Identification in dynamic networks: identifiability and experiment design issues*

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Abstract— **Recently attention has been paid to the identification of networks of linear time-invariant dynamical systems. One of the problems of interest is the identification of a particular transfer function within the network on the basis of available measurements. This raises the question of which signals need to be measured and which external excitation signals need to be present in the network in order to yield a consistent estimate of this desired transfer function. This paper examines the properties of the estimated transfer function in terms of bias error and variance error, for different models of the network. The main contribution is the derivation of sufficient richness conditions on the external signals for the consistent identification of the desired transfer function.**

I. INTRODUCTION

The identification of networks of linear time-invariant dynamical systems has been the subject of much recent attention. The problem contains two aspects. The first is the identification of the topology of the whole network; this consists of identifying the interconnection structure of the subsystems from the available data. Some recent contributions towards the identification of the network topology can be found in [5], [8], [3], [2].

The second aspect is the identification of a particular transfer function or a family of transfer functions within the network, assuming that its interconnection structure is known. The major contribution on this aspect of the problem can be found in [10]. A recent contribution focuses on the design of optimal input signals for the identification in interconnected systems, where the optimality criterion depends on the specific application [6]. The results of [10] and [6] heavily rely on the observation that, when it is desired to identify a particular transfer function within an interconnected system, the problem can be reconfigured as a closed-loop system with the 'to be identified' transfer function in the open-loop path, and the relevant parts of the network acting as a feedback loop.

In this paper, we shall adopt the network model formalism developed in [10]. In that paper the authors have derived a number of results for the consistent identification of a particular transfer function within a known network, using three different closed-loop identification methods, whose properties they have compared for the accomplishment of this task. A key assumption for their results is that the vector consisting of all the node signals in the network are informative. This assumption is crucial for the consistent estimation of the desired transfer functions, but it is an assumption on the internal signals rather than an assumption on the experiment design. In other words, by measuring all internal signals one can evaluate whether they are informative by checking the positivity of the corresponding spectral density matrix, but nothing is said about how to make these data informative by proper experiment design, i.e. by imposing conditions on the externally applied signals. The major contribution of this paper is to fill this gap.

We first show that the network model of [10] can be transformed in a number of equivalent descriptions, leading to different ways of estimating the desired transfer function, or transfer functions. We briefly describe the bias errors obtained with these different network models. We then focus on the direct identification of the desired transfer functions, and we develop expressions that relate the richness of the external signals that enter the network to the informativity of the data vector consisting of the vector of node signals. These expressions also allow us to compute the covariance of the estimated parameter vectors and transfer functions, and to compare the effect of different experiment design scenarios on these covariances. For the sake of simplicity our analysis is developed on a 3-node network; we show why it is representative of a general network with L nodes, and we actually formulate some of our results for this general L-node network. Finally, a simple 3-node example serves to illustrate the usefulness and practicality of our results.

The content of the paper is as follows. The problem is stated in section II. We then present different configurations of the 3-node network in section III. In section IV we illustrate the informativity problem with a simple example. Three different models for the 3-node network, leading to three ways of estimating the desired transfer functions, are presented in section V, where the bias errors obtained from these models are discussed. Our main results are in section VI where we provide conditions on the network for the external signals to provide informative data. An extension of our main result for general L-node networks is presented in section VII.

II. PROBLEM STATEMENT

We adopt the network structure and notations of [10]. Thus, we consider that the network is made up of L nodes, that the outputs of each node, also called node signals,

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are denoted $\{w_1(t), \dots, w_L(t)\}$. These node signals are related to each other and to the external excitation signals $r_j, j = 1, \dots, L$ and the noise signals $v_j, j = 1, \dots, L$ by the following network equations:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \dots & G_{1L} \\ G_{21} & 0 & \ddots & G_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1} & G_{L2} & \dots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix}$$
$$= G(q)w + r + v \tag{1}$$
$$= (I - G)^{-1}(r + v) \tag{2}$$

with the following properties:

- G_{ij} are proper but not necessarily strictly proper transfer functions. Some of them may be zero, indicating that there is no direct link from w_j to w_i.
- There is a delay in every loop going from one w_j to itself.
- all node signals $w_j, j = 1, \dots, L$ are measurable.
- r_i are quasi-stationary external excitation signals that are available to the user in order to produce informative experiments for the identification of some or all of the G_{ij}. Some or all of the r_j are possibly zero.
- v_j are unmeasured stationary noise signals with rational spectral density that can be modeled as $v_j = H_j(q)e_j$ where $H_j(q)$ is a monic stable and inversely stable modeling filter and e_j is white noise with variance λ_j. Some or all of the v_j are possibly zero.
- the external excitation signals r_i are assumed to be uncorrelated with all noise signals $v_j, j = 1, \dots, L$.
- q^{-1} is the delay operator.

In this paper we consider the problem of estimating a particular transfer function, say G_{ij}, from the available nodes signals w_i and the excitation signals r_i. Without loss of generality, we shall assume that the transfer function to be identified is G_{12} in the first row of the network model above. We shall first show that the identification of G_{12} can be obtained through different ways of reconfiguring the network model (1) into an equivalent subnetwork that relates G_{12} to available signals.

First we show that for the estimation of G_{12} the model (1) can always be rewritten as a classical feedback system with $L-1$ inputs and one output, i.e. a Multiple Input Single Output (or MISO) feedback system. We can then split up the vector w into

$$w = \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \tag{3}$$

where $\tilde{w}_1$ is the $L-1$ vector defined by $\tilde{w}_2 \triangleq [w_2 \dots w_L]^T$. Correspondingly, we split up the matrix $G(q)$ into the 4-block matrix

$$G = \begin{bmatrix} 0 & \begin{pmatrix} G_{12} & \dots & G_{1L} \end{pmatrix} \\ \begin{pmatrix} G_{21} \\ \vdots \\ G_{L1} \end{pmatrix} & \begin{pmatrix} 0 & \ddots & G_{2L} \\ \ddots & \ddots & \vdots \\ G_{L2} & \dots & 0 \end{pmatrix} \end{bmatrix} \tag{4}$$

which we denote as

$$G(q) = \begin{bmatrix} 0 & K_1(q) \\ K_2(q) & K_3(q) \end{bmatrix} \tag{5}$$

We can now rewrite the initial network description (1) as a MISO feedback system as follows. First we rewrite (1) as

$$\begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & K_1(q) \\ K_2(q) & K_3(q) \end{bmatrix} \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} + \begin{bmatrix} r_1 \\ \tilde{r}_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ \tilde{v}_2 \end{bmatrix} \tag{6}$$

where the vectors r and v of (1) have been split conformably with the split of w in (3). Next we rewrite (6) in the traditional form of a MISO feedback system:

$$w_1(t) = K_1(q)\tilde{w}_2(t) + r_1(t) + v_1(t) \tag{7}$$
$$\tilde{w}_2(t) = [I - K_3(q)]^{-1}\{K_2(q)w_1(t) + \tilde{r}_2(t) + \tilde{v}_2(t)\} \tag{8}$$

We observe that, if our objective is to identify the transfer functions of the first row of G, then we can apply all the well known results about the identifiability, the excitation requirements, as well as the bias and variance properties for the identification of the open loop path of a multivariable feedback system to the system (7)-(8). Necessary and sufficient conditions for the identifiability of $K_1(q)$, $K_2(q)$ and $K_3(q)$ can, e.g., be found in [1] and [7]. Necessary and sufficient conditions on the richness of the excitation signals for closed loop systems have been given in [4] for the single input single output (SISO) case. In the present paper these conditions are extended to the MISO system in which any network can be transformed, as we have just shown.

For the network described by (1), or alternatively (6), several questions are to be addressed:

- Assuming that only G_{12} needs to be estimated, what is the best procedure for such estimation, in the case where the other transfer functions are known, and in the case where they are unknown but of no interest? The choice of the "best procedure" involves criteria such as bias error and variance error.
- If G_{12} can be estimated using different combinations of measured signals, do some of them lead to better properties for the estimated $\widehat{G}_{12}$?
- What are the excitation conditions (on r and on v) that allow for the estimation of G_{12}?

Some of these questions have been addressed in recent papers [3], [10].

III. The 3-node network

We examine the different questions raised in Section II on the basis of a 3-node network. The reason for this choice is as follows. We have shown that the L-node network can always be transformed into a MISO feedback system with a "regulator" of size $(L-1) \times 1$. Taking $L = 2$ leads to a SISO feedback system that is not representative of the situations encountered in the L-node network because of the triangular structure of the matrix $G(q)$ in (5) and the fact that $K_1(q), K_2(q)$ and $K_3(q)$ are all scalar. The first network structure of modest size that is representative of the general L-node network is the 3-node network. It allows one to address and analyze all questions raised in the previous section, and come up with solutions that are representative of the larger networks. In section VII we shall actually make some specific extensions to a general L-node network.

Thus, consider the following 3-node network:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & G_{32} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{9}$$

We first rewrite it in the form:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \left\{ \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\} \tag{10}$$

where the 3×3 matrix $M(q)$ is defined as follows:

$$M = \frac{1}{\Delta} \begin{bmatrix} 1 - G_{23}G_{32} & G_{12} + G_{13}G_{32} & G_{13} + G_{12}G_{23} \\ G_{21} + G_{23}G_{31} & 1 - G_{13}G_{31} & G_{23} + G_{21}G_{13} \\ G_{31} + G_{32}G_{21} & G_{32} + G_{31}G_{12} & 1 - G_{12}G_{21} \end{bmatrix} \tag{11}$$

and where $\Delta(q) \stackrel{\Delta}{=} \det(I - G)$:

$$\begin{aligned} \Delta(q) &= 1 - G_{12}G_{23}G_{31} - G_{21}G_{13}G_{32} \\ &\quad - G_{13}G_{31} - G_{23}G_{32} - G_{12}G_{21} \end{aligned} \tag{12}$$

We observe that $\Delta(\infty) = 1$ by the conditions on the absence of algebraic loops enunciated in Section II.

The questions we shall address on the network (9) are:

1) what is the best procedure to identify the first row $[G_{12} \quad G_{13}]$?
2) assuming we only want to identify G_{12}, what is the best procedure in the case where either G_{13} is known (this is a typical case if G_{13} is a regulator), or where G_{13} is unknown but not useful to be estimated?
3) what are the sufficient richness conditions on the external signals r_i and v_i, for $i = 1, 2, 3$, that allow a consistent estimation of G_{12} when G_{13} is known, or when G_{13} is unknown?
4) how do the external signals impact on the variance of the estimated $\hat{G}_{12}$?

IV. A 3-NODE MOTIVATING EXAMPLE

In this section we examine a simple 3-node example that illustrates the major issues raised above, in terms of identifiability, richness of the external signals, bias and variance.

We consider the following 3-node example:

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix} 0 & a_1q^{-1} + a_2q^{-2} & bq^{-1} \\ q^{-1} & 0 & 0 \\ 0 & cq^{-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &\quad + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \end{aligned} \tag{13}$$

where the parameters a_1, a_2, b, c are unknown parameters to be identified from available signals. For simplicity, we assume that the noises, if they exist, are all white noises with variances λ_1, λ_2 and λ_3, respectively, and that they are uncorrelated.

Now assume first that $r_3 = 0$ and $e_3 = 0$. Suppose we want to estimate $G_{12}(q) \stackrel{\Delta}{=} a_1q^{-1} + a_2q^{-2}$ and $G_{13}(q) = bq^{-1}$ from the available signals. A direct approach would be to use the first equation:

$$w_1 = (a_1q^{-1} + a_2q^{-2})w_2 + bq^{-1}w_3 + r_1 + e_1 \tag{14}$$

and to estimate the parameters a_1, a_2 and b with a direct prediction error approach, using w_1 as the ouput and w_2, w_3 and r_1 as the inputs of this MISO system. However, we show that these parameters cannot be uniquely identified.

Replacing b in (14) by $b = b_1 + b_2$ and then substituting w_3 in the term $b_1q^{-1}w_3$ by its expression in the third equation of (13) we obtain:

$$w_1 = [a_1q^{-1} + (b_1c^0 + a_2)q^{-2}]w_2 + b_2q^{-1}w_3 + r_1 + e_1 \tag{15}$$

where c^0 is the coefficient appearing in the last equation of the true system. We now observe that the one-step ahead prediction errors for (14) and (15) are identical for all combinations $b = b_1 + b_2$ since $w_3 = c^0q^{-1}w_2$. Thus, using the signals w_1, w_2, w_3, r_1 with the model (14) will not lead to a unique mimimum of the prediction error criterion.

Suppose we now add a nonzero excitation signal r_3 so that the third equation becomes

$$w_3 = c^0q^{-1}w_2 + r_3 \tag{16}$$

Equation (15) is then replaced by

$$w_1 = [a_1q^{-1} + (b_1c^0 + a_2)q^{-2}]w_2 + b_2q^{-1}w_3 + r_1 + b_1q^{-1}r_3 + e_1 \tag{17}$$

If we now use the signals w_1, w_2, w_3, r_1 and r_3, with a model structure that is able to represent all FIR functions in (17), then all parameters can be consistently estimated.

Finally, we note that if the true $G_{13} = b^0q^{-1}$ is known, then $G_{12}(q, \theta)$ can be consistently estimated from w_1, w_2, w_3, r_1 even if $r_3 = 0$, because it makes the predictor for $\hat{w}_1(t|t-1)$ unique: see (14) with $b = b^0$. This example illustrates how the presence or absence of external signals, and the knowledge or lack of knowledge of other transfer functions may decide whether or not G_{12} can be identified.

V. MODELS FOR THE 3-NODE NETWORK

We show now that the identification of G_{12} can be performed using alternative models that relate G_{12} to measured signals, and we discuss the bias properties that result from the use of these alternative models.

The first and most obvious idea for the identification of G_{12} is to use the first equation of (9):

$$w_1 = G_{12}w_2 + G_{13}w_3 + H_1e_1 + r_1, \tag{18}$$

and to identify G_{12} using the signals w_1, w_2, w_3 and, possibly, r_1 if it exists. Since w_2 and w_3 may be correlated through the other equations of (9), this consists of the identification of a closed loop MISO model with input signals w_2, w_3 and possibly r_1, with a possible excitation with an unknown disturbance $v_1 = H_1(q)e_1$. We call this model the **network model**.

Another model for the identification of G_{12} is obtained from the closed loop equations (10), which we call the **joint model**. From the model (10) we can define the following (1×2) and (2×2) submatrices

$$\bar{M}_1 \triangleq [M_{12} M_{13}], \text{ and } \bar{M}_2 \triangleq \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \quad (19)$$

Then straightforward calculations show that

$$[G_{12} \quad G_{13}] = \bar{M}_1 \bar{M}_2^{-1} \quad (20)$$

The identification of G_{12} using this joint model uses the signals w_1, w_2, w_3 as well as r_2 and r_3, if they exist.

An alternative model for the identification of G_{12}, that we shall call **hybrid model**, is obtained as follows. Substitute w_2 in the first equation (9) by its expression in the second equation; this yields:

$$\begin{aligned} w_1 &= G_{12}[G_{21}w_1 + G_{23}w_3 + r_2 + v_2] + G_{13}w_3 + r_1 + v_1 \\ &= G_{12}G_{21}w_1 + [G_{13} + G_{12}G_{23}]w_3 \\ &\quad + r_1 + G_{12}r_2 + v_1 + G_{12}v_2 \end{aligned}$$

Equivalently:

$$\begin{aligned} w_1 &= \frac{1}{1 - G_{12}G_{21}}[(G_{13} + G_{12}G_{23})w_3 + r_1 + G_{12}r_2 \\ &\qquad + v_1 + G_{12}v_2] \end{aligned} \quad (21)$$

We rewrite this equation as follows, with the appropriate definitions, and we obtain the following hybrid model:

$$w_1 = L_{11}r_1 + L_{12}r_2 + L_{13}w_3 + N_{11}v_1 + N_{12}v_2 \quad (22)$$

and we observe that the desired G_{12} can be expressed as

$$G_{12}(q) = \frac{L_{12}(q)}{L_{11}(q)}. \quad (23)$$

Thus, we observe that with this hybrid model, G_{12} can be estimated using the signals w_1, w_3, r_1, r_3. The word hybrid is used for this model because, when viewed as a MISO model, it uses a combination of internal and external signals.

The models (18), (10) and (21) provide us with three different ways to perform the identification of G_{12}, using different signals. However, the estimates obtained from these different models will have different bias properties.

Bias properties for the estimates of G_{12}

Comparing the estimation of G_{12} from the three models discussed above, using their corresponding signals, we can make the following observations.

1) The unbiased estimation of G_{12} from the signals w_1, w_2, w_3, r_1 in (18) requires the simultaneous estimation of a fully parametrized model for G_{13} and for H_1. By fully parametrized, we mean that the model sets chosen for G_{13} and for H_1 must be able to represent the true transfer functions.
2) As shown by (20), the use of the joint model (10) for the unbiased estimation of G_{12} requires an unbiased estimation, and hence a full parametrization, of the last two columns of M. However, it does not require the identification of the noise models H_1, H_2 or H_3, i.e. an Output Error model for (10) will deliver an unbiased estimate of G_{12} even with colored noises. Note also that this identification uses the signals w_1, w_2, w_3, r_2, r_3.
3) The unbiased estimation of G_{12} using the hybrid model (21) is based on (23). It requires an unbiased estimation, and hence a full parametrization, of the transfer functions L_{11}, L_{12}, L_{13} in (21), but G_{12} will be unbiased even if the noise models are not estimated.

VI. Direct identification in the 3-node network

We now study the direct identification of G_{12} using signals w_1, w_2, w_3 in (18). We have shown above that the identification of G_{12} may require the identification of the vector $[G_{12} \quad G_{13}]$ and, possibly also, of H_1. For the purpose of analysing the effect of different excitation scenarios on the estimates, we adopt the following model structure for the parametrization of G_{12}, G_{13} and H_1, as proposed in [9]:

$$\begin{aligned} \mathcal{M} \;=\; \Big\{ & G_{12}(\alpha), G_{13}(\alpha,\beta), H_1(\alpha,\beta,\gamma), \\ & \theta = \begin{pmatrix} \alpha^T & \beta^T & \gamma^T \end{pmatrix}^T \in D_\theta \subset \mathcal{R}^{n_\theta} \Big\} \end{aligned} \quad (24)$$

where $G_{12}(\alpha)$, $G_{13}(\alpha, \beta)$ and $H_1(\alpha, \beta, \gamma)$ are rational transfer functions, $\theta \in \mathcal{R}^{n_\theta}$ is the vector of model parameters, and D_θ is a subset of admissible values for θ. Note that this parametrization covers a wide range of model structures, including all the standard ones. As for the true system, we shall denote by G_{ij}^0 the true transfer functions in (9), and by M_{ij}^0 the true transfer functions in (10). Throughout our further analysis, we make the following assumption.

Assumption 1: The true subsystem $\mathcal{S}_1$ is contained in the model structure $\mathcal{M}$ for some $\theta_0 = (\alpha_0^T, \beta_0^T, \gamma_0^T)^T \in D_\theta$.

The example has shown that the consistent identification of G_{12} in (18) depends on the network structure and on the signals. Proposition 2 in [10] provides conditions for the consistent identification of G_{12} in (18), but the crucial informativity condition is expressed in terms of the positivity of the spectrum $\Phi(\omega)$ of the vector w. This is not a very practical condition in terms of experiment design since the w_i are internal variables, subject to feedback.

Here we examine the experimental conditions that lead to an unbiased estimate of G_{12} under various conditions on the presence or absence of the external signals, the richness of these signals, the structure of the network, the knowledge one may or may not have about certain transfer functions.

The one-step ahead prediction error for $w_1(t)$ in (18) is

$$\begin{aligned} \varepsilon_1(t,\theta) &\triangleq w_1(t) - \hat{w}_1(t|t-1,\theta) \\ &= \frac{1}{H_1(\theta)}[w_1(t) – G_{12}(\alpha)w_2(t) – G_{13}(\alpha,\beta)w_3(t) – r_1(t)] \end{aligned}$$

Provided the model structures are identifiable and the data informative, then the parameter vector $\hat{\theta}^N$ converges asymptotically to the true θ^0, and the per sample asymptotic covariance matrix is given by $P_\theta = \lambda_1[I(\theta^0)]^{-1}$ where $I(\theta)$ is the information matrix defined by

$$I(\theta) = \bar{E}[\psi(t,\theta)\psi^T(t,\theta)] \quad (25)$$

and the pseudoregressor vector is defined as $\psi(t,\theta) \triangleq \frac{\partial \varepsilon_1(t,\theta)}{\partial \theta}$. The pseudoregressor vector $\psi(t,\theta)$ will have full rank at θ^0, and thus also the information matrix, if $w(t)$ is informative, meaning that it is *persistently exciting of sufficient order*. This condition seems like a technicality that will be generally satisfied, but in a network this is far from true. Its satisfaction rests on the network structure, on the complexity of the transfer functions G_{ij}, and on the external signals $r(t)$ and $v(t)$ - their existence and richness. Necessary and sufficient conditions for the nonsingularity of the information matrix have been given for open and closed loop SISO systems in [4]. Here we extend these results to a MISO system with a multivariable feedback loop.

Following the same procedure as in [4] we shall express the pseudoregressor vector as a function of the external signals r_1, r_2, r_3 and the white noise sources e_1, e_2, e_3:

$$\psi(t,\theta) = V_r(q,\theta)\begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix} + V_e(q,\theta)\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \quad (26)$$

where $V_r(q,\theta)$ and $V_e(q,\theta)$ are $d \times 3$ matrices of transfer functions, d being the dimension of the parameter vector θ. Remember that the network may be such that some signals $r_i(t)$ or $e_i(t)$ may not be present. In the definition of V_r and V_e in expression (26) it is understood that, if some of the signals r_i or e_i are zero for the network under investigation, the corresponding columns of V_r and/or V_e are zero. It then follows from the analysis of [4] that the pseudoregressor $\psi(t,\theta)$ can have full rank (and hence $I(\theta) > 0$) if and only if there exists no vector $\mu \in \Re^d$ with $\mu \neq \mathbf{0}$ such that

$$\mu^T [V_r(q,\theta) \quad V_e(q,\theta)] = \mathbf{0}. \quad (27)$$

and the available signals r_j can be made sufficiently rich. Stated otherwise, the necessary condition (27) means that the columns of the matrix $[V_r(q,\theta) \quad V_e(q,\theta)]$ cannot have a common left nullspace. It is important to note that this condition depends entirely on the network structure and on the parametrization of the transfer functions that are estimated. We refer to [4] for details and extensions on the specific richness conditions for the signals r_i.

The task now is to express these vectors in terms of the parameters of the model and of the true system. Obtaining these expressions will serve two purposes:

- rank conditions on the vectors contained in V_r and V_e, and richness conditions on the applied signals r_1, r_2 and r_3 will enable to ensure informativity of the data set $w(t)$, under various scenarii, e.g. presence or not of some of these external signals
- these expressions will determine the covariance of the parameters $\{\alpha, \beta, \gamma\}$ and the variance of $G_{12}(q, \hat{\alpha}^N)$ under various scenarii.

To compute $V_r(q,\theta)$ and $V_e(q,\theta)$ we define the partial derivatives $\nabla_1 \triangleq \frac{\partial G_{12}(\alpha)}{\partial \theta}$, $\nabla_2 \triangleq \frac{\partial G_{13}(\alpha,\beta)}{\partial \theta}$, $\nabla_3 \triangleq \frac{\partial H_1(\theta)}{\partial \theta}$, and we observe that they take the following form:

$$\nabla_1 = \begin{bmatrix} \frac{\partial G_{12}}{\partial \alpha} \\ 0 \\ 0 \end{bmatrix}, \ \nabla_2 = \begin{bmatrix} \frac{\partial G_{13}}{\partial \alpha} \\ \frac{\partial G_{13}}{\partial \beta} \\ 0 \end{bmatrix}, \ \nabla_3 = \begin{bmatrix} \frac{\partial H_1}{\partial \alpha} \\ \frac{\partial H_1}{\partial \beta} \\ \frac{\partial H_1}{\partial \gamma} \end{bmatrix} \quad (28)$$

Some lenghty calculations then show that V_r and V_e can be expressed as follows:

$$V_r(q,\theta) = \frac{1}{H_1(\theta)}[V_1 \quad V_2 \quad V_3] \quad (29)$$

$$V_e(q,\theta) = \frac{1}{H_1(\theta)}[\nabla_3 + H_1^0 V_1 \quad H_2^0 V_2 \quad H_3^0 V_3] \quad (30)$$

$$V_1 = \nabla_1 M_{21}^0 + \nabla_2 M_{31}^0 \quad (31)$$

$$V_2 = \nabla_1 M_{22}^0 + \nabla_2 M_{32}^0 \quad (32)$$

$$V_3 = \nabla_1 M_{23}^0 + \nabla_2 M_{33}^0 \quad (33)$$

We thus have the following result for the 3-node network.

Theorem 6.1: Consider the identification of G_{12}, G_{13} and H_1 in the network (9) using a direct prediction error method applied to (18). Then the data $w(t)$ can be made informative by proper choice of external excitation signals for the consistent estimation of the parameters α, β, γ in $\mathcal{M}$ in (24) only if the columns of $V_r(q,\theta)$ and $V_e(q,\theta)$ in (29)-(30) have no common left nullspace.

Proof: The proof follows immediately from the condition (27) applied to this 3-node model structure. ■

If the condition is satisfied, then informative data are obtained either from the existing noise signals, or by adding sufficiently rich excitation signals r_i.

Now we observe that the left nullspaces of the last two columns of V_r are, respectively, identical to the left nullspaces of the last two columns of V_e, since multiplication by H_2^0 and H_3^0 does not change the ranks of the corresponding columns. Thus, we have the following result.

Theorem 6.2: Suppose one wants to identify G_{12}, G_{13} and H_1 in the 3-node network (9). If the configuration of the network and its external signals is such that the data are not informative, i.e. $I(\theta)$ is singular, then adding an external reference signal, r_2 or r_3, at a node where there is a noise disturbance will not improve the informativity of the data.

Proof: The proof follows immediately from the expressions of the last two columns of V_r and V_e in (29) and (30). ■

We now return to the example of Section IV to illustrate the usefulness of the expressions of V_r and V_e for the generation of informative data $w(t)$. We have $H_1 = 1$, $\alpha = (a_1 \ \ a_2)^T, \beta = b$ and $\theta = (a_1 \ \ a_2 \ \ b)^T = (\alpha^T \ \ \beta)^T$. $\nabla_3 = \mathbf{0}$, $\nabla_1 = \begin{bmatrix} q^{-1} & q^{-2} & 0 \end{bmatrix}^T$ and $\nabla_2 = \begin{bmatrix} 0 & 0 & q^{-1} \end{bmatrix}^T$. Applying the formulas (31)-(33) we get

$$V_1(q,\theta) = \begin{bmatrix} q^{-2} & q^{-3} & c^0 q^{-3} \end{bmatrix}^T$$

$$V_2(q,\theta) = \begin{bmatrix} q^{-1} & q^{-2} & c^0 q^{-2} \end{bmatrix}^T$$

$$V_3(q,\theta) = \begin{bmatrix} b^0 q^{-3} & b^0 q^{-4} & q^{-1} - a_1^0 q^{-3} - a_2^0 q^{-4} \end{bmatrix}^T$$

From these expressions it is clear that $Ker(V_1) = Ker(V_2) = \{\mu \in \Re^3 : \mu = k[0 \quad c^0 \quad -1]^T \ \forall k \in \mathbb{R}\}$, while $Ker(V^3) = \{\mathbf{0}\}$. This shows why the data in the network

(13) are not informative for the identification of G_{12} and G_{13} when $r_3 = e_3 = 0$. A remarkable result of our analysis of sufficient richness is the following.

Conclusion for the example: The consistent identification of the transfer functions $G_{12}(\theta)$ and $G_{13}(\theta)$ in the network (13) of Section IV is possible if either $e_3 \neq 0$, or r_3 is sufficiently rich of order 3 (see [4] for details). $G_{12}(\theta)$ and $G_{13}(\theta)$ can be consistently identified (i.e. the data $w(t)$ are informative) if all external signals in the network (13) are zero except either r_3 or v_3.

Comment: The condition above shows that only the external signals entering in the third node can ensure informativity of the data $w(t)$. This does not mean that the other signals are useless. They will contribute to a decrease in the variance of the estimated transfer functions.

VII. Results for the L-node network

Some of the results derived for the 3-node network in Section VI can be easily generalized to the general L-node network of (1). The expressions $V_r(q,\theta)$ and $V_e(q,\theta)$ of (29) and (30) can be generalized to a L-node network as follows.

Lemma 7.1: Suppose one identifies the transfer functions of the first row of a L-node network by a prediction error method that minimizes a norm of $\varepsilon(t) = w_1(t) - \hat{w}_1(t|t-1)$. Then the pseudoregressor that defines $I(\theta)$ has the form

$$\psi(t,\theta) = V_r(q,\theta) \begin{bmatrix} r_1(t) \\ \vdots \\ r_L(t) \end{bmatrix} + V_e(q,\theta) \begin{bmatrix} e_1(t) \\ \vdots \\ e_L(t) \end{bmatrix} \quad (34)$$

where V_r and V_e are expressed as follows:

$$V_r(q,\theta) = \frac{1}{H_1(\theta)}[V_1 \quad V_2 \quad \dots \quad V_L] \quad (35)$$

$$V_e(q,\theta) = \frac{1}{H_1(\theta)}[\nabla_{H_1} + H_1^0 V_1 \;\; H_2^0 V_2 \;\; \dots \;\; H_L^0 V_L] \quad (36)$$

with $\nabla_{H_1} \triangleq \frac{\partial H_1(\theta)}{\partial \theta}$.

Proof: The proof is a straightforward extension of the derivations that led to (29) and (30) with an appropriate redefinition of the vectors V_i in (31)-(33) to account for the presence of the additional terms in the equation for $w_1(t)$. ■

Now observe that the left nullspaces of the $L-1$ rightmost columns of V_r and V_e are identical, because multiplication by the noise models H_i in the columns of V_e does not change the ranks of the corresponding columns. Thus, we have the following interesting result.

Theorem 7.1: Suppose one wants to identify the nonzero transfer functions $G_{ij}, j = 1, \dots, L$ of any row i of G in (1) and assume that the existing external signals $\{r_j\}$ and $\{v_j\}$ are not informative for the identification of these transfer functions with the existing configuration, i.e. the information matrix $I(\theta)$ is singular, and hence these transfer functions cannot be identified. Then adding an external reference signal, say r_k, at a node where there is a noise disturbance v_k will not improve the informativity of the data.

Proof: The proof, for the identification of the transfer functions G_{1j}, follows immediately from the fact that the left nullspaces of V_{e_k} and V_{r_k} are identical, as shown in Lemma 7.1. Extension of the result to the elements of any row i of G is obtained by a permutation of the positions in (35) and (36) and proper adjustment of the definitions of the columns V_i. ■

VIII. Conclusions

We have studied the identification of a particular transfer function in an interconnected network. A critical condition for the consistent identification of such transfer function is that the data be informative. Guaranteeing such informativity condition in a network is by no means trivial, as we have indicated with a simple example. We have used a 3-node network as a representative model for the identification of a transfer function in a network, because any network can be transformed into a MISO feedback system and the 3-node network is the simplest MISO feedback system. For this 3-node network, we have derived novel informativity conditions expressed in terms of the external signals and the structure of the network. These conditions have led to a simple result for the case of a general L-node network.

The expressions we have derived for the transfer of excitation from external signals to node signals also allow one to compute the variance of the estimated transfer function, and to compare its expression under various experimental conditions. These lead to interesting design considerations in terms of the choice of signals to be applied when the objective is to reduce the variance of the estimated transfer function below some threshold. For lack of space, this analysis is left for a future paper.

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