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Controller validation for stability and performance based on a frequency domain uncertainty region obtained by stochastic embedding†

Xavier Bombois(1), Michel Gevers(1), Gérard Scorletti(2)

(1) CESAME, Université Catholique de Louvain
Bâtiment EULER, 4 av. Georges Lemaitre, B-1348 Louvain-la-Neuve, Belgium
Tel: +32 10 472596, Email: {Bombois, Gevers}@csam.ucl.ac.be
(2) LAP ISMRA
6 boulevard du Maréchal Juin, F-14050 Caen Cedex, France
Tel: +33 2 31452712, Email: gerard.scorletti@greyc.ismra.fr

Abstract

This paper presents a robustness analysis for an uncertainty set deduced from stochastic embedding techniques and made up of ellipsoids at each frequency in the Nyquist plane. Our robustness analysis focuses on the validation of a controller both for robust stability and for robust performance, over all systems in such frequency domain uncertainty region. Our validation procedure for stability ensures that the controller stabilizes all systems in this nonstandard uncertainty set. Our validation procedure for performance computes the worst case performance over all closed loop systems made up of the controller and all plants in the frequency domain uncertainty region.

1 Introduction

This paper is part of our continuing effort to connect time-domain prediction error identification and robustness theory [3, 2]. In our previous papers, we have analysed the robustness properties of an uncertainty set $\mathcal{D}$ delivered by classical prediction error identification methods and to which the true system G_0 was known to belong with some prescribed probability. This uncertainty set $\mathcal{D}$ was defined as a set of parametrized rational transfer functions whose parameter vector lies in an ellipsoidal confidence region. Such uncertainty region results naturally from a prediction error identification experiment when the system is assumed to be in the model set. The uncertainty set is thus entirely defined by covariance errors on the parameters. This restriction can be relaxed using the stochastic embedding approach of [5] to construct uncertainty regions that then take into account both bias and variance errors in the estimated transfer functions. In the present paper, we develop robust analysis tools for such uncertainty regions obtained using a stochastic embedding technique.

Uncertainty region $\mathcal{L}$. Stochastic embedding techniques developed in e.g. [5] allow one to design a frequency domain uncertainty region around a possibly biased identified model with fixed denominator (such as FIR or Laguerre models). This uncertainty region $\mathcal{L}$ contains the *stable* unknown true system G_0 at a certain probability level and is made up of ellipsoids at each frequency in the Nyquist plane around the frequency response of the identified model. These uncertainty ellipsoids are designed using the assumption that the unmodelled dynamics of the true system can be considered as a stochastic process and that the parameters that describe the second-order properties of this stochastic process can be estimated from the data. This parameter estimation step is achieved using a maximum likelihood technique. One of the contributions of the present paper is to extend the stochastic embedding technique to closed-loop identification.

Controller validation for stability. Our validation procedure for stability ensures that a given controller C stabilizes all systems in an uncertainty region $\mathcal{L}$ obtained by stochastic embedding. Robust stability theory developed in e.g. [9, 7, 6] provides some necessary and sufficient conditions for the stabilization, by some given controller C, of all plants in an uncertainty region that is defined in a general LFT (linear fractional transformation) framework. Our contribution with the proposed stability validation procedure is to show that one can rewrite the closed-loop connection of the controller C and all plants in such uncertainty region $\mathcal{L}$ as a particular LFT where the uncertainty part is a transfer vector whose frequency response is real. In that particular LFT, the (real) stability radius can be

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computed exactly, using the result presented in [6, 7]. In [8], the authors present an LFT description of the closed-loop connection of the controller C and all plants in an uncertainty region $\mathcal{L}$, where the ellipsoids at each frequency are approximated by a mixed perturbation set. The main advantage of our LFT description is that it exactly represents the closed-loop connection of the controller C and all plants in the uncertainty region $\mathcal{L}$ without any approximation.

Controller validation for performance. Our validation procedure for performance computes the worst case performance over all closed loop systems made up of a given controller C and all plants in $\mathcal{L}$. We provide an exact computation, using a LMI based optimization problem, of the worst case performance of a closed-loop made up of the controller C and a system in the uncertainty region $\mathcal{L}$. The performance of a particular loop made up of the controller C and a plant in $\mathcal{L}$ is here defined as the largest singular value of a weighted version of the matrix containing the four closed-loop transfer functions of this loop. Our definition of the worst case performance is thus very general and, by an appropriate choice of the weights, allows one to derive most of the commonly used worst case performance measures such as e.g. the largest modulus of the sensitivity function. In [1], the authors compute the largest modulus of the sensitivity functions for plants in $\mathcal{L}$, using the fact that the sensitivity function is the inverse of the distance between the Nyquist curve of the so-called open-loop transfer GC and the point -1. Besides the fact that our performance criterion is more general than the one used in [1], the advantage of our approach is that we propose a convex optimization problem to compute the worst case performance as opposed to the graphical test of [1].

Paper outline. In Section 2, we show how to design an uncertainty region $\mathcal{L}$ from a stochastic embedding procedure in open loop and in closed loop. We then give the general expression of the uncertainty region $\mathcal{L}$. In Section 3, we give the procedure of validation for stability. In Section 4, the concept of worst case performance level is introduced, and the LMI-based optimization problem developed for its computation is given. The procedures for validation for stability and for performance are illustrated by an example in Section 5. Finally, some conclusions are given in the last section.

2 Stochastic embedding and uncertainty region $\mathcal{L}$

In the sequel, we assume that the true *open-loop* system is linear and time-invariant, with a rational input-output transfer function G_0: $y = G_0(z)u + v$, where $v = H(z)e$ is additive noise and e is white noise. We further assume that G_0 is stable[1]. We also define the following vector.

Definition 1 (The RI vector $\delta(z)$) *Let $\Delta(z)$ be a stable transfer function. We define the RI vector $\delta(z)$ as follows:*

$$\delta(z) = \begin{pmatrix} Re(\Delta(z)) \\ Im(\Delta(z)) \end{pmatrix}. \tag{1}$$

where Re and Im denote the real and imaginary part, respectively. Note that the frequency response $\delta(e^{j\Omega})$ of $\delta(z)$ is, at each frequency, **real***: $\delta(e^{j\Omega}) \in \mathbf{R}^{2\times 1}$ $\forall\Omega$.*

2.1 Classical stochastic embedding in open-loop

We first recall the general assumptions used in stochastic embedding techniques.

Assumption 1 ([5]) *The key assumption in stochastic embedding is that the true system $y = G_0 u + v$ can be decomposed in the following expression :*

$$y = G(z, \theta_0)u + G_\Delta(z)u + H(z)e \tag{2}$$

where $G(z,\theta_0) \in H_\infty$ is a transfer function with fixed denominator, parametrized by a vector $\theta_0 \in \mathbf{R}^{k\times 1}$, $G_\Delta(z) \in H_\infty$ represents the (possibly infinite) unmodelled dynamics that is assumed to be a stochastic process with zero mean, independent of the additive noise $v = H(z)e$. It is further assumed that the impulse response coefficients η_k of $G_\Delta(z) = \sum_{k=0}^{\infty} \eta_k z^{-k}$ have a variance that dies at an exponential rate : $\mathcal{E}(\eta_k^2) = \alpha\lambda^k$ ($\mathcal{E}(\eta_k) = 0$), but α and λ need not to be known. As a consequence, $G_\Delta(z)$ can be approximated by $G_\Delta(z) \simeq \sum_{k=0}^{L} \eta_k z^{-k}$ for a given L.

The design of a frequency domain uncertainty region $\mathcal{L}_{OL}$ is divided in two steps. The first step consists of identifying a model $G(z, \hat{\theta}_N)$ of the true input-output dynamic $G_0 = G(z,\theta_0) + G_\Delta(z)$ using N time-domain data $[y\ u]$ collected on the true system G_0. In a second step, the total error at each frequency between the true G_0 and the identified model $G(z, \hat{\theta}_N)$ is embedded at a certain probability level in an ellipsoid in the Nyquist plane. This embedding is made possible using the stochastic assumptions on the unmodelled dynamics $G_\Delta(z)$ and on the additive noise $v = H(z)e$. The ellipsoid that embeds the total error at a certain frequency is thus a function of the frequency, of the stochastic parameters describing $G_\Delta(z)$ (i.e. α and λ) and of the stochastic parameters γ describing v. These parameters can be estimated from the data $[y\ u]$ using a maximum likelihood technique. This is summarized in the following proposition.

Proposition 1 ([5]) *Using the stochastic embedding procedure of [5] based on N input-output data, and un-*

[1] It is possible to get rid of this assumption in our stochastic embedding approach for closed loop systems, but we choose also there to assume G_0 stable in order to shorten this paper.

der Assumption 1, the stable true system G_0 is contained at a chosen probability level $\psi(\chi^2_{ol})$ in a frequency domain uncertainty region $\mathcal{L}_{OL}$ made up of ellipsoids $U_{OL}(\Omega)$ at each frequency in the Nyquist plane:

$$\mathcal{L}_{OL} = \left\{ \begin{array}{c} G_{in}(z) \mid G_{in}(z) = G(z, \hat{\theta}_N) + \Delta(z) \\ \text{with } \Delta(z) \in H_\infty \text{ and } \delta(e^{j\Omega}) \in U_{OL}(\Omega) \end{array} \right\} \tag{3}$$

$$U_{OL}(\Omega) = \{\delta \in \mathbf{R}^{2\times 1} \mid \delta^T P(\Omega, \hat{\alpha}_N, \hat{\lambda}_N, \hat{\gamma}_N)\delta < \chi^2_{ol}\} \tag{4}$$

$G(z, \hat{\theta}_N) \in H_\infty$ is the identified model with fixed denominator based on the N data. $\delta(z)$ is defined in (1). $P(\Omega, \hat{\alpha}_N, \hat{\lambda}_N, \hat{\gamma}_N) \in \mathbf{R}^{2\times 2}$ is a function of the frequency, of the estimated stochastic parameters of the unmodelled dynamics (i.e. $\hat{\alpha}_N$ and $\hat{\lambda}_N$), and of the estimated stochastic parameters of the additive noise (i.e. $\hat{\gamma}_N$). The value χ^2_{ol} is chosen such that $\psi(\chi^2_{ol}) = Pr(\chi^2(k+L) \leq \chi^2_{ol})$ with $\chi^2(k+L)$ the chi-square probability distribution with $k+L$ degrees of freedom.

2.2 Stochastic embedding in closed loop

Let us now consider a controller K which forms a stable closed loop with the stable true system G_0. We consider here the closed-loop transfer function T_0 between the reference r and the output y. We then have to assume that K and K^{-1} are stable [4]. Similar procedures exist for the other three closed-loop transfer functions. Let us thus collect N experimental data $[r\ y]$ on the closed loop composed of the true system G_0 and this stabilizing controller K:

$$y = \frac{G_0 K}{1 + G_0 K} r + \frac{H}{1 + G_0 K} e = T_0 r + \tilde{v} \tag{5}$$

As the loop $[K\ G_0]$ is stable, it is possible to use the stochastic embedding technique presented in Section 2.1 to design a frequency domain uncertainty region $\mathcal{L}_T$ containing T_0. For this purpose, (5) is rewritten in a way similar to (2):

$$y = T(z, \xi_0) r + T_\Delta(z) r + \tilde{v} \tag{6}$$

where T_0 is decomposed into a model $T(z, \xi_0)$ with fixed denominator and the unmodelled dynamics $T_\Delta(z)$. Using Proposition 1, we may deduce the uncertainty region $\mathcal{L}_T$ containing T_0 at a probability level $\psi(\chi^2_{cl})$:

$$\mathcal{L}_T = \left\{ \begin{array}{c} T_{in}(z) \mid T_{in}(z) = T(z, \hat{\xi}_N) + \Delta(z) \\ \text{with } \Delta(z) \in H_\infty \text{ and } \delta(e^{j\Omega}) \in U_{CL}(\Omega) \end{array} \right\} \tag{7}$$

$$U_{CL}(\Omega) = \{\delta \in \mathbf{R}^{2\times 1} \mid \delta^T P(\Omega, \hat{\alpha}_N, \hat{\lambda}_N, \hat{\gamma}_N)\delta < \chi^2_{cl}\} \tag{8}$$

where $\delta(z)$ is defined in (1), $\hat{T}(z) \triangleq T(z, \hat{\xi}_N) \in H_\infty$ is the identified model with fixed denominator, $\hat{\alpha}_N$ and $\hat{\lambda}_N$ are here the stochastic parameters linked to $T_\Delta(z)$, and $\hat{\gamma}_N$ the stochastic parameters linked to $\tilde{v}$. The set $\mathcal{L}_T$ is a set of closed-loop transfer functions. The corresponding set of open-loop transfer functions is now constructed. As $G_0 = T_0/(K(1-T_0))$, the open-loop transfer function $G_{in}(z)$ corresponding to $T_{in}(z)$ is given by:

$$G_{in}(z) = \frac{1}{K} \times \frac{T_{in}(z)}{1 - T_{in}(z)}. \tag{9}$$

In particular, the nominal open-loop model $G(z, \hat{\xi}_N)$ corresponding to $\hat{T} = T(z, \hat{\xi}_N)$ is given by: $G(z, \hat{\xi}_N) = T(z, \hat{\xi}_N)/(K(1 - T(z, \hat{\xi}_N)))$. As we assume that the true system G_0 is stable, we also assume that this open-loop model $G(z, \hat{\xi}_N)$ is stable. The set $\mathcal{L}_{CL}$ of open-loop plants G_{in} corresponding to the set $\mathcal{L}_T$ of closed-loop transfer functions T_{in} is:

$$\mathcal{L}_{CL} = \left\{ \begin{array}{c} G_{in}(z) \mid G_{in}(z) = \frac{G(z,\hat{\xi}_N) + \frac{1+KG(z,\hat{\xi}_N)}{K}\Delta}{1+(-1-KG(z,\hat{\xi}_N))\Delta} \\ \text{with } \Delta(z) \in H_\infty \text{ and } \delta(e^{j\Omega}) \in U_{CL}(\Omega) \end{array} \right\} \tag{10}$$

Properties of $\mathcal{L}_{CL}$. According to Proposition 1, the true closed-loop transfer function T_0 lies in $\mathcal{L}_T$ with probability $\psi(\chi^2_{cl})$. As a consequence, the true system $G_0 = T_0/(K(1 - T_0))$ lies in the frequency domain uncertainty region $\mathcal{L}_{CL}$ with the same probability.

2.3 General structure of the uncertainty regions obtained from stochastic embedding techniques

In the following proposition, we show that $\mathcal{L}_{OL}$ and $\mathcal{L}_{CL}$ can be described using the same generic expression $\mathcal{L}$. The form of this generic expression has been chosen to ease the subsequent robustness analysis.

Proposition 2 *Consider the true open-loop dynamics G_0. The uncertainty regions $\mathcal{L}_{OL}$ and $\mathcal{L}_{CL}$ given in (3) and (10), respectively, and containing G_0 at a certain probability level have the general form of a frequency domain uncertainty region $\mathcal{L}$ where the uncertainty part is the RI vector $\delta(z)$ (see Definition 1).*

$$\mathcal{L} = \left\{ \begin{array}{c} G(z, \delta(z)) \mid G(z, \delta(z)) = \frac{\hat{G}(z) + Z_N(z)\delta(z)}{1 + Z_D(z)\delta(z)} \\ \text{with } \delta(e^{j\Omega}) \in U(\Omega) \end{array} \right\} \tag{11}$$

$$U(\Omega) = \{\delta(e^{j\Omega}) \in \mathbf{R}^{2\times 1} \mid \delta(e^{j\Omega})^T R(\Omega) \delta(e^{j\Omega}) < 1\} \tag{12}$$

where $R(\Omega)$ are symmetric positive definite matrices $\in \mathbf{R}^{2\times 2}$. These matrices are different at each frequency Ω; $Z_N(z)$ and $Z_D(z)$ ($Z_N(z)$ and $Z_D(z) \in H_\infty$) are row vectors of length 2 containing known transfer functions; $\hat{G}(z) \in H_\infty$ is a known transfer function that can be considered as the center of $\mathcal{L}$.

3 Controller validation for stability for $\mathcal{L}$

Consider an uncertainty region $\mathcal{L}$ whose generic structure is given in (11). We now say that a controller C, designed from any model G_{mod} of the true system G_0, is *validated for stability* if it stabilizes all models in this uncertainty region $\mathcal{L}$ (and therefore also the true system G_0). The model G_{mod} may e.g. be the center $\hat{G}(z)$ of the uncertainty region $\mathcal{L}$, or it may be given a-priori. Our contribution in this section is to show that the uncertainty region $\mathcal{L}$ is amenable to classical robust stability analysis. Indeed, we present a way to describe the set of closed-loop connections of all plants in $\mathcal{L}$ with the "to be validated controller" C as a set of loops $[M_{\mathcal{L}}(z)\ \delta(z)]$ where the uncertainty part $\delta(z)$ is the RI vector of (11) and for which we can deduce a necessary and sufficient robust stabilization condition, since the stability radius of such set of loops is exactly and explicitly computable [7].

Theorem 3 *Consider an uncertainty set $\mathcal{L}$ of the form (11) and a controller $C(z) = X(z)/Y(z)$*[2] *that stabilizes the center of that set, $\hat{G}(z)$. Then all models in $\mathcal{L}$ are stabilized by $C(z)$ if and only if, at each frequency Ω,*

$$\mu(M_{\mathcal{L}}(e^{j\Omega})T^{-1}(\Omega)) \leq 1. \tag{13}$$

$M_{\mathcal{L}}(z)$ is defined as

$$M_{\mathcal{L}}(z) = -(Z_D + \frac{X(Z_N - \hat{G}Z_D)}{Y + \hat{G}X}). \tag{14}$$

$T(\Omega)$ is a square root of the matrix $R(\Omega)$ defining $U(\Omega)$: $R(\Omega) = T^T(\Omega)T(\Omega)$ whereby $\delta(e^{j\Omega}) \in U(\Omega) \Leftrightarrow |T(\Omega)\delta(e^{j\Omega})|_2 < 1$. $\mu(M)$ is called the (real) stability radius and is equal to $|M|_2$ if $Im(M) = 0$ and to

$$\sqrt{|Re(M)|_2^2 - \frac{(Re(M)Im(M)^T)^2}{|Im(M)|_2^2}} \text{ if } Im(M) \neq 0 \tag{15}$$

Proof. A similar scheme as for the proof of Theorem 4 in [3] shows that $[M_{\mathcal{L}}(z)\ \delta(z)]$ is equivalent to the closed-loop connection of the controller C and one plant $G(z, \delta(z))$ in $\mathcal{L}$. Our problem of testing if the controller C stabilizes all the plants in the uncertainty region $\mathcal{L}$ is therefore equivalent to testing if the set of loops $[M_{\mathcal{L}}(z)\ \delta(z)]$ are stable for all $\delta(z)$ such that $\delta(e^{j\Omega})$ lies in the uncertainty domain $|T(\Omega)\delta(e^{j\Omega})|_2 < 1$. Since $M_{\mathcal{L}}(z)$ lies in H_∞, the set of loops $[M_{\mathcal{L}}(z)\ \delta(z)]$ are stable for all $\delta(z)$ such that $\delta(e^{j\Omega}) \in \mathbf{R}^{2\times 1}$ lies in the uncertainty domain $|T(\Omega)\delta(e^{j\Omega})|_2 < 1$ if and only if, at each frequency Ω,

$$1 - M_{\mathcal{L}}(e^{j\Omega})\delta(e^{j\Omega}) \neq 0 \tag{16}$$

for all $\delta(e^{j\Omega})$ such that $|T(\Omega)\delta(e^{j\Omega})|_2 < 1$. A final normalisation shows that expression (16) is equivalent with the statement (13). Indeed if, at each frequency Ω, we define a *real* vector $\phi(e^{j\Omega}) \triangleq T(\Omega)\delta(e^{j\Omega})$, then (16) is equivalent with:

$$1 - M_{\mathcal{L}}(e^{j\Omega})T^{-1}(\Omega)\phi(e^{j\Omega}) \neq 0 \tag{17}$$

for all $\phi(e^{j\Omega})$ such that $|\phi(e^{j\Omega})|_2 < 1$. Using the result in e.g. [7], this last expression is equivalent with (13). □

[2] $X(z)$ and $Y(z)$ are the polynomials of the numerator and the denominator of $C(z)$

4 Controller validation for performance

In this section, we show that we can evaluate the worst case performance in the uncertainty region $\mathcal{L}$, i.e. the worst level of performance of a closed loop made up of the connection of the considered controller and any plant in $\mathcal{L}$. The worst case performance in $\mathcal{L}$ is of course a lower bound for the closed-loop performance achieved with the true system. We then say that a controller is validated for performance if this worst case performance in $\mathcal{L}$ remains below some threshold. There is no unique way of defining the performance of a closed-loop system. However, most commonly used performance criteria can be derived from some norm of a frequency weighted version of the stability matrix $H(G, C)$ of the closed-loop system $[G\ C]$ made up of G in feedback with the controller C.

Definition 2 *Given a plant G and a stabilizing controller C, the stability matrix $H(G, C)$ of the closed loop $[G\ C]$ is given by:*

$$H(G,C) = \begin{pmatrix} \frac{GC}{1+GC} & \frac{G}{1+GC} \\ \frac{C}{1+GC} & \frac{1}{1+GC} \end{pmatrix}. \tag{18}$$

4.1 The general criterion

The worst case performance criterion over all plants in an uncertainty region $\mathcal{L}$ will be similarly defined as the worst possible norm, over all plants in $\mathcal{L}$, of a frequency weighted version of the stability matrix $H(G(z, \delta(z)), C)$, where $G(z, \delta(z))$ is any plant in $\mathcal{L}$ and C is the "to-be-validated" controller C.

General Criterion. Consider an uncertainty region $\mathcal{L}$ given by (11). Consider also a controller $C(z)$ that is validated for stability. The general criterion $J_{WC}(\mathcal{L}, C, W_l, W_r, \Omega)$ measuring the worst case performance level is defined at a frequency Ω as follows:

$$J_{WC} = \max_{G(z,\delta(z))\in\mathcal{L}} \sigma_1\left(W_l H(G(e^{j\Omega}, \delta(e^{j\Omega})), C(e^{j\Omega}))W_r\right) \tag{19}$$

where $W_l(z) = diag(W_{l1}, W_{l2})$ and $W_r(z) = diag(W_{r1}, W_{r2})$ are diagonal weights that allow one to define specific worst case performance levels and where $\sigma_1(A)$ denotes the largest singular value of A. Note that J_{WC} is a frequency function : it defines a template.

4.2 Computation of the general criterion

We now present a procedure for the computation of the general criterion $J_{WC}(\mathcal{L}, C, W_l, W_r, \Omega)$ at a given frequency Ω.

Theorem 4 *Consider an uncertainty region $\mathcal{L}$ defined in (11) and a controller $C(z) = X(z)/Y(z)$. The general criterion J_{WC} defined in (19) is equal to $\sqrt{\gamma_{opt}}$, where γ_{opt} is the optimal value of γ for the following standard convex optimization problem involving LMI* *constraints evaluated at the frequency Ω:*

$$\begin{array}{lll} \textit{minimize} & \gamma & \\ \textit{over} & \gamma,\ \tau & \\ \textit{subject to} & \tau \geq 0 \textit{ and} & \end{array} \qquad (20)$$

$$\begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}^*) & Re(a_{22}) \end{pmatrix} - \tau \begin{pmatrix} R(\Omega) & 0 \\ 0 & -1 \end{pmatrix} < 0$$

where $a_{11} = (Z_N^* W_{l1}^* W_{l1} Z_N + Z_D^* W_{l2}^* W_{l2} Z_D) - \gamma(Q Z_1^* Z_1)$, $a_{12} = Z_N^* W_{l1}^* W_{l1} \hat{G} + W_{l2}^* W_{l2} Z_D^* - \gamma(Q Z_1^*(Y + \hat{G}X))$, $a_{22} = \hat{G}^* W_{l1}^* W_{l1} \hat{G} + W_{l2}^* W_{l2} - \gamma(Q(Y + \hat{G}X)^*(Y + \hat{G}X))$, $Z_1 = XZ_N + YZ_D$ *and* $Q = 1/(X^* W_{r1}^* W_{r1} X + Y^* W_{r2}^* W_{r2} Y)$.

Proof. The proof of this theorem follows the same scheme as the proof of Theorem 7 in [3]. □

5 Example

Let us consider the same true system G_0 as in [5]: $y = G_0 u + e$ with $G_0 = (0.0355z^{-1} + 0.0247z^{-2})/(1 - 1.2727z^{-1} + 0.3329z^{-2})$, and e a white noise with a variance equal to 0.005. The sampling time is 1 second. We simulate this system collecting 50 data. As in [5], we choose a second order Laguerre model $G(z, (\theta_1\ \theta_2)^T)$ as model structure:

$$G(z, \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}) = \frac{0.91\ \theta_1 z^{-1}}{1 - 0.82z^{-1}} + \frac{\theta_2 z^{-1}(0.73 - 0.89z^{-1})}{(1 - 0.82z^{-1})^2}.$$

Using the 50 data, the identified parameters are: $\hat{\theta}_1 = 0.1129$ and $\hat{\theta}_2 = -0.0689$. The second order Laguerre model $G(z, (\hat{\theta}_1, \hat{\theta}_2)^T)$ is chosen as model G_{mod} for the control design. From this model G_{mod}, we have designed a controller with a phase advance : $C(z) = (5.2314 - 3.8667z^{-1})/(1 - 0.6z^{-1})$. With this controller, the designed closed-loop $[G_{mod}\ C]$ has a stability margin of 85 degrees. The cut-off frequency Ω_c is equal to 0.5. Before applying this controller $C(z)$ to the true system, we verify whether it achieves satisfactory behaviour with all plants in an uncertainty region $\mathcal{L}_{OL}$ (and therefore also with the true system G_0). The uncertainty region $\mathcal{L}_{OL}$ is constructed using the classical stochastic embedding assumptions and the procedure described in Section 2.1. The maximum likelihood estimation of α, λ and σ^2 delivers: $\hat{\alpha} = 19.96$, $\hat{\lambda} = 0.002$ and $\hat{\sigma}^2 = 0.006$. These values allow us to design a frequency domain uncertainty region $\mathcal{L}_{OL}$ made up of ellipsoids at each frequency in the Nyquist plane. The desired probability for the presence of G_0 in $\mathcal{L}_{OL}$ is here chosen equal to 0.9. This uncertainty region is represented in Figure 1. It is to be noted that there are a few frequencies where $G_0(e^{j\Omega})$ lies slightly outside this region. This phenomenon can be explained by the nonlinear optimization that delivers the estimate of the stochastic parameters, by the very few data used to design the uncertainty regions, but also by the chosen probabilistic framework.

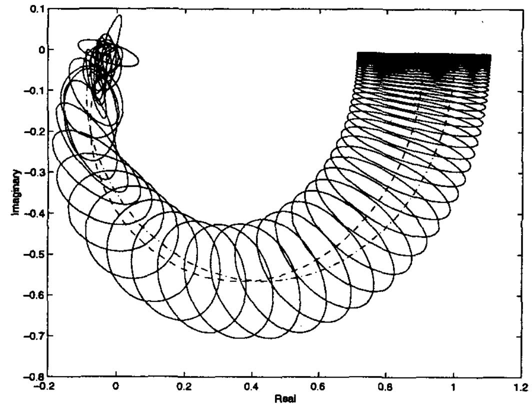

Figure 1: Ellipsoids of $\mathcal{L}_{OL}$ at each frequency, $G_{mod}(e^{j\Omega})$ (dashed) and $G_0(e^{j\Omega})$ (dashdot) in the Nyquist plane

Validation of C for stability. The uncertainty region $\mathcal{L}_{OL}$ having been constructed, we can use the procedure presented in Section 3 to check whether C stabilizes all plants in $\mathcal{L}_{OL}$. For this purpose, we construct the row vector $M_{\mathcal{L}_{OL}}(z)$ defined in Theorem 3 and we compute the corresponding stability radius $\mu(M_{\mathcal{L}_{OL}}(e^{j\Omega})T^{-1}(\Omega))$ at all frequencies. The maximum over all frequencies in $[0\ \pi]$ is $0.4577 < 1$; thus, we conclude that $C(z)$ stabilizes all plants in $\mathcal{L}_{OL}$.

Validation of C for performance. In order to verify that C gives satisfactory performance with all plants in $\mathcal{L}_{OL}$, we compute, at each frequency, the largest modulus $t_{\mathcal{L}_{OL}}(\Omega, S)$ for the sensitivity function "S" over all plants in $\mathcal{L}_{OL}$. This can be done by computing $J_{WC}(\mathcal{L}_{OL}, C, W_l, W_r, \Omega)$ using Theorem 4 with the particular weights $W_l = W_r = diag(0, 1)$. The worst case modulus of all sensitivity functions over $\mathcal{L}_{OL}$ is represented in Figure 2. It is compared with the sensitivity function of the designed closed loop $[G_{mod}\ C]$ and that of the achieved closed loop $[G_0\ C]$. From $t_{\mathcal{L}_{OL}}(\Omega, S)$, we can find that the worst case static error ($= t_{\mathcal{L}_{OL}}(0, S)$) resulting from a constant disturbance of unit amplitude is equal to 0.2889, whereas this static error is 0.2438 in the designed closed-loop and 0.2267 in the achieved closed loop. Using $t_{\mathcal{L}_{OL}}(\Omega, S)$, we can also see that the bandwidth of $\Omega_c = 0.5$ in the designed closed-loop is almost preserved for all closed loops with a plant in $\mathcal{L}_{OL}$ since $t_{\mathcal{L}_{OL}}(\Omega, S)$ is equal to 1 at $\Omega_c \simeq 0.33$. The difference between the resonance peak of the designed sensitivity function (i.e. $\max_\Omega \parallel S(G_{mod}, C) \parallel = 1.1626$) and the worst case resonance peak achieved by a plant in $\mathcal{L}_{OL}$ (i.e. $\max_\Omega t_{\mathcal{L}_{OL}}(\Omega, S) = 2.45$) also remains small. Note that the actually achieved resonance peak (i.e. $\max_\Omega \parallel S(G_0, C) \parallel$) is equal to 1.3930. A last remark is that the actually achieved sensitivity function

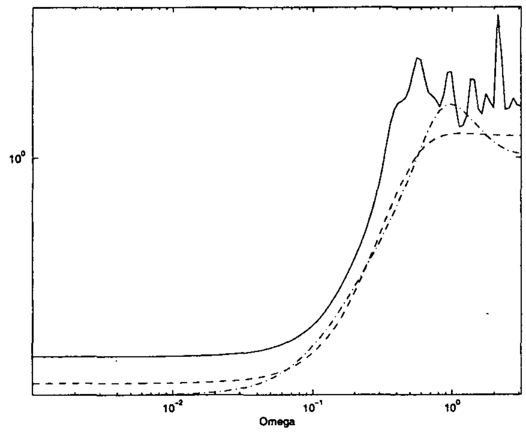

Figure 2: worst case $t_{\mathcal{L}_{OL}}(\Omega, S)$ (solid) and modulus of the designed sensitivity function $S(G_{mod}, C)$ (dashed) and actually achieved sensitivity function $S(G_0, C)$ (dashdot)

is slightly above the template $t_{\mathcal{L}_{OL}}(\Omega, S)$ at a few frequencies. This is due to the fact that $G_0(e^{j\Omega})$ lies slightly outside $\mathcal{L}_{OL}$ at those frequencies. We may therefore conclude that the controller C is validated for performance since the difference between the nominal and worst case performance level remains very small at every frequency. With such stability and performance analysis results, one would confidently apply the controller to the real system, assuming that the nominal performance is judged to be satisfactory.

6 Conclusions

In this paper, we have developed a robustness analysis for the frequency domain uncertainty regions $\mathcal{L}$ delivered by classical stochastic embedding in open loop and by a stochastic embedding procedure in closed loop that we have introduced. We have shown that the stochastic embedding procedure for the estimation of the total Mean Square Error leads, in the open-loop and in the closed-loop case, to an uncertainty region $\mathcal{L}$ that has a general structure, in which the uncertainty part is modelled by a vector of length 2 representing the real and imaginary parts of the dynamic uncertainty. This specific form allows one to deduce necessary and sufficient conditions for both stability and performance robustness. We have proposed a procedure that verifies the stabilization of all plants in $\mathcal{L}$ by a given controller. This procedure uses the general LFT framework of the loops made up of a plant in $\mathcal{L}$ and the controller C. We have also proposed a procedure that computes the worst case performance achieved by the controller C and all plants in $\mathcal{L}$. This computation boils down to an optimization problem involving LMI constraints.

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