A Measure of Robust Stability for an Identified Set of Parametrized Transfer Functions

X. Bombois, M. Gevers, and G. Scorletti

***Abstract*—In this paper, we define a measure of robustness for a set of parameterized transfer functions as delivered by classical prediction error identification and that contains the true system at a prescribed probability level. This measure of robustness is the worst case Vinnicombe distance between the model and the plants in the uncertainty region. We show how it can be computed exactly using LMI-based optimization. In addition, we show that this measure is directly connected to the size of the set of controllers that are guaranteed to stabilize all plants in the uncertainty region, i.e., the smaller the worst case Vinnicombe distance for an uncertainty region, the larger the set of model-based controllers that are guaranteed to stabilize all systems in this uncertainty region.**

***Index Terms*—Control systems, identification, robustness, stability, uncertainty.**

I. Introduction

This paper is part of our continuing investigation in order to reconcile time-domain prediction error identification and robustness theory [1]–[3]. Our approach is based on the ellipsoidal uncertainty region $\mathcal{D}$ delivered by prediction error identification [9], [1], [3] and that contains the so-called true system at a certain probability level. In [3], we have shown how to validate a controller with respect to all plants contained in such uncertainty set $\mathcal{D}$. The controller validation procedure developed in [3] consists of a necessary and sufficient condition that guarantees the stabilization of all plants in $\mathcal{D}$ by the "to-be-validated" controller and in the exact computation of the worst case performance achieved by this controller over all systems in $\mathcal{D}$.

The results of [3] pertain to the validation of a specific controller with respect to a specific uncertainty region. In this paper, we address the question of selecting an uncertainty region among a set of possible regions. Indeed, different identification experiments lead to different uncertainty regions $\mathcal{D}^{(i)}$. Some of those may be better tuned toward robust controller design than others. Roughly speaking we shall say that an uncertainty region $\mathcal{D}^{(1)}$ is better tuned for robust control design than $\mathcal{D}^{(2)}$ if the controller set $\mathcal{C}^{(2)}$ that is guaranteed to robustly stabilize $\mathcal{D}^{(2)}$ is a subset of the controller set $\mathcal{C}^{(1)}$ that is guaranteed to robustly stabilize $\mathcal{D}^{(1)}$.[1] This paper therefore treats the problem of choosing one particular uncertainty region $\mathcal{D}^*$ among all the available ones and this according to robust stability criteria.

Uncertainty Region: Prediction error identification delivers an estimated model $\hat{G}$ for the true plant G_0 and provides us with tools for the estimation of an uncertainty region (see, e.g., [9]). If the parametric structure is sufficiently complex to represent the true system, then $\hat{G}$ is asymptotically unbiased, and the covariance matrix of the parameter estimates allows one to construct a parametric uncertainty region U containing the parameters of the true system G_0 at a certain probability level that we can fix at, say, 95%. The uncertainty region U in the parameter space defines an equivalent uncertainty region $\mathcal{D}$ in the space of transfer functions with $\hat{G}$ as its center. This uncertainty region $\mathcal{D}$ can be obtained for both open-loop identification and indirect closed-loop identification. It is clear that different identification experiments lead to different uncertainty regions $\mathcal{D}^{(i)}$.

In our approach, we directly deduce from the measured data a set of *parameterized* transfer functions. Our approach therefore differs significantly from other methods presented in the literature that consist of validating [6], [4], [11], [8] or designing [10], [7] uncertainty regions containing the true system. In [6], [4], [11], and [8] and references therein, a method is proposed to decide whether a postulated region with bounded uncertainties is consistent with measured input–output data (the so-called model invalidation concept): [6] and [4] deal with frequency domain data while [11] and [8] tackle the same problem with time-domain data. This model invalidation concept has been extended to the concept of controller invalidation in [12]. Our approach here is different in that we do not have to choose an *a priori* structure for the uncertainty region; rather our uncertainty region $\mathcal{D}$ is derived from the data collected on the true system and is consistent with these data if the model structure is unbiased and if the residuals between the predicted and actual output are a white noise process (see [9] for more details).

Our approach also differs significantly from the approach used in traditional *set membership identification* ([10] and references therein), where a hard bound assumption is made on the noise and a known upper bound is required on the impulse response of the true system, leading to the identification of an uncertainty set around a nominal model. In [7], a method to identify an additive uncertainty region with a stochastic noise assumption is presented, but a known prior bound on the true system impulse response is again required. Furthermore, the approach presented in [7] is restricted to linearly parameterized models, such as FIR models. In our approach, rational transfer functions with denominator uncertainty can be used. In addition, no prior assumptions are required on the magnitude of the noise and of the impulse response. The only important restriction in this paper is that we assume that the system is in the model set and that the uncertainty sets are therefore entirely defined by covariance errors on the parameters.

Choice of a Particular Uncertainty Region $\mathcal{D}^*$: The choice of one particular uncertainty region $\mathcal{D}^*$ among several possible ones is based on the computation, for each $\mathcal{D}^{(i)}$, of the worst case (i.e., the largest) Vinnicombe distance [13], [14] between a model $G_{\rm mod}$ and the plants in $\mathcal{D}^{(i)}$. Here $G_{\rm mod}$ is the model that will be used for control design. It need not be any of the full-order identified models $\hat{G}^{(i)}$ that lie at the center of the uncertainty regions $\mathcal{D}^{(i)}$. It is typically a low-order model that lies within all uncertainty regions. Our first contribution is to show that this worst case Vinnicombe distance can be exactly computed using an optimization problem involving linear matrix inequality (LMI) constraints [5]. Our second contribution is to show that the smaller the worst case Vinnicombe distance between the model $G_{\rm mod}$ and all plants in some $\mathcal{D}^{(i)}$, the larger is the controller set that is guaranteed to robustly stabilize $G_{\rm mod}$ and $\mathcal{D}^{(i)}$.

The choice of the Vinnicombe metric to characterize the amount of uncertainty (i.e., the distance) between the model and the plants in $\mathcal{D}^{(i)}$ is motivated by the fact that this metric generally leads to the least conservative robust stability results.

Paper Outline: In Section II, the general expression of the parametric uncertainty regions delivered by classical prediction error identification is presented. In Section III, we define the worst case Vinni-

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[1] We shall say that a controller set is guaranteed to robustly stabilize an uncertainty region $\mathcal{D}$ if all controllers in that set are guaranteed to stabilize all plants in $\mathcal{D}$.

combe distance between a given model and all systems in an uncertainty region $\mathcal{D}$. In the same section, we explain how we can compute this distance using a convex optimization problem involving LMI constraints. In Section IV, we present a procedure, based on the worst case Vinnicombe distance, that allows one to choose the uncertainty region that is best tuned for robust control design with respect to the model. This procedure is illustrated by an example in Section V. Finally, some conclusions are given in Section VI.

II. Uncertainty Region Delivered by Prediction Error Identification

In this section, we give the general expression of the uncertainty regions delivered by classical prediction error identification, assuming that unbiased model structures are used [9]. This general expression, valid for both open-loop and indirect closed-loop identification, is summarized in the following proposition, where we assume that the true *open-loop* system is linear and time-invariant, with a rational input-output transfer function G_0 such that $y = G_0 u + v$, where v is additive noise; see [1] and [3] for more details.

Proposition 1: Consider $\delta \in \mathbf{R}^{k\times 1}$, the real parameter vector of the parameterized transfer function set, $G_0 = G(\delta_0)$, the true open-loop system, and $G(\hat{\delta})$, the full order identified model obtained either "directly" by open-loop identification or "indirectly" by indirect closed-loop identification. The uncertainty region $\mathcal{D}$ containing $G(\delta_0)$ at a certain probability level has the following general form:

$$\mathcal{D} = \left\{ G(\delta) \,|\, G(\delta) = \frac{Z_N \delta}{1 + Z_D \delta} \text{ and } \delta \in U = \left\{ \delta | (\delta - \hat{\delta})^T R (\delta - \hat{\delta}) < \chi^2 \right\} \right\} \tag{1}$$

where $R \in \mathbf{R}^{k\times k}$ is the inverse of the covariance matrix of $\hat{\delta}$, χ^2 is determined by the desired probability level, and $Z_N(z)$ and $Z_D(z)$ are known row vectors of size k.

In the sequel, when we say that the true system lies in the uncertainty region $\mathcal{D}$, it means that the true system lies in the uncertainty region $\mathcal{D}$ at a certain probability level, which is a function of χ^2.

III. A Robust Stability Measure for $\mathcal{D}$

In the previous section, we have given the general expression of the uncertainty regions $\mathcal{D}$ obtained with open-loop and closed-loop identification. It is clear that different identification experiments (i.e., open-loop or closed-loop identification, different measured data, ...) lead to different identified parameter vectors, different covariance matrices, and therefore also different sets of systems $\mathcal{D}^{(i)}$. Let us thus consider that t experiments have delivered t different uncertainty sets $\mathcal{D}^{(i)}, i = 1 \ldots t$, which all have the general structure given in (1). Let us also consider that we have a (possibly low-order) model $G_{\rm mod}$ from which we design a control law for the true system G_0. The problem we tackle in this paper is to find the uncertainty set $\mathcal{D}^*$, among the t possible ones, that is best tuned for *robust control design based on* $G_{\rm mod}$, i.e., the uncertainty set for which the $G_{\rm mod}$-based set of robustly stabilizing controllers is largest. Once this "best" uncertainty set $\mathcal{D}^*$ is chosen, a controller can be designed from $G_{\rm mod}$, and we can then use the results of [3] to test if the designed controller does indeed stabilize all plants in the chosen set $\mathcal{D}^*$. The robust stability measure for $\mathcal{D}$ is the worst case Vinnicombe distance. It is based on the Vinnicombe metric (or ν-gap metric) that defines a distance between two transfer functions. This metric will therefore be defined and its nice properties with respect to robust stability are first presented.

A. The Vinnicombe Metric

The distance $\delta_\nu(G_1, G_2)$ between two systems G_1 and G_2 has been defined by Vinnicombe [13] who calls it the ν-gap between G_1 and G_2, as shown in (2) at the bottom of the page, where $W(G_1, G_2) = wno(1 + G_1^* G_2) + \eta(G_2) - \tilde{\eta}(G_1)$. Here $\eta(G)$ [respectively, $\tilde{\eta}(G)$] denotes the number of poles of G outside (respectively, outside and on) the unit circle, and $wno(G)$ denotes the winding number about the origin of $G(z)$ as z follows the standard Nyquist D-contour. If $W(G_1, G_2) = 0$, $\delta_\nu(G_1, G_2)$ is, according to (2), the supremum of $\kappa(G_1(e^{j\Omega}), G_2(e^{j\Omega}))$ over all frequencies. This quantity $\kappa(G_1(e^{j\Omega}), G_2(e^{j\Omega}))$ is the chordal distance between the projections of $G_1(e^{j\Omega})$ and $G_2(e^{j\Omega})$ onto the Riemann sphere of unit diameter [13].

As said in Section I, this ν-gap has nice properties with respect to robustness analysis. Indeed, the size of the set of controllers that are guaranteed to stabilize both G_1 and G_2 is related to $\delta_\nu(G_1, G_2)$ [13] as shown in the following proposition.

Proposition 2 [13]: Let us consider a nominal plant G_1 and a perturbed plant G_2 and denote $\delta_\nu(G_1, G_2)$ the ν-gap between these two plants. Then, a controller C stabilizing G_1 also stabilizes G_2 if this controller lies in the controller set $\{C | b_{G_1,C} > \delta_\nu(G_1, G_2)\}$ where $b_{G_1,C} = \min_\Omega \kappa(G_1(e^{j\Omega}), -1/C(e^{j\Omega}))$ is the generalized stability margin of the stable loop $[G_1 C]$.

This proposition shows that the smaller the ν-gap between the nominal plant G_1 and the perturbed plant G_2, the larger is the set of controllers stabilizing G_1 that also stabilize G_2.

B. The Worst Case Vinnicombe Distance

The nice stability property presented in the previous section shows that the $G_{\rm mod}$-based controller set that is guaranteed to robustly stabilize $\mathcal{D}$ will be large, if the largest Vinnicombe distance between $G_{\rm mod}$ and any plant in $\mathcal{D}$ remains small. We call this "largest Vinnicombe distance" the worst case Vinnicombe distance $\delta_{WC}(G_{\rm mod}, \mathcal{D})$ between $G_{\rm mod}$ and the set $\mathcal{D}$.

Definition 1: Consider an uncertainty region $\mathcal{D}$ having the structure given in (1) and a model $G_{\rm mod}$. The worst case Vinnicombe distance $\delta_{WC}(G_{\rm mod}, \mathcal{D})$ is given by

$$\delta_{WC}(G_{\rm mod}, \mathcal{D}) = \max_{G_{\rm in} \in \mathcal{D}} \delta_\nu(G_{\rm mod}, G_{\rm in}). \tag{3}$$

Another important quantity is now defined: the worst case chordal distance. This quantity, whose computation is the result of a convex optimization problem involving LMI constraints as will be shown in Section III-C, will allow us to give an alternative expression for $\delta_{WC}(G_{\rm mod}, \mathcal{D})$.

Definition 2: At a particular frequency Ω, we define $\kappa_{WC}(G_{\rm mod}(e^{j\Omega}), \mathcal{D})$ as the maximum chordal distance between

$$\delta_\nu(G_1, G_2) = \begin{cases} \max_\Omega \kappa\left(G_1\left(e^{j\Omega}\right), G_2\left(e^{j\Omega}\right)\right) = \max_\Omega \dfrac{|G_1 - G_2|}{\sqrt{1 + |G_1|^2}\sqrt{1 + |G_2|^2}}, & \text{if } W(G_1, G_2) = 0 \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

$G_{\text{mod}}(e^{j\Omega})$ and the frequency responses of all plants in $\mathcal{D}$ at this frequency

$$\kappa_{WC}\Big(G_{\text{mod}}\Big(e^{j\Omega}\Big),\mathcal{D}\Big)=\max_{G_{\text{in}}\in\mathcal{D}}\kappa\Big(G_{\text{mod}}\Big(e^{j\Omega}\Big),G_{\text{in}}\Big(e^{j\Omega}\Big)\Big) \tag{4}$$

This last quantity can now be used to give an alternative expression of the worst case Vinnicombe distance. This is done in the following lemma, which is an extension of a property presented in [14, p. 66].

Lemma 1: If $W(G_{\text{mod}}, G_{\text{in}}) = 0$ for one plant $G_{\text{in}} \in \mathcal{D}$, then the worst case Vinnicombe distance $\delta_{WC}(G_{\text{mod}}, \mathcal{D})$ defined in (3) can also be expressed in the following way using the worst case chordal distance:

$$\delta_{WC}(G_{\text{mod}},\mathcal{D})=\max_{\Omega}\kappa_{WC}\Big(G_{\text{mod}}\Big(e^{j\Omega}\Big),\mathcal{D}\Big) \tag{5}$$

where $\kappa_{WC}(G_{\text{mod}}(e^{j\Omega}), \mathcal{D})$ is defined in (4).

Proof: The winding number condition may be omitted in (5). Indeed, assume there exists one $G_1 \in \mathcal{D}$ for which $W(G_{\text{mod}}, G_1) \neq 0$, i.e., $\delta_\nu(G_{\text{mod}}, G_1) = 1$. Since $\mathcal{D}$ is a connected set, then there always exists a piecewize continuous application ϕ of [0 1] to plants in $\mathcal{D}$ such that $\phi(0) = G_{\text{in}}$ and $\phi(1) = G_1$. As $W(G_{\text{mod}}, G_{\text{in}}) = 0$ and $W(G_{\text{mod}}, G_1) \neq 0$, there exists a $G_2 = \phi(\lambda) \in \mathcal{D}$ such that $(1 + G^*_{\text{mod}}(e^{j\Omega_0})G_2(e^{j\Omega_0}) = 0)$ and therefore such that $\kappa(G_{\text{mod}}(e^{j\Omega_0}), G_2(e^{j\Omega_0})) = 1$ for some frequency Ω_0. So, $\delta_{WC}(G_{\text{mod}}, \mathcal{D}) = 1$ □

Remark: If $G_{\text{mod}} \in \mathcal{D}$, we always have $W(G_{\text{mod}}, G_{\text{mod}}) = 0$ and therefore (5) is always valid.

C. Computation of the Worst Case Chordal Distance

In the previous subsection, we have defined the worst case Vinnicombe distance between the model G_{mod} and all plants in an uncertainty region $\mathcal{D}$ having the general structure (1). Now we give a procedure to compute this worst case Vinnicombe distance $\delta_{WC}(G_{\text{mod}}, \mathcal{D})$. According to Lemma 1, this is equivalent to finding a procedure to compute the worst case chordal distance $\kappa_{WC}(G_{\text{mod}}(e^{j\Omega}), \mathcal{D})$ defined in (4), since $\delta_{WC}(G_{\text{mod}}, \mathcal{D})$ is the maximum over all frequencies of the worst case chordal distance. In the following theorem, we show that the computation of the worst case chordal distance $\kappa_{WC}(G_{\text{mod}}(e^{j\Omega}), \mathcal{D})$ at a particular frequency Ω can be formulated as a convex optimization problem involving LMI constraints [5].

Theorem 1: Consider the model G_{mod} and an uncertainty region $\mathcal{D}$ having the general structure given in (1). The worst case chordal distance $\kappa_{WC}(G_{\text{mod}}(e^{j\Omega}), \mathcal{D})$ at frequency Ω is equal to $\sqrt{\gamma_{\text{opt}}}$ where γ_{opt} is the optimal value of γ in the following standard convex optimization problem involving LMI constraints:

$$\begin{aligned}
&\text{minimize} && \gamma\\
&\text{over} && \gamma,\tau\\
&\text{subject to} && \tau\geq 0\\
&\text{and} && \begin{pmatrix}\text{Re}(a_{11}) & \text{Re}(a_{12})\\ \text{Re}(a_{12}^*) & \text{Re}(a_{22})\end{pmatrix}\\
& && -\tau\begin{pmatrix}R & -R\hat{\delta}\\ (-R\hat{\delta})^T & \hat{\delta}^T R\hat{\delta}-\chi^2\end{pmatrix}\leq 0
\end{aligned} \tag{6}$$

with

$$\begin{aligned}
a_{11} &= (Z_N^*Z_N - Z_N^*xZ_D - Z_D^*x^*Z_N + Z_D^*x^*xZ_D)\\
&\quad -\gamma(Z_N^*QZ_N + Z_D^*QZ_D)\\
a_{12} &= -Z_N^*x + Z_D^*x^*x - \gamma(Z_D^*Q)\\
a_{22} &= x^*x-\gamma Q, \qquad Q = 1 + x^*x \text{ and } x = G_{\text{mod}}(e^{j\Omega}).
\end{aligned}$$

Proof: If we denote the frequency response at Ω of any plant in $\mathcal{D}$ by $G(\delta, \Omega)$, then a convenient way to state the problem of computing the worst case chordal distance at the frequency Ω is as follows [see (4)]:

minimize γ such that

$$\kappa\Big(G_{\text{mod}}\Big(e^{j\Omega}\Big),G(\delta,\Omega)\Big)^2\leq\gamma \text{ for all } G(\delta,\Omega)\in\mathcal{D}.$$

Using the $\mathcal{S}$ procedure [5], this is equivalent with (6). See full version of [1] for details. □

IV. Robustness-Oriented Choice of $\mathcal{D}^*$ Among a Set of $\mathcal{D}^{(i)}$

In the previous section, the notion of worst case Vinnicombe distance between a model G_{mod} and an uncertainty region $\mathcal{D}^{(i)}$ has been introduced and a procedure has been given to compute this distance. This worst case Vinnicombe distance can be considered as a robustness measure of $\mathcal{D}^{(i)}$ with respect to robustly stable controller design based on the model G_{mod}. Indeed, we show now that the smaller $\delta_{WC}(G_{\text{mod}}, \mathcal{D}^{(i)})$, the larger is the G_{mod}-based controller set that is guaranteed to robustly stabilize $\mathcal{D}^{(i)}$. We first establish a link between $\delta_{WC}(G_{\text{mod}}, \mathcal{D}^{(i)})$ and the size of the "G_{mod}-based controller set that is guaranteed to robustly stabilize $\mathcal{D}^{(i)}$" and then we use this property to compare two different uncertainty regions.

Proposition 3: Consider an uncertainty region $\mathcal{D}^{(i)}$ having the structure given by (1) and a model G_{mod}. All controllers C that stabilize G_{mod} and that lie in the set

$$\mathcal{C}\Big(G_{\text{mod}},\mathcal{D}^{(i)}\Big)=\Big\{C\,|\,b_{G_{\text{mod}},C}>\delta_{WC}\Big(G_{\text{mod}},\mathcal{D}^{(i)}\Big)\Big\}$$

are guaranteed to stabilize all plants in the uncertainty region $\mathcal{D}^{(i)}$.

Proof: This proposition is a direct consequence of Proposition 2 and of Definition 1. □

Theorem 2: Consider two uncertainty regions $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$. If $\delta_{WC}(G_{\text{mod}}, \mathcal{D}^{(1)}) < \delta_{WC}(G_{\text{mod}}, \mathcal{D}^{(2)})$, then $\mathcal{C}(G_{\text{mod}}, \mathcal{D}^{(2)}) \subset \mathcal{C}(G_{\text{mod}}, \mathcal{D}^{(1)})$.

Theorem 2, which directly results from Proposition 3, gives us guidelines to choose the uncertainty region that is best tuned to robust controller design. These guidelines are summarized in the following proposition.

Proposition 4: Consider t uncertainty regions $\mathcal{D}^{(i)}$ obtained from t different identification experiments and a model G_{mod}. Then the uncertainty region $\mathcal{D}^*$ that generates the largest set $\mathcal{C}(G_{\text{mod}}, \mathcal{D}^{(i)})$ $(i = 1\ldots t)$ of robustly stabilizing controllers is the uncertainty region

$$\mathcal{D}^*=\arg\min_{\mathcal{D}^{(i)}}\delta_{WC}\Big(G_{\text{mod}},\mathcal{D}^{(i)}\Big). \tag{7}$$

Remarks:

1) The choice of G_{mod} for the control design is an important feature. Indeed, we analyze the robustness properties of the uncertainty regions $\mathcal{D}^{(i)}$ with respect to controllers designed from G_{mod} (and stabilizing it). If the smallest worst case Vinnicombe distance between G_{mod} and the different $\mathcal{D}^{(i)}$ remains "large," then the chosen model G_{mod} is not appropriate for a control design procedure for G_0 because the actual $\delta_\nu(G_{\text{mod}}, G_0)$ may be too large. A better model G_{mod} must then be chosen: for example, the center of one of the uncertainty regions $\mathcal{D}^{(i)}$.
2) The set $\mathcal{C}(G_{\text{mod}}, \mathcal{D}^{(i)})$ contains all controllers that stabilize the uncertainty set $\{G|\delta_\nu(G_{\text{mod}}, G) < \delta_{WC}(G_{\text{mod}}, \mathcal{D}^{(i)})\}$ that embeds $\mathcal{D}^{(i)}$. Thus, there may be additional controllers outside the set $\mathcal{C}(G_{\text{mod}}, \mathcal{D}^{(i)})$ that stabilize all models in $\mathcal{D}^{(i)}$; in that sense, our analysis is conservative. However, since we typically choose G_{mod} within all $\mathcal{D}^{(i)}$, we essentially introduce the same conservatism for each $\mathcal{D}^{(i)}$ and therefore our procedure remains valid for the selection of the best $\mathcal{D}^{(i)}$.

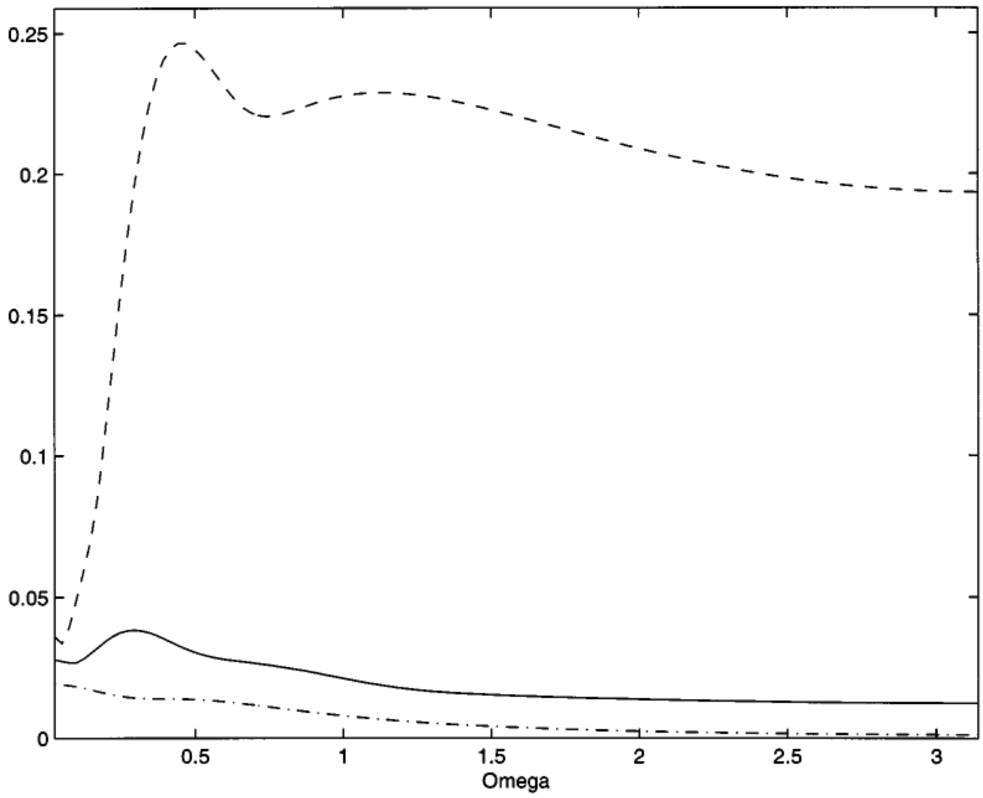

Fig. 1. $\kappa_{WC}(G_{\rm mod}(e^{j\Omega}), \mathcal{D}^{OL})$ (dashed), $\kappa_{WC}(G_{\rm mod}(e^{j\Omega}), \mathcal{D}^{CL})$ (solid) and $\kappa(G_{\rm mod}(e^{j\Omega}), G_0(e^{j\Omega}))$ (dashdot) at each frequency.

V. A Simulation Example

Let us consider the following true system G_0 and the following model $G_{\rm mod}$ of this true system

$$y = G_0 u + e = \frac{0.1047z^{-1} + 0.0872z^{-2}}{1 - 1.5578z^{-1} + 0.5769z^{-2}}\, u + e$$

and

$$G_{\rm mod} = \frac{0.1060z^{-1} + 0.0928z^{-2}}{1 - 1.5308z^{-1} + 0.5467z^{-2}}$$

where e is a white noise of variance 0.1. The actual ν-gap between G_0 and $G_{\rm mod}$ is $\delta_\nu(G_0, G_{\rm mod}) = 0.0193$. We perform one identification of G_0 in open loop and one identification in closed loop (with the controller $K = (1.27 - 1.04z^{-1})/(1 - 0.6z^{-1})$ in the loop) leading to two different uncertainty regions, each of which contains G_0 with probability 0.95. We call these two uncertainty regions $\mathcal{D}^{OL}$ and $\mathcal{D}^{CL}$, respectively. In order to decide which of these uncertainty regions is best tuned for control design with respect to the model $G_{\rm mod}$, we compute the measure of robustness of these two uncertainty regions with respect to $G_{\rm mod}$, i.e., $\delta_{WC}(G_{\rm mod}, \mathcal{D}^{OL})$ and $\delta_{WC}(G_{\rm mod}, \mathcal{D}^{CL})$. For this purpose, we first compute the worst case chordal distances at each frequency for $\mathcal{D}^{OL}$ and $\mathcal{D}^{CL}$ using the LMI tools developed in Section III-C. The worst case chordal distances at each frequency $\kappa_{WC}(G_{\rm mod}(e^{j\Omega}), \mathcal{D}^{OL})$ and $\kappa_{WC}(G_{\rm mod}(e^{j\Omega}), \mathcal{D}^{CL})$ are represented in Fig. 1 where they are compared with the actual chordal distance $\kappa(G_{\rm mod}(e^{j\Omega}), G_0(e^{j\Omega}))$ between $G_{\rm mod}$ and G_0. According to Lemma 1 and since $W(G_{\rm mod}, \hat{G}^{OL}) = W(G_{\rm mod}, \hat{G}^{CL}) = 0$ ($\hat{G}^{OL}$ and $\hat{G}^{CL}$ are the centers of $\mathcal{D}^{OL}$ and $\mathcal{D}^{CL}$, respectively), we can derive the worst case Vinnicombe distances from the worst chordal distances as follows:

$$\delta_{WC}\left(G_{\rm mod}, \mathcal{D}^{OL}\right) = \max_{\Omega} \kappa_{WC}\left(G_{\rm mod}\left(e^{j\Omega}\right), \mathcal{D}^{OL}\right) = 0.2464$$
$$\delta_{WC}\left(G_{\rm mod}, \mathcal{D}^{CL}\right) = \max_{\Omega} \kappa_{WC}\left(G_{\rm mod}\left(e^{j\Omega}\right), \mathcal{D}^{CL}\right) = 0.0384.$$

Therefore, by Proposition 4, the set $\mathcal{C}(G_{\rm mod}, \mathcal{D}^{CL})$ of controllers stabilizing $G_{\rm mod}$ that robustly stabilizes $\mathcal{D}^{CL}$ is much larger than the set $\mathcal{C}(G_{\rm mod}, \mathcal{D}^{OL})$ that robustly stabilizes $\mathcal{D}^{OL}$. To illustrate this statement, let us design two controllers from the model $G_{\rm mod}$. These two controllers are given as follows with the achieved generalized stability margins:

$$C_1 = \frac{1.8464 - 1.3647z^{-1}}{1 - 0.4545z^{-1}} \qquad b_{G_{\rm mod}, C_1} = 0.2861$$
$$C_2 = 3 \quad b_{G_{\rm mod}, C_2} = 0.0653.$$

We directly see that the controller C_1 is guaranteed to stabilize the plants in the two uncertainty regions since it belongs to both guaranteed sets of stabilizing controllers $\mathcal{C}(G_{\rm mod}, \mathcal{D}^{OL})$ and $\mathcal{C}(G_{\rm mod}, \mathcal{D}^{CL})$ defined in Proposition 3. Indeed, $b_{G_{\rm mod}, C_1} > \delta_{WC}(G_{\rm mod}, \mathcal{D}^{OL}) > \delta_{WC}(G_{\rm mod}, \mathcal{D}^{CL})$. However, the controller C_2 belongs to $\mathcal{C}(G_{\rm mod}, \mathcal{D}^{CL})$ only: C_2 therefore stabilizes all the plants in $\mathcal{D}^{CL}$. As $C_2 \notin \mathcal{C}(G_{\rm mod}, \mathcal{D}^{OL})$, it is not guaranteed, by Proposition 3, to stabilize all plants in $\mathcal{D}^{OL}$. Proposition 3 only gives a sufficient condition. To check whether C_2 actually stabilizes all plants in $\mathcal{D}^{OL}$, we use the "necessary and sufficient" test developed in [3]. This test fails, and therefore C_2 does not stabilize all plants in $\mathcal{D}^{OL}$ whereas it does stabilize all plants in $\mathcal{D}^{CL}$ by Proposition 3.

VI. Conclusions

We have proposed a measure of robust stability for a set of parameterized transfer functions as delivered by prediction error identification. This measure is the largest Vinnicombe distance between the nominal model and all plants in the uncertainty region. We have shown that this measure is exactly computable using LMI-based optimization. We have also shown that the smaller the worst case Vinnicombe distance between the model and an uncertainty region, the larger is the set of model-based controllers that robustly stabilize all plants in the uncertainty region. This measure therefore gives us guidelines to select the uncertainty region that is best tuned for robust stability analysis among all available ones. To illustrate the impact of our results in terms of the connection between identification and robust control, we return to the example above. With our robust stability measure for uncertainty sets, we were able to conclude that the $G_{\rm mod}$-based controller set that is guaranteed to robustly stabilize $\mathcal{D}^{CL}$ is much larger than the set that is guaranteed to robustly stabilize $\mathcal{D}^{OL}$. Hence, in terms of identification for control, the closed-loop identification design that led to the uncertainty set $\mathcal{D}^{CL}$ is a much better experiment design than the open-loop design that led to $\mathcal{D}^{OL}$. The results of this paper have thus allowed us to establish a connection between identification design and stability robustness of the controllers resulting from such design.

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Random Spherical Uncertainty in Estimation and Robustness

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***Abstract*—A theorem is formulated that gives an exact probability distribution for a linear function of a random vector uniformly distributed over a ball in n-dimensional space. This mathematical result is illustrated via applications to a number of important problems of estimation and robustness under spherical uncertainty. These include parameter estimation, characterization of attainability sets of dynamical systems, and robust stability of affine polynomial families.**

***Index Terms*—Estimation, random noise, robust stability, uncertain systems.**

I. INTRODUCTION

Traditionally, different fields of control theory exploit various models for the uncertainty. For instance, in parameter and state estimation, the standard approach deals with random (specifically, Gaussian) perturbations, and least squares and Kalman filtering are the most popular tools for estimation under such assumptions. Later, the model of unknown-but-bounded perturbations was developed, which led to ellipsoidal techniques for estimation [14], [8], [10].

On the other hand, the models of parametric uncertainty in control theory are basically deterministic, e.g., see [1], [2], and [5] devoted to robust stability and performance of uncertain linear systems. One of the drawbacks of such models is that the admissible ranges for the uncertainty that satisfy performance specifications are calculated against the worst case uncertainty, which may happen very rarely in practice. Also, the computational complexity of the methods often grows exponentially in the dimension of the uncertainty vector.

In practical applications, it is quite often the case that hard bounds on the uncertainty are not known. Instead, certain probabilistic characteristics for the uncertain parameters are available, the conclusions are obtained in the form of confidence estimates, and the solution often involves Monte Carlo simulations; see [13], [9], [4], and [7]. Along with low computational complexity, the main benefit is a considerable enhancement of admissible uncertainty domains in exchange of a small probability risk that the deterministic specifications are violated. The results obtained so far relate to independent random variables.

Following the probabilistic approach, in this paper we work with an important class of dependent random parametric uncertainty, namely, with the uniform distribution on a ball in l_2-norm. There are several reasons for such an uncertainty model. First, if the uncertainty is supposed to be of a stochastic nature, the l_2-constraint is associated with a bound on the total energy of random noise; in that case, the random l_2 uncertainty can be thought of as a bridge between probabilistic models and unknown-but-bounded models (with ellipsoidal models of uncertainty as conventional tools), e.g., see Section IV, where the deterministic result is enhanced via its probabilistic counterpart. On the other hand, for the parametric uncertainty, the l_2 model is quite natural, since often, the information about the uncertain parameters is derived from

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