tilde{G}_N(e^{i\omega})) \simeq \frac{n}{N} \frac{\Phi_v}{\Phi_u} = \frac{n}{N} \Phi_v \frac{|G_0|^2}{D - |H_0|^2} \tag{4.10}$$

where $|H_0| \geq |H_0^*|$ (see (4.4)). The conclusion is that, if the system has minimum phase and if the output power is constrained, one should use the minimum variance control law (4.8) to identify $G(e^{i\omega})$. The benefit over open loop identification is indicated by a comparison between (4.10) and (4.9).

4.2 Simulation

Suppose the objective is to use the model $\hat{G}_N(q)$ to simulate the output with an input signal with spectrum $\Phi_u^*(\omega)$. The question is: What is the optimal input design during identification? The simulated output will be

$$\hat{y}_N(t) = \hat{G}_N(q)u^*(t) \tag{4.11}$$

It differs from the output simulated with the "expected model" $G^*(q)$. The error is

$$\tilde{y}_N(t) = \tilde{G}_N(q)u^*(t) \tag{4.12}$$

The corresponding spectrum is

$$\Phi_{\tilde{y}}(\omega) = |\tilde{G}_N(e^{i\omega})|^2\Phi_u^*(\omega) \tag{4.13}$$

Comparing with (3.11)-(3.13), we have here: $Q_{11}=\Phi_u^*$, $Q_{12}=Q_{21}=Q_{22}=0$.

With constrained input variance, the optimal experiment design is to identify in open loop. This follows from our Preliminary result. With output power constraint, one can show that the solution for $\Phi_{ue}(\omega)$ is identical to that obtained in Section 4.1, and the same conslusions apply: applying a minimum variance controller during identification is optimal if the system has minimum phase.

4.3 Prediction

Suppose the objective is to use the model as a predictor on a new set of input data with possibly different spectra. The deviation between the prediction error $\hat{\varepsilon}_N(t)$ obtained from the model and its expected value $\varepsilon^*(t)$ can be approximated as follows

$$\tilde{\varepsilon}_N(t) \triangleq \hat{\varepsilon}_N(t) - \varepsilon^*(t) = \frac{1}{H_0}\left(\tilde{G}_N(q)u(t)+\tilde{H}_N(q)e(t)\right) \tag{4.14}$$

This means that $Q(\omega)$ in (3.10)-(3.11) takes the form

$$Q(\omega) = \frac{1}{|H_0|^2}\begin{pmatrix} \Phi_u^*(\omega) & \Phi_{ue}^*(\omega) \\ \Phi_{ue}^*(-\omega) & 1 \end{pmatrix} \tag{4.15}$$

where the stars indicate that these are the spectra of the data to which the predictor will be applied. Therefore

$$J(\omega) = \frac{\Phi_u(\omega)+\Phi_u^*(\omega)-2\mathrm{Re}\left(\Phi_{ue}^*(\omega)\Phi_{ue}(-\omega)\right)}{\Phi_u(\omega)-|\Phi_{ue}(\omega)|^2} \tag{4.16}$$

Suppose the predictor is to be applied to an open loop situation, i.e. $\Phi_{ue}^*=0$. Then, under <u>constrained input variance</u>, it is optimal to identify the system in open loop (Ljung, 1984a). However, under <u>constrained output variance</u>, one can show that the optimal strategy is to use feedback, i.e. $\Phi_{ue}^{opt}\neq 0$ even though $\Phi_{ue}^*=0$.

5 THE MINIMUM VARIANCE REGULATOR APPLICATION

An interesting case is where the identified model is to be used for the design of a minimum variance regulator. We shall show that, in the case of a constraint on either the input or the output variance, the optimal strategy is to identify the model under minimum variance control feedback.

5.1 <u>Minimum variance control with input power constraint</u>

Suppose the system has minimum phase. Then the true minimum variance regulator is

$$u(t) = \frac{1-H_0(q)}{G_0(q)}\, y(t) \tag{5.1}$$

In practice, this would be replaced by

$$u(t) = \frac{1-\hat{H}_N(q)}{\hat{G}_N(q)} y(t) \tag{5.2}$$

Instead of $y(t)=e(t)$, this would produce an output

$$y_N(t) \simeq e(t) + \frac{1}{H_0(q)}\{\frac{H_0(q)-1}{G_0(q)}[\tilde{G}_N(q)+G^*(q)-G_0(q)]-$$

$$-[\tilde{H}_N(q)+H^*(q)-H_0(q)]e(t) \tag{5.3}$$

where we have assumed $\tilde{G}_N$, $\tilde{H}_N$, G^*-G_0 and H^*-H_0 to be small. Considering again only the variance contribution to the performance degradation and denoting

$$Z_0 \triangleq \frac{1-H_0}{G_0}$$

yields the following expression for $Q(\omega)$:

$$Q(\omega) = \frac{1}{|H_0|^2} \begin{bmatrix} |Z_0|^2 & Z_0 \\ \bar{Z}_0 & 1 \end{bmatrix} \tag{5.4}$$

where $\bar{Z}_0 \triangleq Z_0(e^{-i\omega})$. Minimising (4.1) subject to (3.18) yields, after lengthy manipulations, the following set of equations:

$$|\Phi_{ue}|^2+|Z_0|^2-2\mathrm{Re}Z_0\Phi_{ue}(-\omega)=\mu(\Phi_u-|\Phi_{ue}|^2) \tag{5.5a}$$

$$\arg \Phi_{ue}(\omega) = \arg Z_0(e^{i\omega}) \tag{5.5b}$$

$$(\Phi_u-|\Phi_{ue}|^2)^2|Z_0|^2=[|Z_0|^2-2\mathrm{Re}(Z_0\Phi_{ue}(-\omega))+\Phi_u]^2|\Phi_{ue}|^2 \tag{5.5c}$$

where μ is a Lagrange multiplier to be adjusted so as to satisfy the constraint (3.18). Notice now that if we take $\Phi_{ue}(\omega)=Z_0(e^{i\omega})$, then the 3 equations are satisfied, provided $\mu=0$. Hence, provided the choice $\Phi_{ue}=Z_0$ does not violate the constraint (3.18), this is the optimal solution. It corresponds to

$$u(t) = Z_0(q)e(t) + w(t) = Z_0(q)y(t) + w(t) \tag{5.6}$$

Therefore a necessary condition for optimality of $\Phi_{ue}=Z_0$ is that

$$\int_{-\pi}^{\pi} |Z_0(e^{i\omega})|^2 d\omega \le C \tag{5.7}$$

5.2 Discussion and further results

a) The exact minimum variance control law is $u(t)=Z_0(q)e(t)$. Therefore, if the intent is to apply this controller to the system, condition (5.7) will always be satisfied. Hence, (5.6) is the optimal design strategy during identification.

b) One might think that the optimal strategy is to adjust the spectrum of w(t) in (5.6) so that the constraint (3.18) is satisfied with equality. As a matter of fact, it is easy to see that, if $\Phi_{ue}=Z_0$, then $J(\omega)=1$ regardless of Φ_u (i.e. of Φ_w). The conclusion is: <u>if one wants to identify a system in view of applying a minimum variance controller to it, the optimal strategy is to identify it under minimum variance control $u(t)=Z_0(q)y(t)$; surprisingly, addition of an external input w(t) does not contribute to a decrease of the error variance.</u>

c) With the optimal feedback design $u(t)=Z_0(q)y(t)$, the spectrum of the error $y_N(t)-y^*(t)$ is given by

$$\frac{N}{n} \phi_y(\omega) \simeq 1 \tag{5.8}$$

It can be shown that this is the minimum of ϕ_y for all Φ_u

and Φ_{ue}. By comparison with open loop identification, it follows from (4.1) and (5.4) that

$$\frac{N}{n} \phi_y(\omega) \approx 1 + \frac{|z_0|^2}{\Phi_u(\omega)} \tag{5.9}$$

Hence, in open loop, the same minimum can only be achieved with infinite input energy.

d) The reason why an external input w(t) does not decrease the error variance is as follows. Suppose the system is described by an ARMAX model

$$A_0(q)y(t) = B_0(q)u(t) + C_0(q)e(t) \tag{5.10}$$

Then the true minimum variance regulator is

$$u(t) = \frac{A_0(q)-C_0(q)}{B_0(q)} \tag{5.11}$$

By applying more input power, the estimates $\hat{A}_N(q)$, $\hat{B}_N(q)$ and $\hat{C}_N(q)$ are improved, but not those of $(\hat{A}_N-\hat{C}_N)/\hat{B}_N$, which is all that is needed to compute the minimum variance regulator (MVR). The simulations will illustrate this point.

5.3 Simulation

The theoretical results have been tested by a sereis of simulations on the following system.

$$S: A(q)y(t) = B(q)u(t) + C(q)e(t) \tag{5.12}$$

with

$$A(q) = 1 - 1.5q^{-1} + 0.7q^{-2}$$

$$B(q) = q^{-1}(1+0.5q^{-1})$$

$C(q) = 1 - q^{-1} + 0.2q^{-2}$

The true MVR is

$$\text{MVR:}\quad (1+0.5q^{-1})u(t)=(-0.5+0.5q^{-1})e(t)+w(t) \qquad (5.13)$$

One aim of the simulations was to check whether the asymptotic results apply to low order models (n small) and for reasonably short data lengths. Five simulations have been performed, numbered 1 to 5 in the sequel. In the first three, the system is identified from I/O data obtained while the MVR (5.16) acts on the system: $\{u(t)\}$ and $\{y(t)\}$ are generated through (5.15)-(5.16), where $\{e(t)\}$ is white Gaussian noise of zero mean and unit variance, WGN(0,1), while $\{w(t)\}$ is WGN($0,\sigma_w$), where σ_w takes on 3 different values. In the last two simulations, the system is identified in open loop: $\{e(t)\}$ is WGN(0,1), $\{u(t)\}$ is WGN($0,\sigma_u$) with 2 different values for σ_u and $\{y(t)\}$ is generated through (5.12). For each simulation, 10 independent runs have been performed using 10 independent sequences $\{e(t)\}$ of 500 points each.

For each of the 10 runs in simulations 1 to 5, the polynomials $\hat{A}_N(q)$, $\hat{B}_N(q)$, $\hat{C}_N(q)$ have been estimated by maximum likelihood estimation using IDPAC, assuming a second order ARMAX model. From these, the estimated MVR parameters have been calculated

$$u(t) = \frac{\hat{A}_N(q)-\hat{C}_N(q)}{\hat{B}_N(q)} = \frac{\hat{\alpha}_1+\hat{\alpha}_2 q^{-1}}{1+\hat{\beta}\ q^{-1}}\ y(t) \qquad (5.14)$$

with

$$\alpha_1 = \frac{a_1-c_1}{b_1},\ \alpha_2 = \frac{a_2-c_2}{b_1},\ \beta = \frac{b_2}{b_1} \qquad (5.15)$$

Table 1 gives the true values of three of the system parameters and of the three regulator parameters, as well as the average (over 10 runs) of their estimates and then standard deviations.

Table 1 confirms several points made in Section 5.2. Simulations 1 to 3 show that an increase in the external signal power increases the accuracy of the a_i, b_i and c_i, but does not affect the accuracy of the regulator parameters. To obtain the same accuracy with open loop identification, one needs to allow for more output variance. In fact, output variances such as those obtained in simulations 1 and 2 cannot be achieved in open loop, since the noise $\{e(t)\}$ by itself contributes an output variance of 1.521.

5.4 Minimum variance regulator with output power constraint

The optimal experiment design for this case is also (5.6), because it yields the smallest possible value for ϕ_y, and because the output spectrum is $\Phi_y(\omega)=\sigma^2+\phi_y(\omega)$.

Simulation number and experimental conditions

	True value	1 C.L $\sigma_w=0.01$	2 C.L $\sigma_w=0.1$	3 C.L $\sigma_w=1$	4 O.L $\sigma_u=10$	5 O.L $\sigma_u=1$
a_1	-1.5	0.546 ± 2.443	-1.459 ± 0.565	-1.509 ± 0.009	-1.499 ± 0.001	-1.506 ± 0.009
b_1	1	-0.987 ± 3.840	1.118 ± 0.565	1.002 ± 0.056	1.000 ± 0.005	0.997 ± 0.058
c_1	-1	0.052 ± 0.666	-0.895 ± 0.479	-1.014 ± 0.049	-1.007 ± 0.035	-1.012 ± 0.031
α_1	-0.5	-0.505 ± 0.026	-0.508 ± 0.037	-0.494 +0.035	-0.493 ± 0.036	-0.436 ± 0.041
α_2	0.5	0.509 ± 0.085	0.517 +0.045	0.500 ± 0.048	0.499 ± 0.051	0.501 ± 0.052
β	0.5	0.484 ± 0.051	0.482 ± 0.099	0.470 ± 0.049	0.502 ± 0.007	0.488 ± 0.088
σ_y^2		1.000	1.034	4.409	1.889	20.401
σ_u^2		1.000	1.013	2.333	100	1

Table 1 True and estimated values (with standard deviations) of a_1, b_1, c_1 and α_1, α_2, β for 5 different experimental conditions, together with output variance and input variance. The estimates are averages over 10 Monte Carlo runs of 500 data each. (Note, C.L.=closed-loop experiment; O.L.=open-loop experiment).

6 CONCLUSIONS

We have exhibited a number of applications where it proves beneficial to identify the system under closed-loop operation. Several of our results are consistent with earlier experiment design results obtained under different sets of assumptions (see e.g. Gustavsson, Ljung and Söderström, 1981; Ng, Goodwin and Söderström 1977). However the results on the minimum variance regulator are new and perhaps surprising. Because the optimal design strategies always depend on the unknown system, one might argue that our results are not very practical. However, they are qualitatively important, and provide further justification for the use of self-tuning regulators.

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