Simultaneous stabilization of three or more plants: conditions on the positive real axis do not suffice

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Abstract

The problem of the simultaneous stabilizability of a finite family of single input single output time invariant systems by a time invariant controller is studied. The link between stabilization and avoidance is shown and is used to derive necessary conditions for the simultaneous stabilization of k plants. These necessary conditions are proved to be, in general, not sufficient. This result also disproves a long-standing conjecture on the stabilizability condition of a single plant with a stable minimum phase controller. The main result is to show that, unlike the case of two plants, the existence of a simultaneous stabilizing controller for more than two plants is not guaranteed by the existence of a controller such that the closed loops have no *real* unstable poles.

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1 Introduction

Do you believe that simple questions always have simple answers? If you do, you may consider with interest the following problem: let $p_1(s) = 0, p_2(s) = \frac{2}{17}\frac{s-1}{s+1}$ and $p_3(s) = \frac{(s-1)^2}{(9s-8)(s+1)}$ be three continuous time rational transfer functions. Is it possible to find a single rational controller $c(s)$ that simultaneously stabilizes $p_i(s)$ $i = 1, 2, 3$ (i.e. such that the closed loop transfer functions $p_i(s)c(s)(1 + p_i(s)c(s))^{-1}$ have no poles in the complex right half plane for $i = 1, 2, 3$)? The question may not look too hard: it merely asks whether or not three plants are *simultaneously stabilizable* by a single controller. At present nobody is capable of answering such a question and this paper is devoted to it.

Let us first state the problem clearly. We restrict our attention to single input single output systems that are described by linear, time invariant, rational but not necessarily proper transfer functions. Each one of these systems is thus represented by an arbitrary real rational function $p_i(s) \in \mathbb{R}(s)$ $i = 1, ..., k$. To control our systems we allow ourselves to use a dynamic but time invariant, rational and not necessarily proper controller $c(s) \in \mathbb{R}(s)$. Finally, our goal is to achieve continuous time closed loop internal stability with the controller. That is, we want that, with the chosen controller $c(s)$, the four transfer functions $p_i c(1 + p_i c)^{-1}, p_i(1 + p_i c)^{-1}, c(1 + p_i c)^{-1}$ and $(1 + p_i c)^{-1}$ have no poles in the extended right half plane. Our question is now: under what conditions on the $p_i(s)$, $i = 1, ..., k$ is it possible to find such a simultaneous stabilizing controller? This problem has been formulated for some years now (see e.g. [33]) and, despite many efforts, it has remained unsolved for $k \geq 3$. It is nowadays commonly referred to as the *simultaneous stabilization problem* and is recognized as one of the hard open problem in linear system theory.

Although this paper does not solve the simultaneous stabilization problem, we provide some fresh angle of attack by introducing the concept of avoidance, we produce a range of new necessary and sufficient conditions and, most importantly, we prove a negative result by showing that, unlike the case $k = 2$, the simultaneous stabilizability question of more than two plants cannot be answered by just checking whether a controller exist such that the closed loop transfer functions have no *real* unstable poles.

We would like to draw the reader's attention to the crucial point that we allow ourselves the use only of a *time invariant* controller. It is not always possible to simultaneously stabilize two or more plants with such a controller. To overcome this limitation, alternative strategies have recently been developed with time varying controllers and we refer the reader to the existing literature for more details on this subject (see for example [24] and references therein). This paper will deal only with the time invariant case.

Historically, the first line of attack on the simultaneous stabilization question was given through the solution of a seemingly unrelated question: 'when is it possible to stabilize a single plant with a *stable* controller?'. This question, known as the *strong stabilization problem*, was fully solved by Youla et al. [39] in a now classical paper. A plant is stabilizable by a stable controller if and only if it has an even number of real unstable poles between each pair of real unstable zeros. Such plants are said to have the *parity interlacing property*. A most remarkable feature of this condition is that it involves only the *real* unstable poles and zeros of the plant.

The link between strong stabilization and simultaneous stabilization of two plants was discovered, and used, by Saeks and Murray [30]. Roughly speaking, two plants p_1 and p_2 are simultaneously stabilizable if and only if the plant $p_1 - p_2$ is strongly stabilizable. Since a tractable condition for strong stabilization is known, this solves the problem of simultaneous stabilization of two plants. This result was further extended to a multi-input multi-output setting by Vidyasagar and Viswanadham [33] where it was shown that from k plants p_i, $i = 1, ..., k$ it is possible to construct $k-1$ plants p_i', $i = 1, ..., k-1$ in such a way that the plants p_i are simultaneously stabilizable if and only if the p_i' are simultaneously stabilizable by a *stable* controller. This equivalence, while theoretically interesting, does not provide a computable test for the simultaneous stabilization of three or more plants since we have no criteria to decide if two or more plants are simultaneously stabilizable by a stable controller.

After these results were obtained, the main contributions to simultaneous stabilization this last decade have been in the form of necessary or sufficient conditions for simultaneous stabilization (but never necessary *and* sufficient conditions). *At present no tractable necessary and sufficient conditions exist for simultaneous stabilization except for the case of two plants.*

The simultaneous stabilization of two plants is equivalent, as stated above, to the stabilization of a single plant by a stable controller. This idea can be extended to the simultaneous stabilization of three plants. Modulo an avoidance condition, the simultaneous stabilization of three plants is equivalent to the stabilization of a single plant by a stable controller whose inverse is also stable. Such a controller is called a *unit* controller. The problem of finding a condition under which a plant p can be stabilized by a unit controller can thus be seen as an intermediate step towards the solution of the simultaneous stabilization of three plants. It is easy to see that a necessary condition is that both p and p^{-1} must have the parity interlacing property, i.e. p must have an even number of real unstable poles between each pair of real unstable zeros and vice versa. Such plants are referred to as having the *even interlacing property*. It is shown in [36] that this even interlacing property condition is also sufficient to ensure that the plant p is stabilizable by a stable controller with no *real* unstable zeros. Note that such controller may have complex unstable zeros, so that the result of [36] does not prove that the even interlacing property of a plant p is sufficient for stabilization by a unit controller. This even interlacing property also ensures that there exists a unit controller such that the closed loop transfer function has no *real* unstable poles. In the same vein, [37] and [38] gives a condition on three plants p_1, p_2 and p_3 under which it is possible to find a single controller such that none of the closed loop transfer functions have *real* unstable poles.

In the first part of this paper, we shall pursue this line of thinking and we shall give a thorough study of the question: 'given k plants p_i, $i = 1, ..., k$, under what condition is it possible to find a single controller such that none of the closed loop transfer functions have *real* unstable poles?'. The motivations to develop such results are threefold. First, the conditions obtained are tractable, which is seldom the case for simultaneous stabilization questions. Second, such conditions remain necessary when the closed loop transfer functions are constrained not only to have no *real* unstable poles but no unstable poles at all. They are therefore necessary conditions for simultaneous stabilization. Third, it is known that these conditions are also sufficient for the strong stabilization of a single plant and for the simultaneous stabilization of two plants. A single plant is stabilizable by a stable controller if and only if there exists a stable controller such that the closed loop transfer function has no *real* unstable poles, and two plants p_1 and p_2 are simultaneously stabilizable if and only if there exists a single controller such that the closed loop transfer functions associated to p_1 and to p_2 have no real unstable poles. By analogy it was hoped (see, for example, the conclusion in [37]) that this property would extend to the simultaneous stabilization problem for three or more plants. As we shall see at the very end of this paper, this is unfortunately not the case.

The main contributions of this paper have been briefly described above. The layout is as follows. We introduce, in Section 3, the simultaneous stabilization problem as an avoidance problem in the complex plane. We shall show that k plants are simultaneously stabilizable if and only if there exists a controller that avoids, in a way that we shall define, the k plants in the complex right half plane. We shall see that this reinterpretation in terms of avoidance (i.e. non-intersection) of functions gives powerful new insights into stabilization and simultaneous stabilization problems. We shall use these insights in Section 4 where, after a quick review of some known results, we answer the question: 'given k plants p_i, $i = 1, ..., k$, under what condition is it possible to find a single controller such that the closed loop transfer functions associated to each plant have no *real* unstable poles'. Strikingly, we shall see that under a weak assumption this can be achieved if and only if for each *pair of plants* there exists such stabilizing controllers. The fulfillment of these conditions can be checked by using the parity interlacing property so that we have a tractable test to answer the above question. Finally, in Section 5 we present some negative results. We first show that the even interlacing property is not sufficient for stabilizability of a plant by a unit controller. It then follows that the condition given in [37] and presented in ourSsection 4 is not sufficient either for simultaneous stabilization of three plants.

2 Notations

$\mathbb{R}[s]$ is the set of real polynomials. $\mathbb{R}(s)$ is the set of real rational functions. $\mathbb{C}_\infty$ is the extended complex plane, $\mathbb{C} \cup \{\infty\}$, adequately topologized, and $\mathbb{R}_\infty$

is the extended real line, $\mathbb{R} \cup \{\infty\}$. D is the open unit disc $\{s \in \mathbb{C} :| s |< 1\}$. Ω is some chosen subset of $\mathbb{C}_\infty$. We shall assume throughout this paper that Ω is closed in the Riemann sphere topology, that it is symmetric with respect to the real axis and that it contains at least one value of the extended real line $\mathbb{R}_\infty$ but not the whole extended real line $\mathbb{R}_\infty$. Ω is to be thought of as the complement in $\mathbb{C}_\infty$ of a region of stability. Classical examples of regions Ω are the closed unit disc $\overline{D} = \{s \in \mathbb{C} :| s |\leq 1\}$ and the extended closed right half plane $\mathbb{C}_{+\infty} = \{s \in \mathbb{C} : Real(s) \geq 0\} \cup \{\infty\}$ which correspond, respectively, to the complement in $\mathbb{C}_\infty$ of the discrete and continuous time stability regions. We define $I = \overline{D} \cap \mathbb{R}_\infty = [-1, 1]$ and $\mathbb{R}_{+\infty} = \mathbb{C}_{+\infty} \cap \mathbb{R}_\infty = [0, \infty]$. The subsets $\overline{D}, \mathbb{C}_{+\infty}, I$ and $\mathbb{R}_{+\infty}$ all satisfy the assumptions on Ω. A real rational function $f(s) \in \mathbb{R}(s)$ is Ω-*stable* if it has no poles in Ω. $S(\Omega)$ is the set of all Ω-stable rational functions. We use $U(\Omega)$ to denote the set of functions in $S(\Omega)$ whose inverse are in $S(\Omega)$ and we call such rational functions Ω-*units*. Finally, to shorten the notations, we define $U = U(\mathbb{C}_{+\infty})$ and $S = S(\mathbb{C}_{+\infty})$.

3 Stabilization as avoidance

The equivalence between the solvability of the simultaneous stabilization problem of 2 plants and conditions of interpolation by real rational functions was pointed out by various authors (see [39], [16], [18], [11] and [21]). By a few algebraic manipulations it is possible to show that the problem of stabilizing two plants simultaneously is equivalent to one of finding a stable rational function having a stable inverse that interpolates a set of values at a set of points in the right half plane. This interpretation of the problem has the advantage of giving a geometrical insight to the problem. However, this equivalence does not carry over when the number of plants is greater than or equal to three. It is in no known way possible to formulate the simultaneous stabilization question of three or more plants in terms of an interpolation problem. In this section we develop a different view of the problem which we call an 'avoidance' approach. Roughly speaking a controller stabilizes a set of k plants if and only if it avoids, in a sense that we will define, the k plants in the extended right half plane. By the end of this section we hope that we will have convinced the reader that stabilization and avoidance are different names for the same mathematical question. We refer the reader to [6] or [7] for more details on avoidance concepts applied to simultaneous stabilization problems.

3.1 Internal stabilization

We shall throughout this paper consider a controller to be within a unity feedback loop with the plant and we shall adopt the following usual definition of stability for this closed loop configuration.

Definition 3.1. *A controller $c(s) \in \mathbb{R}(s)$ is an internal stabilizer of (or internally stabilizes) a plant $p(s) \in \mathbb{R}(s)$ if the four transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}$, $c(s)(1+p(s)c(s))^{-1}$, $p(s)(1+p(s)c(s))^{-1}$ and $(1+p(s)c(s))^{-1}$ belong to S (i.e. they have no poles with nonnegative real part).*

These four unpractical conditions for internal stability can elegantly be condensed into a single one by using the so-called 'factorization approach' described in [31] and [30]. We give hereafter a short introduction to this approach and refer the interested reader to [31] for more details. In the sequel we will always understand 'internal stability' when writing 'stability'.

It is easy to check that the set S of stable rational functions is a commutative ring. The invertible elements (or units) in the ring S are the stable real rational functions whose inverse are stable, that is the real rational functions with no poles nor zeros in $\mathbb{C}_{+\infty}$. We have denoted this set by U. Two elements of S are called *coprime* if they have no common zeros in $\mathbb{C}_{+\infty}$. It can be proved (see [31] p. 10) that S is an Euclidean ring and hence, if $a(s), b(s) \in S$ are coprime, then there exists $x(s), y(s) \in S$ such that $a(s)x(s) + b(s)y(s) = 1$. Such an identity is called a *Bezout identity*. Finally, the field of fractions of S is $\mathbb{R}(s)$. All this together shows that if $p(s) \in \mathbb{R}(s)$ then there exists $n_p(s), d_p(s) \in S$ and $x(s), y(s) \in S$ such that $p(s) = \frac{n_p(s)}{d_p(s)}$ and $n_p(s)x(s) + d_p(s)y(s) = 1$ (such a fractional decomposition of $p(s)$ will be called a *coprime decomposition*). This is the only property of S that we will need in this paper. It provide us the following result (see [31] p. 45 for a proof). For conciseness, we sometimes drop the reference to the complex variable s when writing rational functions.

Theorem 3.2. *Let $p, c \in \mathbb{R}(s)$ and let $p = \frac{n_p}{d_p}$ and $c = \frac{n_c}{d_c}$ be any coprime decompositions of p and c. Then c stabilizes p if and only if $n_c n_p + d_c d_p \in U$.*

As a corollary of this Theorem 3.2, we may formulate the simultaneous stabilization problem under the following form.

Corollary 3.3. *Let $p_i \in \mathbb{R}(s)$, $i = 1, ..., k$ and let $p_i = \frac{n_i}{d_i}$ be any coprime decomposition of p_i, $i = 1, ..., k$. Then p_i are simultaneously stabilizable if and only if there exist $n_c, d_c \in S$ such that $n_c n_i + d_c d_i \in U$, $i = 1, ..., k$.*

A controller $c \in \mathbb{R}(s)$ is stable if it has no poles in $\mathbb{C}_{+\infty}$, in other words it is stable if for every coprime decomposition $c = \frac{n_c}{d_c}$ we have $d_c \in U$. The controller c is stable and inverse stable (we have called such functions units) if both n_c and d_c are in U. In the next section we will need the following natural definition:

Definition 3.4. *Let $p \in \mathbb{R}(s)$ and let $p = \frac{n_p}{d_p}$ be any coprime decomposition of p in S. The plant p is strongly stabilizable (i.e. stabilizable by a stable controller) if and only if there exist $n_c \in S$ and $d_c \in U$ such that $n_c n_p + d_c d_p \in U$. The plant p is unit stabilizable (i.e. stabilizable by a stable controller whose inverse is stable) if and only if there exist $n_c \in U$ and $d_c \in U$ such that $n_c n_p + d_c d_p \in U$.*

Note that the definition above is independent of the choice of the coprime decompositions. Theorem 3.2 and Corollary 3.3 above are proved for the case where the stability region is the extended closed right half plane $\mathbb{C}_{+\infty}$. It may, however, be useful to define the concept of stability in a more general framework. First, this allows to treat continuous and discrete time stability questions in a general setting and, secondly, the use of stability regions different from the extended right half plane may be justified for practical purposes (see [32] for example). The generalisation goes exactly along the same line. Let Ω be a closed subset of the extended complex plane $\mathbb{C}_\infty$ satisfying the assumptions given in Section 2. $S(\Omega)$ is the set of real rational functions with no poles in Ω, and $U(\Omega)$ is the set of invertible elements of $S(\Omega)$. Then the above results on the ring S carry over, namely $S(\Omega)$ is an Euclidean commutative ring whose field of fractions is $\mathbb{R}(s)$. Since these were the only properties that are needed to prove Theorem 3.2 and Corollary 3.3, these results remain valid for a general stability region Ω. Let us state this clearly.

Definition 3.5. *Let Ω be a subset of $\mathbb{C}_\infty$. A controller $c(s) \in \mathbb{R}(s)$ is an internal Ω-stabilizer of a plant $p(s) \in \mathbb{R}(s)$ if and only if the four transfer* *functions* $p(s)c(s)(1 + p(s)c(s))^{-1}, c(s)(1 + p(s)c(s))^{-1}, p(s)(1 + p(s)c(s))^{-1}$ *and* $(1 + p(s)c(s))^{-1}$ *belong to* $S(\Omega)$*) (i.e. they have no poles in* Ω*).*

We then have.

Corollary 3.6. *Let* Ω *be a subset of* $\mathbb{C}_\infty$ *as described in Section 2. Let* $p_i \in \mathbb{R}(s)$, $i = 1, ..., k$ *and let* $p_i = \frac{n_i}{d_i}$ *be any coprime decompositions of* p_i *in* $S(\Omega)$, $i = 1, ..., k$. *Then* p_i *are simultaneously* Ω*-stabilizable if and only if there exist* $n_c, d_c \in S(\Omega)$ *such that* $n_c n_i + d_c d_i \in U(\Omega)$, $i = 1, ..., k$.

Proof. Goes exactly along the same line as Theorem 3.2 and Corollary 3.3. See [31]. ■

It is clear that if Ω' is a subset of Ω, then an Ω-stabilizing controller of a plant p is also an Ω'-stabilizing controller (since if the transfer functions have no poles in Ω then they have no poles in Ω'). In particular, if we define $\omega = \Omega \cap \mathbb{R}_\infty$ then an Ω-stabilizing controller is also an ω-stabilizing controller. ω-stabilizability is thus a necessary condition for Ω-stabilizability. This necessary condition will play a crucial role in Section 4. First we show the link between stability and avoidance.

3.2 Avoidance

Functions in $\mathbb{R}(s)$ go from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ and have the additional property that they take extended *real* values on $\mathbb{R}_\infty$. It is therefore easy to represent their behaviour on $\mathbb{R}_\infty$ with a two-dimensional graphic. On the other hand we need four dimensions to represent their behaviour on the complex plane. With these representations in mind we may figure out where two plants $p_1(s), p_2(s) \in \mathbb{R}(s)$ possibly intersect on the $\mathbb{R}_\infty$ axis, that is the set of values $s_0 \in \mathbb{R}_\infty$ for which $p_1(s_0) = p_2(s_0)$. It is still easy to define, but more difficult to represent geometrically, the points in $\mathbb{C}_\infty \backslash \mathbb{R}_\infty$ where two plants intersect. We give a formal definition for this.

Definition 3.7. *Let* $p_1(s), p_2(s) \in \mathbb{R}(s)$, *let* Ω *be a subset of* $\mathbb{C}_\infty$ *and let* $p_i(s) = \frac{n_i(s)}{d_i(s)}$ *be any coprime decompositions of* $p_i(s)$ *in* $S(\Omega)$, $i = 1, 2$. $s_0 \in \Omega$ *is a point of intersection of multiplicity n between* $p_1(s)$ *and* $p_2(s)$ *if* $n_1(s)d_2(s) - n_2(s)d_1(s) \in S(\Omega)$ *has a zero of multiplicity n at* s_0. $p_1(s)$ *avoids* $p_2(s)$ *in* Ω *if* $p_1(s)$ *and* $p_2(s)$ *have no points of intersection in* Ω.

Note that the points of intersection in Ω between $p_1(s)$ and $p_2(s)$ (and thus avoidance in Ω) do not depend on the particular choice of the coprime factorizations in $S(\Omega)$.

To illustrate the Definition 3.7, consider, for example, $p_1(s) = \frac{s+1}{s^2}$ and $p_2(s) = \frac{5s-1}{s^2(s+1)}$. The points of intersection between $p_1(s)$ and $p_2(s)$ in $\mathbb{C}_{+\infty}$ are at $s_0 = 1, s_0 = 2$ and $s_0 = \infty$ with multiplicity one and at $s_0 = 0$ with multiplicity two.
Again it is clear that if $\Omega' \subset \Omega \subset \mathbb{C}_\infty$, then p_1 avoids p_2 in Ω implies that p_1 avoids p_2 in Ω'. In particular, if two rational functions avoid each other on a subset Ω of $\mathbb{C}_\infty$, then they do so on $\omega = \mathbb{R}_\infty \cap \Omega$.
The link between stabilization and avoidance is shown in the next theorem.

Theorem 3.8. *Let $p(s), c(s) \in \mathbb{R}(s)$. Then the controller $c(s)$ Ω-stabilizes $p(s)$ if and only if $-c^{-1}(s)$ avoids $p(s)$ in Ω (or, equivalently, if and only if $-p^{-1}(s)$ avoids $c(s)$ in Ω).*

Proof. Let $p(s) = \frac{n_p(s)}{d_p(s)}$ and $c(s) = \frac{n_c(s)}{d_c(s)}$ be coprime decompositions of $p(s)$ and $c(s)$ in $S(\Omega)$. By Theorem 3.2, $c(s)$ Ω-stabilizes $p(s)$ if and only if $n_p(s)n_c(s) + d_p(s)d_c(s) \in U(\Omega)$. This last condition is satisfied if and only if $n_p(s)n_c(s) + d_p(s)d_c(s) \in S(\Omega)$ has no zeros in Ω or, alternatively, if and only if $-c(s)^{-1}$ avoids $p(s)$ in Ω. ■

As a trivial consequence, notice that the plants which are Ω-stabilizable by a real constant feedback gain are precisely those that avoid a real value on Ω. With Theorem 3.8. we can formulate the general simultaneous stabilization problem of k plants in the form of an avoidance problem.

Corollary 3.9. *Let $p_i \in \mathbb{R}(s)$, $i = 1, ..., k$. The plants p_i are simultaneously Ω-stabilizable if and only if there exists a $q(s) \in \mathbb{R}(s)$ such that $q(s)$ avoids $p_i(s)$ in Ω, $i = 1, ..., k$, in which case $c(s) = -q^{-1}(s)$ is a Ω-stabilizing controller.*

The problem of the simultaneous Ω-stabilization of k plants thus has an easily understandable geometric interpretation. We are given a set of rational functions defined on a region Ω of the extended complex plane and we ask whether it is possible to find a rational function which avoids them all on Ω. If this is possible then the plants are simultaneously Ω-stabilizable. Now, as we pointed out above, if it is possible to find a rational function that avoids k rational functions on Ω then the same function avoids them all on $\omega = \mathbb{R}_\infty \cap \Omega$. *The existence of an ω-avoiding rational function is thus a necessary condition for simultaneous Ω-stabilization.* It is this necessary ω-stabilizability condition that we analyse in the next section.

When dealing with general stability regions Ω the terminology, as well as the notations, get somewhat heavy. For our purpose a large class of such stability regions are in fact equivalent: all the results contained in this paper are valid for general closed simply connected stability regions. In what follows we concentrate on 'canonical' simply connected stability regions, in Section 4 we deal with $\mathbb{C}_{+\infty}$ and in Section 5 we analyse counterexamples in $\overline{D}$.

4 Stabilization on the real axis: the search for necessary conditions.

The problem is simple. We examine k real rational functions on the interval $\mathbb{R}_{+\infty} = [0, \infty]$ where they are real valued. They may have poles as well as zeros on $\mathbb{R}_{+\infty}$. Their behaviour can easily be represented on a two dimensional graph as functions from $\mathbb{R}_{+\infty}$ to $\mathbb{R}_\infty$. Now we ask the question: 'Is it possible to find a real rational function, with perhaps poles and zeros in $\mathbb{R}_{+\infty}$, which avoids this set of functions on the interval $\mathbb{R}_{+\infty}$?'. In view of Corollary 3.9 this question is equivalent to the following: 'Given a set of plants $p_i(s) \in \mathbb{R}(s)$, $i = 1, ..., k$, when is it possible to find a single controller $c(s) \in \mathbb{R}(s)$ such that $p_i c(1+p_i c)^{-1}, p_i(1+p_i c)^{-1}, c(1+p_i c)^{-1}$ and $(1+p_i c)^{-1}$ have no *real* unstable poles?'. Obviously, this is a weaker requirement than simultaneous stabilization where poles in the whole $\mathbb{C}_{+\infty}$ are to be avoided. We argued in the introduction of this paper the interest of this question.

4.1 Stabilization of two plants and strong stabilization

In this section we will need the following well-known definitions:

Definition 4.1. *Let $p(s) \in \mathbb{R}(s)$. $p(s)$ has the parity interlacing property if $p(s)$ has an even number (counting multiplicities) of poles between each* *pair of zeros in $\mathbb{R}_{+\infty}$. p(s) has the even interlacing property if both $p(s)$ and $p^{-1}(s)$ have the parity interlacing property.*

An alternative way of defining this is by means of a graph. Let $z_i, p_j \in \mathbb{R}_{+\infty}$ $(i = 1, ..., l)$ $(j = 1, ..., m)$ be the l zeros and m poles of a plant $p(s) \in \mathbb{R}(s)$ in $\mathbb{R}_{+\infty}$. The plant $p(s)$ has the parity interlacing property if and only if the succession of its poles and zeros on $\mathbb{R}_{+\infty}$, as s increases from zero to infinity, corresponds to a possible path in Graph 1.1. In the same vein, $p(s)$ has the even interlacing property if and only if the succession of its poles and zeros on $\mathbb{R}_{+\infty}$ corresponds to a possible path in Graph 1.2. For example, the succession of poles and zeros on $\mathbb{R}_{+\infty}$ of $p(s) = \frac{s-1}{(s-3)(s-2)s}$ follows the following pattern: PZPPZ and hence it has the parity interlacing property but not the even interlacing property since the succession PZPPZ corresponds to a possible path in Graph 1.1 but not in Graph 1.2. The same kind of figure will be used in Theorem 4.13 to describe a $\mathbb{R}_{+\infty}$-stabilizability condition for three plants, but first we analyse the two-plant case.

Theorem 4.2. *Let $p(s) \in \mathbb{R}(s)$. If there exists a stable controller that $\mathbb{R}_{+\infty}$-stabilizes p(s), then p(s) has the parity interlacing property.*

Proof. Let $c(s)$ be a stable $\mathbb{R}_{+\infty}$-stabilizing controller of $p(s)$. Then by Theorem 3.8, $c(s)$ avoids $-p^{-1}(s)$ on $\mathbb{R}_{+\infty}$. Since $c(s)$ is stable it also avoids ∞ on $\mathbb{R}_{+\infty}$. Suppose, to get a contradiction, that $p(s)$ has an odd number of poles between two zeros on $\mathbb{R}_{+\infty}$. Then $-p^{-1}(s)$ has an odd number of zeros between two poles on $\mathbb{R}_{+\infty}$. But then $c(s)$ has to avoid both a rational function $-p^{-1}(s)$ which has an odd number of zeros between two of its poles on $\mathbb{R}_{+\infty}$ and ∞ on $\mathbb{R}_{+\infty}$. This is impossible and hence $p(s)$ has an even number of poles between two zeros on $\mathbb{R}_{+\infty}$. ■

A stronger version of Theorem 4.2 can easily be obtained. Indeed, it can be shown that the parity interlacing property is in fact sufficient for $\mathbb{R}_{+\infty}$-stabilizability by a stable controller. This last result in turn is contained under a stronger form in the next theorem.

Theorem 4.3. *Let $p(s) \in \mathbb{R}(s)$. There exists a stable controller that stabilizes p(s) if and only if p(s) has the parity interlacing property.*

The proof of this fundamental fact was first given in [39]. The reader may find both an elementary and an advanced proof in [31].

Corollary 4.4. *A plant is stabilizable by a stable controller if and only if it is $\mathbb{R}_{+\infty}$-stabilizable by a stable controller.*

Proof. Use the two previous theorems together with the fact that a stabilizing controller is also $\mathbb{R}_{+\infty}$-stabilizing. ■

Using this last Corollary, we stress in the next theorem a fundamental property of simultaneous stabilization of two plants: if there exist a controller such that the closed loop transfer functions associated to each plant have no *real* unstable poles then there exist a controller that simultaneously stabilize the two plants. More formally:

Theorem 4.5. *Two plants are simultaneously stabilizable if and only if they are simultaneously $\mathbb{R}_{+\infty}$-stabilizable.*

Proof. Let p_1 and $p_2 \in \mathbb{R}(s)$ and let $p_i = \frac{n_i}{d_i}$ be any coprime decompositions, $i = 1, 2$. By Theorem 3.2, p_i are simultaneously stabilizable if and only if there exist $n_c, d_c \in S$ such that $n_c n_i + d_c d_i \in U$, $i = 1, 2$. Since n_1, d_1 are coprime, there exists $x, y \in S$ such that $n_1 x + d_1 y = 1$. Any controller $c = \frac{x+rd_1}{y-rn_1}$ where $r \in S$ is a stabilizing controller of p_1. In fact, it can easily be proved (see [31] or [13]) that any stabilizing controller of p_1 can be written in the form $c = \frac{x+rd_1}{y-rn_1}$ for some $r \in S$. Therefore p_1 and p_2 are simultaneously stabilizable if and only if $(x+rd_1)n_2 + (y-rn_1)d_2 = xn_2 + yd_2 + r(d_1n_2 - d_2n_1) \in U$ for some $r \in S$. If $xn_2 + yd_2 = 0$ then p_1 and p_2 are both simultaneously stabilizable and simultaneously $\mathbb{R}_{+\infty}$-stabilizable and so we rule out this case. Assume that $xn_2 + yd_2 \neq 0$. By Definition 3.4 the equation above has a solution if and only if the plant $q = \frac{d_1n_2 - d_2n_1}{xn_2+yd_2}$ is strongly stabilizable. We could have derived exactly the same computations for $\mathbb{R}_{+\infty}$-stabilizability by replacing S by $S(\mathbb{R}_{+\infty})$ and U by $U(\mathbb{R}_{+\infty})$ in our derivations. We have thus also that p_1 and p_2 are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if $q = \frac{d_1n_2 - d_2n_1}{xn_2+yd_2}$ is strongly $\mathbb{R}_{+\infty}$-stabilizable. But now, by applying Corollary 4.4 the theorem is proved. ■

In the proof we show that p_1 and p_2 are simultaneously stabilizable if and only if $q = \frac{d_1n_2 - d_2n_1}{xn_2+yd_2}$ is strongly stabilizable. This intermediate plant q is constructed with the coprime decompositions $p_i = \frac{n_i}{d_i}$, $i = 1, 2$ together with the solutions $x, y \in S$ of $n_1x + d_1y = 1$. It does not have an 'intuitively clear' interpretation. With a weak additional condition on the poles of p_1, and p_2, it is possible to put this equivalence between simultaneous stabilization of two plants and strong stabilization of a single plant in a new and more obvious form. Roughly speaking, two plants are simultaneously stabilizable if and only if their difference is strongly stabilizable.

Theorem 4.6. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, 2$ and suppose that $p_1(s)$ and $p_2(s)$ have no common poles on $\mathbb{R}_{+\infty}$. Then $p_1(s)$ and $p_2(s)$ are simultaneously stabilizable if and only if $p_1(s) - p_2(s)$ is strongly stabilizable.*

Proof. With p_1 and $p_2 \in \mathbb{R}(s)$, let $p_i = \frac{n_i}{d_i}$ be any coprime decomposition in $S(\mathbb{R}_{+\infty})$, $i = 1, 2$ and let $x, y \in S(\mathbb{R}_{+\infty})$ be such that $n_1x + d_1y = 1$. Then by Corollary 3.6 and Theorem 4.5, p_i are simultaneously stabilizable if and only if there exist $n_c, d_c \in S(\mathbb{R}_{+\infty})$ such that $n_cn_i + d_cd_i \in U(\mathbb{R}_{+\infty})$, $i = 1, 2$. By using the same argument as in the proof of Theorem 4.5, these two equations can be simultaneously fulfilled if and only if $(n_1d_2 - n_2d_1)r + (n_2x + d_2y) \in U(\mathbb{R}_{+\infty})$ has a solution for some $r \in S(\mathbb{R}_{+\infty})$. Such an equation has a solution if and only if $(n_2x + d_2y)$ is non zero and always has the same sign at the zeros of $(n_1d_2 - n_2d_1)$ on $\mathbb{R}_{+\infty}$ (this result is crucial and far reaching, a proof of it can be found in [31], p.38). Under the assumption that d_1 and d_2 do not have any common zeros on $\mathbb{R}_{+\infty}$ and with some additional algebra this last condition can be shown to be equivalent to imposing that d_1d_2 always has the same sign at the zeros of $(n_1d_2 - n_2d_1)$. This in turn is equivalent to imposing $p_1 - p_2$ to be stabilizable by a stable controller. ■

Theorem 4.6 is a stronger form of the results contained in [31] and in [37] which state respectively 'if p_1 is stable, then p_1, p_2 are simultaneously stabilizable if and only if $p_1 - p_2$ is stabilizable by a stable controller' and 'if p_1 and p_2 have no common poles in $\mathbb{C}_{+\infty}$, then they are simultaneously stabilizable if and only if $p_1 - p_2$ is stabilizable by a stable controller'. Both these results are contained in Theorem 4.6 since if p_1 is stable then p_1 and p_2 have no common poles on $\mathbb{C}_{+\infty}$, and if they have no common poles on $\mathbb{C}_{+\infty}$ then they have no common poles on $\mathbb{R}_{+\infty}$.

4.2 Stabilization of three plants and unit stabilization

We now investigate the case of three plants and its link with unit stabilization. In this subsection we consider only the case of plants that do not all intersect at the same point. This assumption is not generic (for example strictly proper plants all intersect at infinity) and will be dropped in Subsection 4.4. We start with a crucial theorem.

Theorem 4.7. *Let $p_i(s) \in \mathbb{R}(s)$ $i = 1, 2, 3$. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection in $\mathbb{C}_{+\infty}$ (i.e. there is no $s_0 \in \mathbb{C}_{+\infty}$ for which $p_1(s_0) = p_2(s_0) = p_3(s_0)$). Let $p_i = \frac{n_i}{d_i}$, $i = 1, 2, 3$ be any coprime decompositions and define $a_{ij} = n_i d_j - n_j d_i$, $(i, j = 1, 2, 3)$. Then $p_i(s)$, $i = 1, 2, 3$ are simultaneously stabilizable if and only if there exist $u_i \in U$, $i = 1, 2, 3$ such that $a_{12}u_3 + a_{23}u_1 + a_{31}u_2 = 0$.*

Proof. Let $x, y \in S$ be solutions of $n_1x+d_1y = 1$ and define $b_i = n_ix+d_iy$, $i = 2, 3$. It is easy to check that $b_2a_{13} - b_3a_{12} = a_{23}$. If $n_in_c + d_id_c = u_i$ for some $n_c, d_c \in S$, $i = 1, 2, 3$, then for these u_i it is easy to check that $a_{12}u_3 + a_{23}u_1 + a_{31}u_2 = 0$ and hence the necessity is proved. For sufficiency, suppose that there exists $u_i \in U$, $i = 1, 2, 3$ such that $a_{12}u_3+a_{23}u_1+a_{31}u_2 = 0$. Using $b_2a_{13} - b_3a_{12} = a_{23}$, we have $a_{12}u_3 + (b_2a_{13} - b_3a_{12})u_1 + a_{31}u_2 = a_{12}(u_3 - b_3u_1) + a_{31}(u_2 - b_2u_1) = 0$. Since there is no $s_0 \in \mathbb{C}_{+\infty}$ for which $p_1(s_0) = p_2(s_0) = p_3(s_0)$, this implies in algebraic terms that a_{12} and a_{31} are coprime. Hence there exists some $r \in S$ for which $a_{31}r = u_3 - b_3u_1$ and $a_{12}r = -u_2 + b_2u_1$. Defining $r' = \frac{r}{u_1}$ we have that $a_{31}r' + b_3 = \frac{u_3}{u_1} \in U$ and $a_{21}r' + b_2 = \frac{u_2}{u_1} \in U$. But now, defining $n_c = x + r'd_1$ and $d_c = y - r'n_1$, the theorem is proved since for these d_c, n_c we have $n_in_c+d_id_c \in U$, $i = 1, 2, 3$. ■

It is known (see [36] or [16]) that, modulo an additional condition, the three plant problem can be reduced to one of finding a single controller which is stable, inverse stable (from here on we will refer to such controllers as unit controllers), and which stabilizes a single plant. Let us make this connection more obvious by using Theorem 4.7.

Theorem 4.8. *Let $p_i \in \mathbb{R}(s)$, $i = 1, 2, 3$ and let $p_i = \frac{n_i}{d_i}$, $i = 1, 2, 3$ be arbitrary coprime decompositions in S. Suppose that p_1 avoids p_2 in $\mathbb{C}_{+\infty}$. Then p_i, $i = 1, 2, 3$ are simultaneously stabilizable if and only if $\frac{n_3d_1-n_1d_3}{n_2d_3-d_2n_3}$ is unit stabilizable i.e. stabilizable by a unit controller.*

Proof. Since p_1 avoids p_2 in $\mathbb{C}_{+\infty}$ we have $n_1d_2 - n_2d_1 = u \in U$. Trivially p_1, p_2 and p_3 have no common point of intersection in $\mathbb{C}_{+\infty}$ since p_1 and p_2 do not intersect in $\mathbb{C}_{+\infty}$. We may thus apply Theorem 4.7. Therefore, p_i, $i = 1, 2, 3$ are simultaneously stabilizable if and only if there exist $u_i \in U$,

$i = 1, 2, 3$ such that $uu_3 + a_{23}u_1 + a_{31}u_2 = 0$. This last equation has a solution if and only if there exists some u_1 and u_2 in U for which $a_{23}u_1 + a_{31}u_2 \in U$ or, equivalently, if and only if $\frac{n_3d_1 - n_1d_3}{n_2d_3 - d_2n_3}$ is unit stabilizable. ■

Contrary to the similar result for strong stabilization of Theorem 4.6 we have no interpretation to propose for $\frac{n_3d_1 - n_1d_3}{n_2d_3 - d_2n_3}$ in terms of the plants p_1, p_2 and p_3.

As an illustration of the theorem, consider the plants $p_1(s) = 1, p_2(s) = -\frac{1}{s}$ and $p_3(s) = -\frac{s-1}{s}$. We can take for coprime decompositions $n_1 = 1, d_1 = 1$, $n_2 = -\frac{1}{s+1}, d_2 = \frac{s}{s+1}$ and $n_3 = -\frac{s-1}{s+1}, d_3 = \frac{s}{s+1}$. p_1 and p_2 have no intersections in $\mathbb{C}_{+\infty}$ since $n_1d_2 - n_2d_1 = 1 \in U$. We can apply Theorem 4.8. Therefore p_i, $i = 1, 2, 3$ are simultaneously stabilizable if and only if $\frac{(2s-1)(s+1)}{s(s-2)}$ is unit stabilizable.

An important special case of the problem of the stabilizability of three plants is therefore equivalent to the stabilizability of a single plant by a unit controller. This can in fact be proven rigorously in the sense that for any plant $p(s)$ it is possible to construct three plants $p_1(s), p_2(s)$ and $p_3(s)$ such that $p(s)$ is stabilizable by a unit controller if and only if $p_i(s)$, $i = 1, 2, 3$ are simultaneously stabilizable. This equivalence is one of the reasons for investigating conditions under which a plant is stabilizable by a unit controller. For the same reason as before, we first examine the condition under which a single plant is $\mathbb{R}_{+\infty}$-stabilizable by a unit controller. This condition is rather simple.

Theorem 4.9. *Let $p(s) \in \mathbb{R}(s)$. There exists a unit controller that $\mathbb{R}_{+\infty}$-stabilizes p(s) if and only if p(s) has the even interlacing property.*

Proof. Necessity is trivial: apply Theorem 4.2 to $p(s)$ and $p^{-1}(s)$. Sufficiency can be shown by modifying slightly the proof of Theorem 3.2 in [36] in which the author proves that a stable controller with no real unstable zeros exist for any plant that satisfy the even interlacing condition. ■

Obviously, the even interlacing property is a necessary condition for stabilization of a plant by a unit controller, since stabilization requires in particular $\mathbb{R}_{+\infty}$-stabilization. By similarity with the strong stabilization condition and with Corollary 4.4, this necessary condition for stabilizability by a unit controller was conjectured to be also sufficient. The conjecture is false, however, and we give a counterexample in Section 5. Before proceeding to this, we investigate additional conditions under which three plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable.

4.3 Alternative conditions for stabilization of three plants

In section 4.2 we have shown the connection between simultaneous stabilizability of three plants and stabilizability of a related plant with a unit controller. Here we provide new conditions for simultaneous stabilizability and $\mathbb{R}_{+\infty}$-stabilizability of three plants. We start with a theorem which is of independent interest. Roughly speaking, it says that three plants are simultaneously stabilizable if and only if there exist three *stable* plants that have pairwise the same intersections in $\mathbb{C}_{+\infty}$ as the original three plants.

Theorem 4.10. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, 2, 3$. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection in $\mathbb{C}_{+\infty}$ (i.e. there is no $s_0 \in \mathbb{C}_{+\infty}$ for which $p_1(s_0) = p_2(s_0) = p_3(s_0)$). Then $p_i(s)$, $i = 1, 2, 3$ are simultaneously stabilizable if and only if there exist $p'_i(s) \in S$, $i = 1, 2, 3$ such that $p_i(s)$ and $p_j(s)$ have pairwise the same intersections in $\mathbb{C}_{+\infty}$ as $p'_i(s)$ and $p'_j(s)$ when $i, j = 1, 2, 3$.*

Proof. Let $p_i = \frac{n_i}{d_i}$, $i = 1, 2, 3$ be arbitrary coprime decompositions and define $a_{ij} = n_i d_j - n_j d_i$ $(i, j = 1, 2, 3)$. Suppose first that there exists $p'_i(s) \in S$, $i = 1, 2, 3$ such that $p_i(s)$ and $p_j(s)$ have pairwise the same intersections in $\mathbb{C}_{+\infty}$ as $p'_i(s)$ and $p'_j(s)$ when $i, j = 1, 2, 3$. In algebraic terms this means that $p'_i - p'_j = u_{ij} a_{ij}$ $(i, j = 1, 2, 3)$ for some units $u_{ij} \in U$ $(i, j = 1, 2, 3)$. Putting $u_1 = u_{23}, u_2 = u_{31}$ and $u_3 = u_{12}$ in Theorem 4.7 we get that $p_i(s)$, $i = 1, 2, 3$ are simultaneously stabilizable. To prove necessity, suppose that $p_i(s)$, $i = 1, 2, 3$ are simultaneously stabilizable and have no common intersection in $\mathbb{C}_{+\infty}$. Again, by Theorem 4.7 there exist $u_i \in U$, $i = 1, 2, 3$ such that $a_{12}u_3 + a_{23}u_1 + a_{31}u_2 = 0$. Take any $r_2 \in S$ and define $r_1 = r_2 + u_3 a_{12} \in S$ and $r_3 = r_2 - u_1 a_{23} \in S$. Then we have that $r_1 - r_2 = a_{12}u_3$, $r_2 - r_3 = a_{23}u_1$, but also $r_3 - r_1 = a_{31}u_2$. And thus $r_i \in S$, $i = 1, 2, 3$ are such that r_i and r_j have pairwise the same intersections in $\mathbb{C}_{+\infty}$ as $p_i(s)$ and $p_j(s)$ for $i, j = 1, 2, 3$. This ends the proof. ■

As we argued in Section 3.2, the results that we obtain in this section are still valid for general regions Ω that satify the assumptions stated in Section 2. In particular, we may derive the counterpart of Theorem 4.10 for the region $\Omega = \mathbb{R}_{+\infty}$.

Theorem 4.11. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, 2, 3$. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection on $\mathbb{R}_{+\infty}$ (i.e. there is no $s_0 \in \mathbb{R}_{+\infty}$ for which $p_1(s_0) = p_2(s_0) = p_3(s_0)$). Then $p_i(s)$, $i = 1, 2, 3$ are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if there exist $p'_i(s) \in S(\mathbb{R}_{+\infty})$, $i = 1, 2, 3$ such that $p_i(s)$ and $p_j(s)$ have pairwise the same intersections on $\mathbb{R}_{+\infty}$ as $p'_i(s)$ and $p'_j(s)$ when $i, j = 1, 2, 3$.*

The interest of this last result is that, while we do not know a tractable test to check the condition in Theorem 4.10, we have one for the condition in Theorem 4.11. The existence of three rational functions with no poles on $\mathbb{R}_{+\infty}$ that 'mimic' the pairwise intersections of three plants on $\mathbb{R}_{+\infty}$ relies on an interlacing property that we state hereafter.

Definition 4.12. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, 2, 3$. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection on $\mathbb{R}_{+\infty}$. Then $p_i(s)$, $i = 1, 2, 3$ have the 3-interlacing property if and only if the succession of their intersections on $\mathbb{R}_{+\infty}$, as s increases from zero to infinity, corresponds to a possible path in Graph 1.3.*

We can now prove our theorem.

Theorem 4.13. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, 2, 3$. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection on $\mathbb{R}_{+\infty}$. Then $p_i(s)$, $i = 1, 2, 3$ are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if they have the 3-interlacing property.*

Proof. Suppose that $p_1(s), p_2(s), p_3(s)$ have no common point of intersection on $\mathbb{R}_{+\infty}$. By Theorem 4.11, $p_i(s)$, $i = 1, 2, 3$ are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if there exist $p'_i(s) \in S(\mathbb{R}_{+\infty})$, $i = 1, 2, 3$ such that $p_i(s)$ and $p_j(s)$ have pairwise the same intersections on $\mathbb{R}_{+\infty}$ as $p'_i(s)$ and $p'_j(s)$ for $i, j = 1, 2, 3$. The fact that $p'_i(s)$ have no poles on $\mathbb{R}_{+\infty}$ implies that not all successions of pairwise intersections are possible, i.e. the succession of intersections between three continous functions from $\mathbb{R}_{+\infty}$ to $\mathbb{R}$ is not arbitrary. We claim that the successions that are possible are precisely those that represent a possible path in Graph 1.3. To prove this, at each point $s_0 \in \mathbb{R}_{+\infty}$ where the p'_i do not pairwise intersect (i.e. $p'_i(s_0) \neq p'_j(s_0)$, $i, j = 1, 2, 3$) we have $p'_i(s_0) > p'_j(s_0) > p'_k(s_0)$ for some $i, j, k = 1, 2, 3$. In this way we can associate, to each point $s_0 \in \mathbb{R}_{+\infty}$ where the plants p'_i do not pairwise intersect, one of the six orderings $p'_1 < p'_2 < p'_3, p'_1 < p'_3 < p'_2, p'_2 < p'_1 < p'_3, p'_2 < p'_3 < p'_1, p'_3 < p'_1 < p'_2$ or $p'_3 < p'_2 < p'_1$. If s_0 and s_1 are two points on $\mathbb{R}_{+\infty}$ such that $p'_i(s)$ have no pairwise intersections on $[s_0, s_1]$, then, because the $p'_i(s)$ are continuous, the ordering at s_0 and s_1 are the same. Hence, the ordering changes precisely at the pairwise intersections of the $p'_i(s)$. For example, the ordering $p'_1 < p'_2 < p'_3$ changes to $p'_1 < p'_3 < p'_2$ after an intersection between p'_2 and p'_3. Notice also that not all changes are admitted, for example $p'_1 < p'_2 < p'_3$ can not be changed to $p'_3 < p'_2 < p'_1$ after a single intersection. Representing the six possible orderings above in a graph together with all possible changes at the intersections yields the Graph 1.3. Necessity is proved. To prove sufficiency, it suffices to show that given a succession of pairwise intersections on $\mathbb{R}_{+\infty}$ that follows a path in Graph 1.3, it is always possible to construct three functions in $S(\mathbb{R}_{+\infty})$ that do not intersect simultaneously on $\mathbb{R}_{+\infty}$ and whose pairwise intersections are the given points. We do not give a technical, and tedious, proof of this here. Instead we outline the sketch of a constructive procedure. First translate the problem onto I by using the usual conformal equivalence. Then construct three continuous functions that satisfy the desired property. By careful use of the fact that polynomials are dense in the set of continuous functions on I, construct three polynomials that also satisfy this property. Notice then that polynomials are members of $S(I)$, so that by using the conformal equivalence again the theorem is proved. ■

The case where the plants do intersect on $\mathbb{R}_{+\infty}$ is analysed in Section 4.4 below.

Theorem 4.13 and the 3-interlacing property are equivalent under a different form to an algebraic condition recently given in [37]. It was obtained independently by the authors. Again, it is a necessary condition for simultaneous stabilizability of three plants since it is necessary and sufficient for $\mathbb{R}_{+\infty}$-stabilizability. In the conclusion of [37] it is conjectured that this condition is also sufficient for stabilizability, but we will prove in Section 5 that this is not true.

To illustrate the use of Theorem 4.13 we analyse an example given in the literature [16]. A natural question when analysing the simultaneous stabilizability of three plants is: 'Given three plants that are simultaneously stabilizable, they are of course pairwise simultaneously stabilizable. Is the converse also true?'. Unfortunately the answer is no. Ghosh provided a counterexample to this: $p_1(s) = \frac{s-7}{s-4.6}, p_2(s) = \frac{s-2}{2s-2.6}$ and $p_3(s) = \frac{s-6}{4.8s-24.6}$ are pairwise simultaneously stabilizable but it is shown in [16] that they are not simultaneously stabilizable. Application of our Theorem 4.13 easily shows that they are not even $\mathbb{R}_{+\infty}$-simultaneously stabilizable. The intersections between p_1 and p_2 are $\sigma_{12} = 1$ and $\sigma_{12} = 9$. For the other two pairwise intersections we get: $\sigma_{23} = 3$ and $\sigma_{23} = 4$, $\sigma_{31} = 7.34$ and $\sigma_{31} = 5.17$. Notice that for these three plants all the intersections happen to be on $\mathbb{R}_{+\infty}$, which is by no means generic. Ordering the succesion of pairwise $\mathbb{R}_{+\infty}$-intersections we get: $\sigma_{12}, \sigma_{23}, \sigma_{23}, \sigma_{31}, \sigma_{31}, \sigma_{12}$. This does not correspond to a possible path in Graph 1.3. Hence $p_i(s)$, $i = 1, 2, 3$, do not have the 3-interlacing property and, by Theorem 4.13, the three plants are not simultaneously stabilizable.

As a final remark on Theorem 4.13, it is worth noting that our 3-interlacing property can be extended to more than 3 plants. If k plants are simultaneously stabilizable, then the same sequence of pairwise intersections on $\mathbb{R}_{+\infty}$ is achievable by the pairwise intersections of k $\mathbb{R}_{+\infty}$-stable plants. This provides a necessary condition for simultaneous stabilization of k plants. We do not develop this further here because we believe that the results contained in the next section overshadow the interest of stabilization conditions of k plants when the plants do not intersect. We end this Section 4 by analysing the case where there exists some $s_0 \in \mathbb{R}_{+\infty}$ such that $p_i(s_0) = w_0$, $i = 1, ..., k$.

4.4 Simultaneous $\mathbb{R}_{+\infty}$ stabilization for intersecting plants

All the conditions in Section 4.2 and 4.3 are for the case where the three plants have no common point of intersection on either $\mathbb{C}_{+\infty}$ or $\mathbb{R}_{+\infty}$. There exists an important special case for which this condition is not satisfied. When the plants are all strictly proper, they all take the value 0 at infinity so that they have a common point of intersection at infinity. It is this special structure which partly motivates the next result, which is the central result of this section. It shows that, for simultaneous $\mathbb{R}_{+\infty}$-stabilizability, the condition are much simpler when the plants have a common point of intersection on

$\mathbb{R}_{+\infty}$. Note that the theorem applies not just to the three plant case but to the general k plants case.

Theorem 4.14. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, ..., k$ and suppose that there exists a value $s_0 \in \mathbb{R}_{+\infty}$ such that the plants intersect at s_0 (i.e. there exists some $s_0 \in \mathbb{R}_{+\infty}$ and some $w \in \mathbb{R}_\infty$ such that $p_i(s_0) = w$, $i = 1, ..., k$). Then the plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if they are pairwise simultaneously $\mathbb{R}_{+\infty}$-stabilizable.*

Proof. Necessity is obvious. We prove sufficiency by showing that, under the assumptions that the k plants $p_i(s)$, $i = 1, ..., k$ intersect at $s_0 \in \mathbb{R}_{+\infty}$ and that they are pairwise simultaneously $\mathbb{R}_{+\infty}$-stabilizable, it is possible to find a rational function $q(s)$ that avoids them all on $\mathbb{R}_{+\infty}$. The result will then follow by Corollary 3.9.

For simplicity we assume that $s_0 = 0$ and we define $w_0 = p_i(s_0) = p_i(0)$; the proof for an arbitrary s_0 goes along the same line. We assume also that $w_0 \neq \infty$. If not, we can redefine $p'_i = \frac{1}{p_i}$ and $w'_0 = 0$. First use the bilinear transformation that maps $\mathbb{C}_{+\infty}$ onto $\overline{D}$. Under this transformation, we get $p'_i(z) = p_i(\frac{(1+z)}{(1-z)})$. Since $p_i(0) = w_0$ we have $p'_i(-1) = w_0$ for $i = 1, ..., k$. In view of this, define $p''_i(z) = p'_i(z) - w_0$. It is clear that $p''_i(z)$ all have a zero at $z_0 = -1$. Also from our assumptions $p''_i(z)$, $i = 1, ..., k$ are real rational and pairwise simultaneously I-stabilizable. To end the proof it remains to show that $p''_i(z)$ are simultaneously I-stabilizable, i.e. that there exists a rational function that avoids $p''_i(z)$ on I.

To see this we define k continuous functions $v_i(z)$ from I to $\mathbb{R}$ by $v_i(z) = \arctan p''_i(z)$, $z \in I$. Here the inverse tangent function has to be taken with 'unwrapped argument', i.e. the function is continuous from $\mathbb{R}_\infty$ to $\mathbb{R}$ as z increases from -1 to 1 by choosing an appropriate branch of the inverse tangent function at the real poles of $p(z)$. Since $p''_i(-1) = 0$, we may chose $v_i(-1) = 0$. Some manipulations show that a rational function $r(z)$ avoids $p''_i(z)$ on I, $i = 1, ..., k$, if and only if $v_i(z) - n\pi < \arctan r(z) < v_i(z) - (n-1)\pi$ $\forall z \in I$, $i = 1, ..., k$ and for some $n \in \mathbb{N}$. In the sequel our objective is to construct such a $r(z)$. We therefore need an intermediate result.

We show that, because $p''_i(z)$ are pairwise simultaneously I-stabilizable, we have $\mid v_i(z) - v_j(z) \mid < \pi$, $\forall z \in I$, $i, j = 1, ..., k$. Suppose, by contradiction, that for some i, j and some $z_0 \in I$ we have $\mid v_i(z_0) - v_j(z_0) \mid \geq \pi$. Then, since $\mid v_i(-1) - v_j(-1) \mid = 0$, and since $v_i(z)$ are continuous, there must exist $z_1 \in [-1, z_0]$ such that $\mid v_i(z_1) - v_j(z_1) \mid = \pi$. But then, given any rational function $r(z)$ and the continuous function $v(z) = \arctan r(z)$ from I to $\mathbb{R}$, there exists some $z_2 \in [-1, z_1]$ such that either $v_i(z_2) - v(z_2) = n\pi$ or $v_j(z_2) - v(z_2) = n\pi$ for some $n \in \mathbb{N}$. Say $v_i(z_2) - v(z_2) = n\pi$. Then $v_i(z_2) = v(z_2) + n\pi$ and, taking the tangent of both sides, $p_i''(z_2) = r(z_2)$. This shows that every rational function intersects either $p_i''(z)$ or $p_j''(z)$ at some $z \in I$. This last statement contradicts the fact that $p_i''(z)$ and $p_j''(z)$ are simultaneously I-stabilizable and so we have proved that $\mid v_i(z) - v_j(z) \mid < \pi$, $\forall z \in I$, $i, j = 1, ..., k$. We now construct a stabilizing controller.

Define $w(z) : z \to min_{i=1,...,k} v_i(z)$. $w(z)$ is a continuous function from I to $\mathbb{R}$. By the above argument, $\mid v_i(z) - v_j(z) \mid < \pi$, $\forall z \in I$, $i, j = 1, ..., k$ and hence $v_i(z) - \pi < w(z) \leq v_i(z)$, $\forall z \in I$, $i = 1, ..., k$. We define $w'(z) = w(z) - \epsilon$ with ϵ sufficiently small so that $v_i(z) - \pi < w'(z) < v_i(z)$, $\forall z \in I$, $i = 1, ..., k$. Some algebraic manipulations, together with the fact that polynomials are uniformly dense in the set of continuous functions from I to $\mathbb{R}$, shows that given $w'(z)$ and $\epsilon > 0$ it is possible to find a rational function $q(z)$ such that $\mid w'(z) - \arctan q(z) \mid < \epsilon$, $\forall z \in I$. But then, for sufficiently small ϵ, we have $v_i(z) - \pi < \arctan q(z) < v_i(z)$, $\forall z \in I$, $i = 1, ..., k$. Taking the tangent of both sides, this last statement clearly shows that $q(z)$ avoids $p_i''(z)$ for $i = 1, ..., k$ and $z \in I$. This in turn implies by Corollary 3.9 that $p_i''(z)$, and hence $p_i'(z)$, are simultaneously I-stabilizable. The equivalence between the simultaneous I-stabilizability of the $p_i'(z)$ and that of the $\mathbb{R}_{+\infty}$-stabilizability of the $p_i(s)$ ends the proof. ■

Using this theorem, the next results are straightforward and their proofs are left to the reader.

Corollary 4.15. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, ..., k$ and suppose that there exists a value $s_0 \in \mathbb{R}_{+\infty}$ such that the plants intersect at s_0. Then the plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if they are pairwise simultaneously stabilizable.*

Corollary 4.16. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, ..., k$ and suppose that $p_i(s)$ have a common pole or a common zero on $\mathbb{R}_{+\infty}$. Then the plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if they are pairwise simultaneously*

$\mathbb{R}_{+\infty}$*-stabilizable.*

Corollary 4.17. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1, ..., k$ be strictly proper (they all have a zero at infinity). The plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable if and only if they are pairwise simultaneously $\mathbb{R}_{+\infty}$-stabilizable.*

Notice that in the above example of Ghosh the plants are not strictly proper.

These are only some of the possible corollaries of Theorem 4.14. Their main common interest is that, contrary to most of the results on simultaneous stabilisation, they provide tractable tests to decide whether k plants are simultaneously $\mathbb{R}_{+\infty}$-stabilizable. Most of the known results on simultaneous stabilization of more than two plants are only restatements of untractable conditions into other untractable conditions. Here we have provided tractable tests since the simultaneous stabilizability of two plants can be tested by using only a finite number of rational operations (see [1]). On the other hand, the drawback of our conditions is that, even though they are necessary and sufficient for $\mathbb{R}_{+\infty}$-stabilizability, they are only necessary conditions for $\mathbb{C}_{+\infty}$-stabilizability. We show in Section 5 that the conditions that we have obtained are in general not sufficient and, as soon as k is greater than two, it is necessary to look at the behaviour of the plants in the whole extended right half complex plane and not just on the extended positive real axis.

5 Stabilization in the complex plane

In the previous section we have found necessary and sufficient conditions for $\mathbb{R}_{+\infty}$-stabilizability of a single plant by a stable controller (parity interlacing property) and by a unit controller (even interlacing property). We have also treated the case of simultaneous $\mathbb{R}_{+\infty}$-stabilization of three or more plants (3-interlacing condition in the case of 3 plants which do not intersect, and pairwise stabilizability in the case of k plants that intersect on $\mathbb{R}_{+\infty}$). All these conditions are, as we have shown, necessary conditions for stabilizability in the usual sense i.e. $\mathbb{C}_{+\infty}$-stabilizability. One of these conditions has also been shown to be sufficient for $\mathbb{C}_{+\infty}$-stabilizability, namely two plants are simultaneously stabilizable if and only if they are simultaneously $\mathbb{R}_{+\infty}$-stabilizable. It was hoped that this property would flow on to the case $k \geq 3$.

The implicit conjecture 'k plants are simultaneously stabilizable if and only if they are $\mathbb{R}_{+\infty}$-stabilizable' has obviously been a driving motivation for many of the partial results on simultaneous stabilization. In this section we will give counterexamples showing that $\mathbb{R}_{+\infty}$-stabilizability does not, in general, imply $\mathbb{C}_{+\infty}$-stabilizability.

For convenience (mainly because $\overline{D}$ is a bounded set), we will give the counterexamples of this section in $\overline{D}$ rather than in $\mathbb{C}_{+\infty}$ i.e. in a 'discrete time' set-up. The discrete-time counterpart of $\mathbb{R}_{+\infty}$ is then $I = \overline{D} \cap \mathbb{R}_{\infty}$. It must be clear, however, that *all our counterexamples have a counterpart in continuous time*. The equivalence can be shown by using the bilinear transformation and we illustrate it for the first theorem.

We start with the easiest counterexample.

Theorem 5.1. *Let $p_1(z) = 0, p_2(z) = \frac{z}{z+2}, p_3(z) = \frac{2z}{z+2}$ and $p_4(z) = \frac{2z}{(z+2)(2-kz)}$ be four dicrete time systems. If $k > e^{26}$ then $p_i(z)$, $i = 1, ..., 4$ are simultaneously I-stabilizable but not simultaneously $\overline{D}$-stabilizable.*

Proof. Recall that $I = [-1, 1]$. The plants have a common point of intersection at $z = 0$ since $p_i(0) = 0$, $i = 1, ..., 4$. It is easy to check that for any k they are pairwise stabilizable and hence, applying Theorem 4.14, they are simultaneously I-stabilizable. It remains to be shown that for $k > e^{26}$ they are not simultaneously $\overline{D}$-stabilizable. Suppose, by contradiction, that for some $k > e^{26}$ the plants are simultaneously $\overline{D}$-stabilizable. Then for this k, and by using the natural coprime decomposition of $p_i(z)$, there must exist $n_c, d_c \in S(\overline{D})$ such that $d_c \in U(\overline{D}), zn_c + (z+2)d_c \in U(\overline{D}), 2zn_c + (z+2)d_c \in U(\overline{D})$ and $2zn_c + (2-kz)(z+2)d_c \in U(\overline{D})$. We define $f = \frac{2zn_c}{d_c(z+2)} + 2 \in S(\overline{D})$. By the above equations it is then clear that $f \in U(\overline{D}), f - 1 \in U(\overline{D})$ and $f - kz \in U(\overline{D})$. The first two equations imply that $f(z) \neq 0$ and $f(z) \neq 1$ for every $z \in D$. In addition $f(z)$ is analytic in D and $f(0) = 2$. By applying Picard-Schottky's theorem ([2], p.19) we have that $\mid f(z) \mid \leq e^{24}$ for every $\mid z \mid \leq \frac{1}{2}$. But then $\mid f(z) \mid < k \mid z \mid$ for $\mid z \mid = \frac{1}{2}$. This last inequality implies by Rouché's theorem [29] that $f - kz$ has a zero in $\{z : \mid z \mid \leq \frac{1}{2}\}$. This leads to a contradiction since $f - kz \in U(\overline{D})$, and thus the theorem is proved. ■

We provide the counterpart for continuous time stability by using the conformal mapping.

Corollary 5.2. *Let* $p_1(s) = 0, p_2(s) = \frac{s-1}{s+1}, p_3(s) = \frac{2(s-1)}{(s+1)}, p_4(s) = \frac{2(s-1)}{(2-k)s+(2+k)}$ *be four continuous time systems. If* $k > e^{26}$ *then* $p_i(s)$, $i = 1, ..., 4$ *are simultaneously* $\mathbb{R}_{+\infty}$*-stabilizable but they are not simultaneously* $\mathbb{C}_{+\infty}$*-stabilizable.*

Proof. The four plants are simultaneously $\mathbb{C}_{+\infty}$-stabilizable if and only if $p_1(z) = 0, p_2(z) = z, p_3(z) = 2z$ and $p_4(z) = \frac{2z}{2-kz}$ are simultaneously $\overline{D}$-stabilizable. This, in turn, implies that the four plants are simultaneously $\mathbb{C}_{+\infty}$-stabilizable if and only if $p_1(z) = 0, p_2(z) = \frac{z}{z+2}, p_3(z) = \frac{2z}{z+2}$ and $p_4(z) = \frac{2z}{(z+2)(2-kz)}$ are simultaneously $\overline{D}$-stabilizable. The impossibility of this is proved in Theorem 5.1. ■

The next counterexample is slightly stronger. It applies to the case of three plants. This result also answers negatively the question addressed in the conclusion of [37].

Theorem 5.3. *Let* n *be a positive integer and let* $p_{1,n}(z) = 0, p_{2,n}(z) = \frac{nz}{z+2}$ *and* $p_{3,n}(z) = -\frac{1}{nz(z+2)}$ *be three discrete time plants. For every* n, $p_{i,n}(z)$, $i = 1, 2, 3$ *are simultaneously* I*-stabilizable. There exists, however, a* n *such that* $p_{i,n}(z)$, $i = 1, 2, 3$ *are not simultaneously* $\overline{D}$*-stabilizable.*

Proof. It can be checked that for any positive integer n these three plants are simultaneously I-stabilizable; this part is left to the reader (the result follows trivially from Theorem 4.13). The fact that they are not simultaneously $\overline{D}$-stabilizable for all n is more difficult to prove. We suppose in the sequel that for every n they are simultaneously $\overline{D}$-stabilizable and we produce a contradiction.

Notice first, since $(z + 2) \in U(\overline{D})$ that $p_{i,n}(z)$ are simultaneously $\overline{D}$-stabilizable for every integer n if and only if $p'_{1,n}(z) = 0$, $p'_{2,n}(z) = nz$ and $p'_{3,n}(z) = -\frac{1}{nz}$ are simultaneously $\overline{D}$-stabilizable for every n. This in turn is possible if and only if for each n there exist $n_{c,n}(z), d_{c,n}(z) \in S(\overline{D})$ such that $d_{c,n}(z) \in U(\overline{D})$, $n_{c,n}(z)nz + d_{c,n}(z) \in U(\overline{D})$ and $n_{c,n}(z) - d_{c,n}(z)nz \in U(\overline{D})$. Since $d_{c,n}(z) \in U(\overline{D})$, we may define $h_n(z) \triangleq \frac{n_{c,n}(z)}{d_{c,n}(z)} \in S(\overline{D})$ to be the solution associated to n. We then have that $h_n(z)nz + 1 \in U(\overline{D})$ and $h_n(z) - nz \in U(\overline{D})$ for every n. In the next part we show that the existence, for every n, of a simultaneous solution $h_n(z)$ to these two equations is impossible.

Since $h_n(z)nz+1 \in U(\overline{D})$ we can define $g_n(z) = \frac{h_n(z)nz-n^2z^2}{h_n(z)nz+1} \in S(\overline{D})$. These functions are analytic in D, they have no zeros in $D\backslash\{0\}$ and they take the value 1 only twice in D, namely at $z = \frac{j}{n}$ and $z = -\frac{j}{n}$. By the generalised form of Montel's normal family criterion ([19], p.70) this implies that the sequence $(g_n(z))$ is a normal family in $D\backslash\{0\}$. Hence, going to a subsequence, we can assume that $g_n(z)$ converges uniformly on compact subsets of $D\backslash\{0\}$. There are only two possible cases: either $g_n(z)$ tends locally uniformly to infinity, or $g_n(z)$ tends locally uniformly to an analytic function in $D\backslash\{0\}$. We show in what follows that both these cases lead to a contradiction.

Case 1. $g_n(z)$ tends locally uniformly to infinity, i.e. the functions $\frac{1}{g_n(z)}$ tend locally to zero on every compact set of $D\backslash\{0\}$. Consider the compact set $\{z :\mid z \mid= \frac{1}{2}\}$. Given $\epsilon > 0$, we have $\mid \frac{1}{g_n(z)} \mid\leq \frac{\epsilon}{2} = \epsilon \mid z \mid$ for every $n \geq n_0(\epsilon)$ and $\mid z \mid= \frac{1}{2}$. By definition of $g_n(z)$ we know that $nzh_n(z)(1-\frac{1}{g_n(z)}) = -(1+\frac{n^2z^2}{g_n(z)})$. Using this equality together with the bounds obtained above we get $\mid \frac{h_n(z)}{n} \mid\leq \frac{\frac{\epsilon}{8}}{\frac{1}{2}(1-\frac{1}{4})}$ for $n \geq n_0(\epsilon)+n_0(\frac{1}{2})$ and $\{z :\mid z \mid= \frac{1}{2}\}$. For some large integer n we thus have $\mid \frac{h_n(z)}{n} \mid< \frac{1}{2}$ when $\mid z \mid= \frac{1}{2}$, i.e. $\mid \frac{h_n(z)}{n} \mid<\mid z \mid$ when $\mid z \mid= \frac{1}{2}$. The functions $\frac{h_n(z)}{n}$ are analytic in $\{z :\mid z \mid\leq \frac{1}{2}\}$ and hence, by Rouché's theorem, $\frac{h_n(z)}{n} - z$ has a zero in $\{z :\mid z \mid\leq \frac{1}{2}\}$ for some integer n. But this contradicts the fact that $h_n(z) - nz \in U(\overline{D})$ and thus case 1 can not occur.

Case 2. $g_n(z)$ tends locally uniformly to an analytic function in $D\backslash\{0\}$. Then $g_n(z)$ are uniformly bounded on compact subsets of $D\backslash\{0\}$. Say $\mid g_n(z) \mid\leq M$ for $\mid z \mid= \frac{1}{2}$. We have defined $g_n(z) = \frac{h_n(z)nz-n^2z^2}{h_n(z)nz+1}$ and thus also $g_n(z) = 1 - \frac{1+n^2z^2}{h_n(z)nz+1}$. This last equation, together with the bound on $g_n(z)$, implies that $\mid \frac{1+n^2z^2}{h_n(z)nz+1} \mid\leq M+1$ for $\mid z \mid= \frac{1}{2}$. This in turn implies that $\mid \frac{n^2}{h_n(z)nz+1} \mid\leq \frac{M+1}{\frac{1}{4}-\frac{1}{n^2}}$ for $\mid z \mid= \frac{1}{2}$ and $n > 3$. The function $\frac{n^2}{h_n(z)nz+1}$ is analytic in D and hence, by the Maximum Modulus Theorem, the bound obtained above holds throughout the disc of radius $\frac{1}{2}$. In particular it holds at $z = 0$ so that we must have $n^2 \leq \frac{M+1}{\frac{1}{4}-\frac{1}{n^2}}$ for $n > 3$. But this inequality is obviously violated when $n > 2\sqrt{M+2}$. A contradiction is obtained and thus Case 2 can not occur. ■

By using Theorem 5.3, we end this paper by providing an example of a plant which has the even interlacing property but which is not $\overline{D}$-stabilizable by a unit controller. Recall that in Section 4.2 we established that a plant $p(z)$ is I-stabilizable by a unit controller if and only if $p(z)$ has the even interlacing property on I.

Theorem 5.4. *Let $p_n(z) = \frac{z}{1+n^2z^2}$. $p_n(z)$ has the even interlacing property for every positive integer n. There exists, however, a n such that $p_n(z)$ is not unit $\overline{D}$-stabilizable.*

Proof. Suppose, by contradiction, that for every integer n there exists a unit $\overline{D}$-stabilizer of $p_n(z) = \frac{z}{1+n^2z^2}$. Then, for every positive integer n, there exist $n_{c,n}, d_{c,n} \in U(\overline{D})$ such that $zn_{c,n} + (1+n^2z^2)d_{c,n} = u_n \in U(\overline{D})$. Since $n_{c,n}, d_{c,n} \in U(\overline{D})$ this implies that $\frac{u_n}{d_{c,n}} = z\frac{n_{c,n}}{d_{c,n}} + (1+n^2z^2) = nz(\frac{n_{c,n}}{nd_{c,n}} + nz) + 1 \in U(\overline{D})$. Define $h_n = \frac{n_{c,n}}{nd_{c,n}} + nz$; then, for every n, h_n defined above is such that $h_n nz + 1 \in U(\overline{D})$ and $h_n - nz \in U(\overline{D})$. This has been proved to be impossible in the proof of Theorem 5.3 and thus the theorem is proved. ■

6 Conclusion

In this paper we have analysed some aspects of the simultaneous stabilization question.

Our first contribution was to show that the problem of internal stabilization of k plants p_i, $i = 1, ..., k$ is equivalent to what we have called an avoidance problem: 'under what condition on $p_i(s)$, $i = 1, ..., k$ is it possible to find $q(s)$ such that $p_i(s) \neq q(s)$, $\forall s \in \mathbb{C}_{+\infty}$, $i = 1, ..., k$'. Our first message is clear: *stabilization = avoidance*. This is only a restatement of the problem; it does not answer any question, but it provides new insights and new proof techniques for the establishement of other results.

The second part dealt with a subproblem of the simultaneous stabilization problem. Given two plants p_1 and p_2, we showed that there exists a controller c such that the closed loop transfer functions associated to p_1 and

p_2 are stable if and only if there exist a controller c such that the closed loop transfer functions associated to p_1 and p_2 have no *real* unstable poles. The same property is proved for the strong stabilization problem. Motivated by these results we have developed in that part a complete answer to the question: 'given k plants p_i, $i = 1, ..., k$ when is it possible to find a single controller c such that all the transfer functions have no real unstable poles?'. Although such a question may sound of limited practical interest, we have given some motivations for it.

The third part gave answers to some of the questions raised in part two and elsewhere. In particular, we showed that, unlike the case of two plants, the existence of a simultaneous stabilizing controller for more than two plants cannot be guaranteed by the existence of a controller such that the closed loop transfer functions have no real unstable poles.

To conclude, let us stress the fact that our results provide a much better understanding of the original simultaneous stabilization problem for more than two plants but that the problem is... still unanswered.

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