Closed-loop identification of MIMO systems: a new look at identifiability and experiment design

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Abstract

This paper addresses a question that has been posed to us: "Is it necessary to excite all reference signals for the identification of a multivariable system operating in closed loop with a linear time-invariant controller?" In this paper we follow a careful re-examination of the notions of identifiability and informative experiments for multi-input multi-output (MIMO) closed-loop systems, which provides a negative answer to this question. Our analysis also allows to establish conditions on the controller complexity that guarantee existence of a unique global minimum of the identification criterion in the absence of external excitation; these conditions extend to the MIMO case conditions that were known for the SISO case. We illustrate our results for 2-input 2-output systems by presenting various possible experiment designs that produce a unique global minimum for the identification criterion.

1 Introduction

This paper re-examines the identifiability of closed-loop systems with a double objective: (i) clarify the differences between several definitions of identifiability and propose one that is realistic from a practical point of view; (ii) for the case of multi-input multi-output (MIMO) systems, establish conditions on the controller complexity and on the external excitation signals that guarantee that

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the asymptotic identification criterion has a unique global minimum at the true parameter value. Our results shed new light on the experiment design question for MIMO systems; in particular they provide an answer to the question raised in the abstract about the necessity (or not) of excitating all external references.

Identifiability of linear systems operating in closed loop was a much studied problem in the 1970's and early 1980's. Results in [24, 25, 21, 1] seemed to fully answer all questions about the identifiability of closed-loop systems under different feedback configurations and for different assumptions about the excitation signals and the measured signals. Recently, there has been a renewed interest in closed-loop identification in the context of *least costly identification experiment for control* [5, 13]. Briefly speaking, this concept refers to achieving a prescribed accuracy at the lowest possible price, measured in terms of the duration of the identification experiment, the perturbation induced by the excitation signal, or any combination of these.

In this context, our recent work has focused on experiment design questions such as the respective contributions of the noise and of the external excitation signal to the accuracy of the parameters and the transfer function estimates [4, 19], and the necessary and sufficient conditions on the controller complexity and on the excitation signals that make the data informative [8, 9, **?**]. The main contribution of this paper, which has its origins in the conference paper [2], is to extend several of the earlier results on identifiability, informativity and uniqueness of the global minimum to MIMO systems by addressing a series of questions along the following lines: what are the exact experimental conditions that are required to make a MIMO system identifiable? under which conditions on the controller complexity can one identify a MIMO system without external excitation? assuming that external excitation is required for reasons of either identifiability or accuracy, is it necessary that all reference inputs be excited in a MIMO system?

In addressing these questions, we have observed the need for a reassessment of the definitions of identifiability that were in use during the 1970's and which apparently still influence today's practice. Indeed, some of these definitions are unnecessarily demanding in terms of a realistic identification setup. For example, the main result in [25] states that in the case of a time-invariant controller a MIMO closed-loop system is *strongly system identifiable* if and only if all reference inputs are excited by persistently exciting signals. On the basis of such results, it is assumed by some in the system identification community that it is necessary to excite all external reference inputs if one wants to identify a system under closed-loop conditions. We show in this paper that this is not the case.

The reason for this is that for all practical purposes the definition of *strong system identifiability* is unnecessarily demanding and has since been replaced by the separate and more realistic concepts of *identifiability of the model structure* and *informativity of the data set* : see e.g. [17]. Furthermore, our analysis will show that a certain accuracy level can be obtained using a variety of possible external excitation schemes including, possibly, no excitation at all, or excitation by just one of the external reference signals.

The definitions of identifiability and informativity given in Section 2 are very general, in that they apply equally well to the situation where the system is in the model set as to the situation where it is not. However, the results of our paper, on convergence to the true system and on experiment design issues, are restricted to the situation where the system is in the model set.

Our first contribution, therefore, is one of clarification. We recall the definitions of identifiability as they have evolved, and adopt the complementary definitions of *identifiability of a model structure*, and *informativity (or richness) of the data set*, which have been prevalent in the engineering community of system identification over the last 20 years. These definitions allow one to separate conditions on the parametric model structure (identifiability), and conditions on the experiment (informativity). The separation of these two concepts shows much more clearly what the user's choices are. When the system is in the model set, the combination of an identifiable model structure and informative experiments yields, asymptotically, a unique global minimum for the identification criterion.

Our second contribution is to show that for MIMO systems operating in closed loop, if the controller is chosen of sufficient complexity with respect to the chosen model structure and if this model structure is globally identifiable, then a unique model can always be obtained without any external excitation, using information from the noise source only. This is an extension to MIMO systems of known results for SISO systems. These allow a significant relaxation of the constraints on the experiment design. In particular, the separate excitation of all external reference signals is not a necessary condition for convergence to a unique global minimum. By adding external excitation at one or several of the reference inputs, one can achieve the desired accuracy in a shorter time, as has been shown in [20].

Our third contribution is to illustrate the possible experiment design choices on a 2×2 closed-loop system. By particularizing our analysis to this case, the respective contributions of each external signal to the overall identification goal become more apparent. This illustrates the general concepts presented and results in guidance for experiment design choices.

The outline of our paper is as follows. In section 2 we present the key concepts of identifiability and of informativity of the data set. Section 3 establishes convergence conditions for the situation where the true system is contained in the model set. A discussion of various experimental conditions that ensure convergence to the true system for two-input two-output systems operating in closed loop is given in Section 4; these are illustrated with a numerical example. Concluding remarks appear in Section 5.

2 Identifiability and Informativity

2.1 Definitions of identifiability

The concept of identifiability has been given different contents in the econometrics and in the engineering literature [22] and the prevailing definition has also evolved over the years. [16] establishes a useful distinction between *consistency-oriented* and *uniqueness-oriented definitions* of identifiability.

Consistency-oriented identifiability deals with the question of whether the parameter estimate $\hat{\theta}_N$ converges to the 'true' parameter θ_0 in some stochastic sense. This definition was prevalent until the late 1970's when system identification was perceived as an exercise in finding the 'true system'. When identification began to be viewed as an exercise in finding the best approximate system within a parametrized model set, convergence to a true θ_0 became meaningless, since the model set may very well not contain a description of the true system, i.e. there is no 'true θ_0'. This is a severe limitation of the consistency-oriented definition of identifiability.

Uniqueness-oriented identifiability [3] deals with the question of whether the model structure is such that the identification criterion has a unique global minimum. While this no longer requires that the true system is in the model set, the existence of a unique minimum imposes conditions both on the model structure and on the data set used for identification.

From a users' point of view, it is important to work with a definition that clearly separates conditions on the model structure, and conditions on the experimental conditions that generate the data. Thus, in the 1980's, the consistency-oriented definitions of identifiability were replaced by two separate concepts : *identifiability of the model structure* and *informativity of the data* [17]. *Identifiability of the model structure* means that the map from parameter to model is injective, i.e. the same model cannot be described by two different parameter vectors. In the engineering literature, the concept of structural identifiability can be traced back to [3]. The choice of an identifiable model structure is entirely a user choice; it does not depend on the experimental conditions or on the identification criterion. *Informativity of the data* relates to the experimental conditions: the input-output data are informative if they cannot be produced by different models within the chosen model set.

These definitions, which we state formally in the remainder of this section, are operational in the following sense. The user first selects a model structure with the knowledge that it is identifiable. He or she then selects experimental conditions that make the data informative with respect to that model structure. The generation of informative data depends on the true system (since it is the true system that generates the data), but also to a large extent on users' choices: open- or closed-loop configuration, complexity of the controller (in a closed-loop setup), richness of the externally applied signals. These definitions do not require that the system is in the model set. However, if the true system can be exactly described within the chosen model structure, then the combination of an identifiable model structure and of informative data yields an identification

criterion which, asymptotically, has a unique global minimum. We now describe the Prediction Error Identification (PEI) setup, and recall the definitions of identifiability and informativity as described in [17, 18].

2.2 The identification setup

We consider that the data are generated by a *true system* $\mathcal{S}$, which is described by

$$y(t) = G_0(q^{-1})u(t) + H_0(q^{-1})e(t) \tag{1}$$

where $G_0(q^{-1})$ is a $p \times m$ causal, rational transfer function matrix, and $H_0(q^{-1})$ is a $p \times p$ stable and inversely stable transfer function matrix; it is also assumed that $H_0(0) = I$. To be precise, we shall define $\mathcal{S} \triangleq [G_0(q^{-1}) \;\; H_0(q^{-1})]$. The signal $y(t) \in \mathcal{R}^p$ is the output of the true plant, $u(t) \in \mathcal{R}^m$ is the control signal, while $e(t) \in \mathcal{R}^p$ is a zero-mean white noise input with diagonal covariance matrix $\Lambda = diag[\lambda_1, \lambda_2, \ldots, \lambda_p]$, with $\lambda_i > 0$. This true system is under feedback control with a stabilizing controller $K(q^{-1})$, a $m \times p$ causal rational transfer function matrix:

$$u(t) = K(q^{-1})[r(t) - y(t)]. \tag{2}$$

where $r(t) \in \Re^p$ is the reference,which is a quasi-stationary signal [18].

The system (1) is identified using a model structure parametrized by a vector $\theta \in \mathcal{R}^d$:

$$y(t) = G(q^{-1}, \theta)u(t) + H(q^{-1}, \theta)e(t) \tag{3}$$

where $G(q^{-1}, \theta)$ and $H(q^{-1}, \theta)$ are, respectively, $p \times m$ and $p \times p$ rational transfer function matrices parametrized by a vector $\theta \in \mathcal{R}^d$. For a given $\theta \in \mathcal{R}^d$, $M(\theta) \triangleq [G(q^{-1}, \theta) \;\; H(q^{-1}, \theta)]$ is called a *model*, while the *model structure* $\mathcal{M}$ is defined as a differentiable mapping from a connected open subset $D_\theta \in \mathcal{R}^d$ to a *model set* $\mathcal{M}^*$:

$$\mathcal{M} : \theta \in D_\theta \longrightarrow M(\theta) = [G(q^{-1}, \theta) \;\; H(q^{-1}, \theta)] \in \mathcal{M}^*. \tag{4}$$

It is assumed that the loop transfer function $G_0(q^{-1})K(q^{-1})$ is strictly causal, i.e. $\lim_{z \to \infty} G_0(z^{-1})K(z^{-1}) = \mathbf{0}$, and that the same holds for $G(q^{-1}, \theta)K(q^{-1}) \;\; \forall \theta \in D_\theta$. For brevity, we will most often drop the argument q^{-1}, thus referring to $G(\theta)$ and $H(\theta)$. The true system belongs to this model set, $\mathcal{S} \in \mathcal{M}^*$, if there is a θ_0 such that $M(\theta_0) = [G_0(q^{-1}) \;\; H_0(q^{-1})]$. In our definitions of identifiability, we shall not necessarily assume that $\mathcal{S} \in \mathcal{M}^*$. A model $M(\theta) = [G(\theta) \;\; H(\theta)]$ uniquely defines the one-step-ahead predictor of $y(t)$ given all input/output data up to time t:

$$\hat{y}(t|t-1, \theta) = W_u(\theta)u(t) + W_y(\theta)y(t), \quad \text{where} \tag{5}$$

$$W_u(\theta) = H^{-1}(\theta)G(\theta), \quad W_y(\theta) = [I - H^{-1}(\theta)]. \tag{6}$$

For later use, we introduce the following notation:

$$W(\theta) \triangleq [W_u(\theta) \;\; W_y(\theta)], \quad z(t) \triangleq \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \tag{7}$$

where $W(\theta)$ denotes the $p \times (m+p)$ rational transfer function matrix. Since there is a one-to-one relationship between $[G(\theta) \;\; H(\theta)]$ and $[W_u(\theta) \;\; W_y(\theta)]$, a model $M(\theta)$ will indistinctly refer to the pair $[G(\theta) \;\; H(\theta)]$ or to the predictor $W(\theta)$. The one-step-ahead prediction error is defined as:

$$\varepsilon(t,\theta) \triangleq y(t) - \hat{y}(t|t-1,\theta) = H^{-1}(\theta)\left[y(t) - G(\theta)u(t)\right]. \tag{8}$$

Using a set of input-output data of length N, the estimate $\hat{\theta}_N$ is calculated via the prediction error criterion [18]:

$$\hat{\theta}_N = \arg\min_{\theta \in D_\theta} \frac{1}{N} \sum_{t=1}^{N} \varepsilon^T(t,\theta)\varepsilon(t,\theta). \tag{9}$$

Under mild technical conditions on the data set this estimate converges w.p. 1 to a value θ^* [18]:

$$\hat{\theta}_N \overset{N\to\infty}{\longrightarrow} \theta^* \triangleq \arg\min_{\theta \in D_\theta} \bar{V}(\theta), \tag{10}$$

with

$$\bar{V}(\theta) \triangleq \bar{E}[\varepsilon^T(t,\theta)\varepsilon(t,\theta)]. \tag{11}$$

where $\bar{E}[f(t)] \overset{\Delta}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[f(t)]$

2.3 Identifiability of a model structure

Here we adopt the definition proposed in [18]; much of the paper serves to explain why this is an operational definition.

Definition 2.1 *A model structure $\mathcal{M}$ is globally identifiable at a value $\theta_1 \in D_\theta$ if, for all $\theta \in D_\theta$, $M(\theta) = M(\theta_1) \Rightarrow \theta = \theta_1$. It is locally identifiable at θ_1 if $M(\theta) = M(\theta_1) \Rightarrow \theta = \theta_1$ for all θ in $|\theta - \theta_1| < \epsilon$ for some $\epsilon > 0$. It is globally identifiable if it is globally identifiable at almost all $\theta \in D_\theta$.*

This definition is a property of the parametrization of $[G(\theta),\ H(\theta)]$. It says nothing about possible convergence to a value θ^*, as defined in (10), or to a "true" parameter value θ_0. The definition does not require that the system is in the model set, but it applies of course also to the situation where $\mathcal{S} \in \mathcal{M}^*$.

We now provide another characterization of local identifiability. With $W(\theta)$ defined as in (7), we introduce:

$$\vec{W}(\theta) \quad \triangleq \quad [W_1(\theta)\ W_2(\theta)\ \ldots W_p(\theta)] \tag{12}$$

where $W_k(\theta)$ denotes the k-th row of $W(\theta)$ with dimension $m+p$. Thus, $\vec{W}$ is a $p(m+p)$ row vector of transfer functions. We also define the $d \times p$ matrix $\psi(t,\theta)$ as:

$$\psi(t,\theta) \triangleq -\frac{\partial \varepsilon(t,\theta)}{\partial \theta} = \frac{\partial \hat{y}(t|t-1,\theta)}{\partial \theta} \tag{13}$$

Using (12), the matrix $\psi(t,\theta)$ can be written as

$$\psi(t,\theta) = \frac{\partial \vec{W}(\theta)}{\partial \theta}\ diag[z(t), z(t), \ldots, z(t)] \tag{14}$$

where $\frac{\partial \vec{W}(\theta)}{\partial \theta}$ is a $d \times p(m+p)$ matrix of transfer functions, while $diag[z(t), z(t), \ldots, z(t)]$ is a $(m+p)p \times p$ block-diagonal matrix with $z(t)$ in the diagonal blocks.

We now introduce the *identifiability Gramian* $\Gamma_1(\theta) \in \mathcal{R}^{d\times d}$:

$$\Gamma_1(\theta) \quad \triangleq \quad \int_{-\pi}^{\pi} \frac{\partial \vec{W}(e^{-j\omega},\theta)}{\partial \theta}\ \frac{\partial \vec{W}^H(e^{-j\omega},\theta)}{\partial \theta}\ d\omega \tag{15}$$

where $\vec{W}^H(e^{j\omega})$ denotes $\vec{W}^T(e^{-j\omega})$. The following result shows an alternative interpretation of local identifiability of a model structure; see problem 4G.4 in [18] for a SISO version of this result.

Theorem 2.1 *A parametric model structure $\mathcal{M}$ is locally identifiable at θ_1 if $\Gamma_1(\theta_1)$ is nonsingular.*

Proof. *For θ close to θ_1 we can write*

$$\vec{W}(e^{-j\omega},\theta) = \vec{W}(e^{-j\omega},\theta_1) + (\theta-\theta_1)^T \frac{\partial \vec{W}(e^{-j\omega},\theta)}{\partial \theta} + \sigma(|\theta-\theta_1|^2) \tag{16}$$

where $\lim_{\theta\to\theta_1} \frac{\sigma(|\theta-\theta_1|^2)}{|\theta-\theta_1|} = 0$. Assume that for θ close to θ_1 we have $W(e^{-j\omega},\theta) = W(e^{-j\omega},\theta_1)$ for all ω. It then follows from (16) that $(\theta-\theta_1)^T \frac{\partial \vec{W}(e^{-j\omega},\theta)}{\partial \theta} + \sigma(|\theta-\theta_1|^2) = 0$ for all ω. Multiplying this equation by its conjugate transpose and integrating over ω yields

$$(\theta-\theta_1)^T \Gamma_1(\theta_1)(\theta-\theta_1) = 0 \tag{17}$$

If $\Gamma_1(\theta_1)$ is nonsingular, this implies $\theta = \theta_1$.

2.4 Informative experiments for a model structure

If a model structure is globally identifiable at some value θ_1, it means that the model $M(\theta_1)$ cannot be represented by any other $M(\theta)$ within the model set. However, not all experiments allow to distinguish between $M(\theta_1)$ and any other model $M(\theta)$. Since we consider that the data are quasi-stationary, an experiment refers to the data generation mechanism which defines the asymptotic properties, and in particular the spectrum, of the joint input-output process $z(t)$ defined in (7); it does not refer to a particular set of N data.

Definition 2.2 *An experiment $z(t)$ is called informative with respect to a model set $\mathcal{M}^*$ if, for any two models $W^{(1)}$ and $W^{(2)}$ in that set,*

$$E\{|[W^{(1)}(q^{-1},\theta) - W^{(2)}(q^{-1},\theta)]z(t)|^2\} = 0 \tag{18}$$

implies

$$W^{(1)}(e^{-j\omega}) = W^{(2)}(e^{-j\omega}) \text{ for almost all } \omega, \tag{19}$$

where $W^{(i)}(q^{-1},\theta)$ are defined from a model by (6)-(7).

The definition means that there cannot be two different models within the model set that give exactly the same predictions, when excited by the same experiment. It is easy to show [18] that an experiment that yields $\Phi_z(\omega) > 0$ for almost all ω is informative for all model structures.

Comments: (i) The definition of informative experiment is with respect to a given model set, not with respect to the true system, which may or may not belong to the model set. In an identification experiment, one typically first selects a globally identifiable model structure; this is a user's choice. Experimental conditions must then be selected that are informative with respect to that structure; this is again a user's choice. However, the data are generated by the true system, in open or in closed loop. Thus, the conditions that make a data set $z(t)$ informative with respect to some model structure depend on the true system, on the feedback configuration, and on the externally applied signals.

(ii) Informative experiments guarantee that no two different models within the set can generate the same predictions. A globally identifiable model structure additionally guarantees that almost all models in the set are represented by a unique parameter vector. The selection of a globally identifiable model structure, together with an experiment that is informative with respect to that model structure then guarantees that, asymptotically, the minimum of the criterion is almost surely unique, since the mapping from θ to $M(\theta)$ is injective at almost every θ.

Now, suppose that the true system belongs to the model set ($\mathcal{S} \in \mathcal{M}^*$), that is, $M(\theta_0) = \mathcal{S}$ for some θ_0; then $\theta^* = \theta_0$. Moreover, if the model structure is globally identifiable at θ_0 and the experiment is informative enough with respect to this model structure, then the parameter error converges to a Gaussian random variable:

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \stackrel{N\to\infty}{\longrightarrow} \mathcal{N}(0, P_\theta), \tag{20}$$

where $\mathcal{N}(a, b)$ represents a normal distribution with mean a and variance b and

$$P_\theta = [I(\theta)]^{-1}|_{\theta=\theta_0}, \tag{21}$$

$$I(\theta) = \bar{E}\left(\psi(t,\theta)\Lambda^{-1}\psi(t,\theta)^T\right), \tag{22}$$

with $\psi(t,\theta)$ defined in (13). The matrix $I(\theta_0)$ is called the information matrix. In the remainder of this paper, we take (22) as the definition of the information matrix $I(\theta)$ for any value of θ; of course, the relationship (21) between the asymptotic covariance matrix and the information matrix holds only at $\theta = \theta_0$.

For large enough N, it is customary to approximate the covariance of the Gaussian random variable $\hat{\theta}_N - \theta_0$ by

$$Cov(\hat{\theta}_N - \theta_0) \approx \frac{1}{N} P_\theta = \frac{1}{N}[I(\theta_0)]^{-1} \tag{23}$$

Thus, for a large enough data set, the inverse of the information matrix is a measure of the accuracy of the estimated parameter vector. We therefore examine in more detail the role of the information matrix. Consider an identification

experiment which generates quasi-stationary data $z(t)$ with a power spectrum $\Phi_z(\omega)$. Combining (22), (14) and the definition of the noise matrix Λ yields:

$$\begin{aligned} I(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial \vec{W}(e^{-j\omega},\theta)}{\partial\theta}\; diag\left[\lambda_1^{-1}\Phi_z(\omega),\ldots,\lambda_p^{-1}\Phi_z(\omega)\right] \frac{\partial \vec{W}^H(e^{-j\omega},\theta)}{\partial\theta} d\omega \\ &= \frac{1}{2\pi}\sum_{k=1}^{p}\lambda_k^{-1}\int_{-\pi}^{\pi} \frac{\partial W_k(e^{-j\omega},\theta)}{\partial\theta}\Phi_z(\omega)\frac{\partial W_k^H(e^{-j\omega},\theta)}{\partial\theta} d\omega \end{aligned} \tag{24}$$

We can now state the following theorem.

Theorem 2.2 *Consider an identification experiment where $\Phi_z(\omega)$ is the spectrum of the data generated by this experiment and $W(q^{-1},\theta)$ is the model structure. Then $I(\theta_1) > 0$ if the following two conditions hold:*
(i) the identifiability Gramian is nonsingular at θ_1: $\Gamma_1(\theta_1) > 0$;
(ii) $\Phi_z(\omega) > 0$ for almost all ω.
Proof: *The result follows immediately by comparing (15) and (24) and using the positivity of $\Phi_z(\omega)$.*

Comments: (i) The positivity of the information matrix at a value θ_1 results from the combination of the two ingredients that are required to insure convergence to a unique minimum of an identification criterion at θ_1: the local identifiability of the model structure at θ_1 and the informativity of the data with respect to that model structure.
(ii) The conditions of Theorem 2.2 are sufficient but they are by no means necessary. As we shall see later, condition (ii) is much stronger than is needed. As for condition (i), expression (24) shows that it can be achieved if the rows of one or several of the matrices $\frac{\partial W_k(e^{-j\omega},\theta)}{\partial\theta}$ are linearly independent.

3 Convergence to the true system

We now consider the situation where the system is in the model set: $\mathcal{S} \in \mathcal{M}^*$. Thus there exists a parameter vector θ_0 such that $W(q^{-1},\theta_0) = W_0(q^{-1})$, where $W_0(q^{-1})$ is the one-step ahead prediction filter corresponding to the true system: $W_0(q^{-1}) \triangleq [H_0^{-1}(q^{-1})G_0(q^{-1}) \quad I - H_0^{-1}(q^{-1})]$. This means, equivalently, that $G(q^{-1},\theta_0) = G_0(q^{-1})$ and $H(q^{-1},\theta_0) = H_0(q^{-1})$. In addition, we assume that the model structure is globally identifiable at θ_0. We combine these two assumptions in a single statement.

Assumption 1 *The model structure $\mathcal{M}$ is such that $M(\theta_0) = \mathcal{S}$ for some θ_0 and it is globally identifiable at θ_0.*

Assumption 1 does not necessarily imply that the model $M(\theta)$ has the same structure as the true system; it can be an overparametrization of $\mathcal{S}$. For example, consider that the true system is a single-input single-output ARX system

described by $(1+0.8q^{-1})y(t) = 0.5u(t-1)+e(t)$. This system is contained in the model set $\mathcal{M}^*$ defined by the following ARMAX structure:

$$M(\theta) : (1+aq^{-1})y(t) = bu(t-1)+(1+cq^{-1})\varepsilon(t),$$

where $\theta \triangleq (a,b,c)$, which is globally identifiable at $\theta_0 = (0.8, 0.5, 0)$. However, the model structure $(1+a_1q^{-1}+a_2q^{-2})y(t) = (b_0+b_1q^{-1})u(t-1)+(1+cq^{-1})\varepsilon(t)$ is not globally identifiable at θ_0. For MIMO systems, overlapping parametrizations whose structure indices add up to the McMillan degree of the true system are identifiable [6, 11]. However, conditions under which this remains true when some structure indices are overestimated are more complicated to establish.

The following result shows the roles of informativity of data and of the information matrix.

Theorem 3.1 *Consider a model structure that obeys Assumption 1, and let $\bar{V}(\theta)$ be defined by (11). Then*
(i) θ_0 is the unique minimum of $\bar{V}(\theta)$ if the data are informative with respect to the model structure $\mathcal{M}$.
(ii) θ_0 is an isolated minimum of $\bar{V}(\theta)$ if $I(\theta_0) > 0$.
Proof: *(i) Let $\hat{y}^0(t)$ denote the optimal predictor for the true system (1), such that for this true system: $y(t) = \hat{y}^0(t) + e(t)$. The asymptotic cost criterion $\bar{V}(\theta)$ can then be written as*

$$\begin{aligned} \bar{V}(\theta) &= \bar{E}[|y(t) - \hat{y}(t|t-1,\theta)|^2] \\ &= \bar{E}[|\hat{y}^0(t) - \hat{y}(t|t-1,\theta)|^2] + tr\Lambda. \end{aligned} \tag{25}$$

Clearly, θ_0 is a global minimum of $\bar{V}(\theta)$, yielding $\hat{y}^0(t) = \hat{y}(t|t-1,\theta_0)$ and $\bar{V}(\theta_0) = tr\Lambda$. Let $\bar{\theta}$ be another global minimum. This implies that $\bar{E}[|\hat{y}(t|t-1,\theta_0) - \hat{y}(t|t-1,\bar{\theta})|^2] = 0$. Since the data is informative with respect to $\mathcal{M}$, this implies $W(\bar{\theta}) = W(\theta_0)$. By the global identifiability assumption at θ_0, this in turn implies that $\bar{\theta} = \theta_0$.
(ii) We compute the first and second derivative of $\bar{V}(\theta)$ - $\bar{V}'$ and $\bar{V}''$.

$$\bar{V}'(\theta) = -2\bar{E}\left[\frac{\partial \hat{y}(t|t-1,\theta)}{\partial \theta}\varepsilon(t,\theta)\right] = 0 \quad at \quad \theta = \theta_0. \tag{26}$$

As for $\bar{V}''(\theta)$, it is a $d \times d$ matrix whose (k,j)-th element is

$$\begin{aligned} \left[\bar{V}''(\theta)\right]_{k,j} &= -2\bar{E}\left\{\sum_{l=1}^{p} \frac{\partial^2 \hat{y}_l(t|t-1,\theta)}{\partial\theta_k \partial\theta_j}\varepsilon_l(t,\theta)\right\} \\ &\quad +2\bar{E}\left\{\frac{\partial \hat{y}(t|t-1,\theta)}{\partial\theta_k}\frac{\partial \hat{y}^T(t|t-1,\theta)}{\partial\theta_j}\right\} \end{aligned} \tag{27}$$

Since $\varepsilon(t,\theta_0) = e(t)$ and since $e(t)$ is uncorrelated with the predictions up to time $t-1$, the first term of (27) is zero at $\theta = \theta_0$. The result then follows, since $\bar{V}''(\theta_0) = I(\theta_0)$.

We can find ourselves in much more restrictive situations when applying consistency-oriented notions of identifiability. Take the classical notion of strong system identifiability [24, 25], which implies that the model estimate must converge to the true system for **all model structures** that can represent the true system. It was shown in [25] that, in the case of a closed-loop experiment with a time-invariant controller, strong system identifiability can be achieved only if all references are excited. This requirement is too strong to be practical, since in practice the user selects a particular model structure.

Informativity of the data depends on the model structure and, in a closed-loop experiment, also on the controller. It can be obtained in a number of different excitation scenarios, as is discussed in the sequel.

3.1 Identification without external excitation: $\mathbf{r = 0}$.

We first show that if one selects a model structure that obeys Assumption 1, and if the identification is performed in closed loop without any external excitation (i.e. $r \equiv 0$) but with a controller $K(q^{-1})$ of "sufficient complexity", then the asymptotic criterion $\bar{V}(\theta)$ has a unique global minimum at the value θ_0 for which $M(\theta_0) = \mathcal{S}$. First we state two technical lemma's that will be needed subsequently.

Lemma 3.1 *[7] Let $D(q^{-1})$ be a row reduced[1] polynomial matrix of full row rank and let $\rho(q^{-1}) = p(q^{-1})D(q^{-1})$ where $p(q^{-1})$ is any polynomial row vector. Then*

$$deg\ \rho(q^{-1}) = max_{j:p_j(q^{-1})\neq 0}[deg\ p_j(q^{-1}) + k_j] \tag{28}$$

where the degree of a vector is defined as the highest degree of any of its elements, $p_j(q^{-1})$ is the j-th entry of $p(q^{-1})$, and k_j is the degree of the j-th row of $D(q^{-1})$. ■

Lemma 3.2 *[15] Consider the Diophantine equation*

$$AX + BY = 0, \tag{29}$$

where A, B, X, Y are polynomial matrices, with X and Y given, and A and B unknown. The general solution of this equation can be written as

$$B = -PX_1,\ A = PY_1, \tag{30}$$

where X_1, Y_1 are a coprime solution of $Y_1X = X_1Y$ and P is an arbitrary polynomial matrix. ■

[1] See [14] for properties of polynomial matrices, such as coprime factorizations, column- or row-reduced matrices, etc.

Now, consider Definition 2.2 and assume that

$$\bar{E}\{|[W(\theta_1) - W(\theta_2)]z(t)|^2\} = 0. \tag{31}$$

Remembering (7), introducing $\Delta W_u(\theta_1, \theta_2) \triangleq W_u(\theta_1) - W_u(\theta_2)$, $\Delta W_y(\theta_1, \theta_2) \triangleq W_y(\theta_1) - W_y(\theta_2)$, and substituting $u(t) = -Ky(t)$, we can rewrite (31) as

$$\bar{E}\{|[\Delta W_y(\theta_1, \theta_2) - \Delta W_u(\theta_1, \theta_2)K]y(t)|^2\} = 0. \tag{32}$$

Since $\Phi_y(\omega)$ consists of filtered white noise, it is generically nonsingular for almost all ω. This implies, using Parseval's theorem and considering ΔW_y, ΔW_u and K as rational transfer function matrices, that

$$\Delta W_y(\theta_1, \theta_2) - \Delta W_u(\theta_1, \theta_2)K = 0. \tag{33}$$

Now let

$$[\Delta W_y(\theta_1, \theta_2) \quad - \Delta W_u(\theta_1, \theta_2)] = D^{-1}\,[B \quad A] \tag{34}$$

be a left coprime polynomial factorization with $D(q^{-1})$ row reduced, and let l_{max} and l_{min} be respectively the largest and smallest row degrees of the polynomial matrix $[B \;\; A]$. We can then state the following theorem.

Theorem 3.2 *Let the true system (1) be identified in closed loop with a feedback controller $u(t) = -K(q^{-1})y(t)$ without external excitation, using a model structure that obeys Assumption 1. Let k_{max} and k_{min} denote, respectively, the maximal and minimal observability indices of any irreducible representation of $K(q^{-1})$. We then have the following results:*
(i) if $k_{min} > l_{max}$ then θ_0 is the unique global minimum of the asymptotic criterion $\bar{V}(\theta)$;
(ii) if $k_{max} < l_{max}$ then θ_0 is NOT the unique global minimum of the asymptotic criterion $\bar{V}(\theta)$.
Proof: *Let $K = XY^{-1}$ be a right coprime factorization. Then, using equation (34), (33) can be written as:*

$$AX + BY = 0 \tag{35}$$

where $A \in \mathbb{R}^{p\times m}[q^{-1}]$, $B \in \mathbb{R}^{p\times p}[q^{-1}]$, $X \in \mathbb{R}^{m\times p}[q^{-1}]$, $Y \in \mathbb{R}^{p\times p}[q^{-1}]$.
By Lemma 3.2, the general solution of (35) is

$$[B \quad A] = P\,[-X_1 \quad Y_1]\,, \tag{36}$$

where $K = Y_1^{-1}X_1$ is a left coprime row reduced factorization of K, P is an arbitrary polynomial matrix in $\mathbb{R}^{p\times m}[q^{-1}]$, while k_{min} is the smallest row degree of the matrix $[-X_1 \;\; Y_1]$ [14]. Therefore:
(i) if $k_{min} > l_{max}$ then by Lemma 3.1 the only solution to (35) is obtained by setting $P = 0$ in (36), which implies $\Delta W_y = 0$ and $\Delta W_u = 0$ by (34). The result (i) then follows from part (i) of Theorem 3.1.
(ii) if $k_{max} \leq l_{max}$ then by Lemma 3.1 there exist non-zero solutions P of (36). With these solutions $\Delta W_y \neq 0$ and $\Delta W_u \neq 0$, which implies that the data are not informative with respect to the model structure.

Notice that the choice of a controller that satisfies the complexity requirement of the Theorem depends only on the model structure chosen by the user and not on the true system. A similar result can be found in [21]. The conditions obtained there are consistent with but slightly different from ours, because they are based on a factorization of the model $G(q^{-1},\theta)$, whereas ours are based on the factorization (34) of $[\Delta W_y(q^{-1},\theta)\ \ \Delta W_u(q^{-1},\theta)]$, which also involves the noise model $H(q^{-1},\theta)$.

Comment: The result of Theorem 3.2 provides an answer to the question raised in the abstract: for a MIMO system it is not necessary to excite all reference signals in order to converge to the unique global minimum θ_0 which represents the true system. The reason why this result may not be widely known lies in the inheritance of another concept of identifiability which prevailed in the seventies, namely that of *strong system identifiability*. In [25] and [12] necessary and sufficient conditions have been derived for *strong system identifiability* of time-invariant MIMO systems; in the case of a time-invariant controller, these require the excitation of all references. Our motivation for writing this paper was to show that, for all practical purposes, strong system identifiability is an unnecessarily strong concept. Indeed, it requires that the parameter estimate $\hat{\theta}_N$ converges asymptotically to a value that represents the true system **for all model structures that contain the true system**. This means in particular that such convergence must hold for models whose polynomial degrees are all allowed to grow unbounded. In practice, the user will always fix an upper bound to the polynomial degrees of his/her model structure.

3.2 Identification with external excitation: $\mathbf{r \neq 0}$

We have just shown that closed-loop identification can be performed without external excitation, provided a controller of sufficient complexity is chosen. However, by doing so one may have to collect many data to arrive at a prescribed level of accuracy. Applying external excitation signals adds contributions to the information matrix $I(\theta)$, and therefore decreases the covariance matrix P_θ for a given data length N.

Condition (31) can be rewritten as follows, using Parseval's theorem:

$$\begin{aligned} tr\{\int_{-\pi}^{\pi} &[\Delta W_u(e^{-j\omega},\theta_1,\theta_2) \quad \Delta W_y(e^{-j\omega},\theta_1,\theta_2)]\Phi_z(\omega) \\ &\times[\Delta W_u(e^{-j\omega},\theta_1,\theta_2) \quad \Delta W_y(e^{-j\omega}\theta_1,\theta_2)]^H d\omega\} = 0. \end{aligned} \tag{37}$$

This implies $\Delta W_u(e^{-j\omega},\theta_1,\theta_2) \equiv 0$ and $\Delta W_y(e^{-j\omega},\theta_1,\theta_2) \equiv 0$ if $\Phi_z(\omega) > 0$ for almost all ω. The conditions under which this holds depend on the experimental conditions, and therefore on the true system $[G_0\ \ H_0]$ and on the controller K. The expression of z as a function of the true closed-loop system is as follows:

$$z(t) \triangleq \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} I & K \\ -G_0 & I \end{bmatrix}^{-1} \begin{bmatrix} K & 0 \\ 0 & H_0 \end{bmatrix} \begin{pmatrix} r(t) \\ e(t) \end{pmatrix} \tag{38}$$

Since G_0K is strictly causal, the first matrix is always nonsingular. Assuming that K has full normal rank, then $\Phi_z(\omega) > 0$ for almost all ω if $\Phi_r(\omega) > 0$ for almost all ω. However, while this is a sufficient condition for the generation of informative data, it is by no means necessary. A complete analysis of the conditions that make closed-loop data, obtained with a time-invariant controller, informative with respect to a MIMO model structure is based on the analysis of the linear dependencies of the matrix (see (37)):

$$[\Delta W_u(e^{j\omega}, \theta_1, \theta_2) \quad \Delta W_y(e^{j\omega}, \theta_1, \theta_2)]\Phi_z(\omega) \tag{39}$$

when the data are generated by (38). Such an analysis for single-input single-output systems has been performed in [9, **?**], where necessary and sufficient conditions for informativity of the data have been given. In multivariable systems this analysis is more complex, and obtaining meaningful necessary conditions for this case is still an open issue. Yet, it is clear from (39) that informativity can be obtained in different scenarios. Informative data can be obtained from noise excitation only (if K is sufficiently complex), from some or all of the reference signals, or from a combination of reference signals and noise excitation. This is further illustrated by detailing the case of two-input two-output systems, which is done in the next Section.

4 Informative experiments for two-input two-ouput systems

We make the simplifying assumption that the noise sources that affect the different components of the measured outputs are uncorrelated. We thus consider that the true system $\mathcal{S}$ is described by

$$\begin{aligned} \mathcal{S} : y(t) &= G_0(q^{-1})u(t) + H_0(q^{-1})e(t) \\ &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} u(t) + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} e(t) \end{aligned} \tag{40}$$

where G_{11}, G_{12}, G_{21} and G_{22} are causal, rational transfer functions that are not necessarily analytic outside the unit circle, and H_1 and H_2 are stable and inversely stable transfer functions. The system $\mathcal{S}$ is controlled by the controller $u(t) = K[r(t) - y(t)]$. The control signal can then be expressed as follows:

$$\begin{aligned} u(t) &= U\left[r(t) - H_0 e(t)\right] \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} [r(t) - H_0 e(t)] \end{aligned} \tag{41}$$

where the input sensitivity function U is $U = K(I + KG_0)^{-1}$, where $I \in \mathcal{R}^{2\times 2}$ is the identity matrix. Assume that we perform an identification of the system (40) using a model structure $\mathcal{M}$ that is able to represent this true system, i.e. $\mathcal{S} = M(\theta_0)$ for some θ_0. For this model structure we have

$$\psi(t, \theta) = [\psi_1(t, \theta) \ \psi_2(t, \theta)], \quad \Lambda = diag(\lambda_1, \lambda_2)$$

$$P_\theta^{-1} \quad = \quad \bar{E}\left(\sum_{i=1}^{2}\psi_i(t,\theta_0)\lambda_i^{-1}\psi_i(t,\theta_0)^T\right). \tag{42}$$

Define, for convenience of notation, $g_{ij}^\theta = \left.\frac{\partial G_{ij}(\theta)}{\partial\theta}\right|_{\theta=\theta_0}$ and $h_i^\theta = \left.\frac{\partial H_i(\theta)}{\partial\theta}\right|_{\theta=\theta_0}$. Combining (40), (41) and (8) yields:

$$\begin{aligned}\psi_i(t,\theta_0) \quad &= \quad H_i{}^{-1}\left[\left(g_{i1}^\theta U_{11}+g_{i2}^\theta U_{21}\right)r_1(t)+\left(g_{i1}^\theta U_{12}+g_{i2}^\theta U_{22}\right)r_2(t)\right.\\ &\quad -\left(g_{i1}^\theta U_{11}H_1+g_{i2}^\theta U_{21}H_1\right)e_1(t)\\ &\quad \left.-\left(g_{i1}^\theta U_{12}H_2+g_{i2}^\theta U_{22}H_2\right)e_2(t)-h_i^\theta e_i(t)\right]\\ &\triangleq \quad \Pi_{i,r_1}r_1(t)+\Pi_{i,r_2}r_2(t)+\Pi_{i,e_1}e_1(t)+\Pi_{i,e_2}e_2(t)\end{aligned} \tag{43}$$

with the obvious definitions for Π_{i,r_1}, Π_{i,r_2}, Π_{i,e_1} and Π_{i,e_2}. Assuming that $r_1(t)$ and $r_2(t)$ are independent, and using Parseval's relationship, from (42) and (43) it is possible to express P_θ^{-1} as

$$\begin{aligned}P_\theta^{-1} \quad &= \quad \frac{1}{2\pi}\left(\sum_{i=1}^{2}\int_{-\pi}^{\pi}\lambda_i^{-1}[\Pi_{i,r_1}\Pi_{i,r_1}^H\Phi_{r_1}+\Pi_{i,r_2}\Pi_{i,r_2}^H\Phi_{r_2}]d\omega\right)\\ &\quad +\frac{1}{2\pi}\left(\sum_{i=1}^{2}\int_{-\pi}^{\pi}\lambda_i^{-1}[\Pi_{i,e_1}\Pi_{i,e_1}^H\lambda_1+\Pi_{i,e_2}\Pi_{i,e_2}^H\lambda_2]d\omega\right).\end{aligned} \tag{44}$$

This expression clearly shows the separate contributions of the two reference excitations and the two noise signals to the information matrix (i.e. the inverse of P_θ). Identification by noise excitation alone is possible if the second term in (44) is positive definite; this depends on the controller satisfying the complexity requirement of Theorem 3.2. More information (and hence more accuracy) is obtained when reference signals are added, but this often carries a price.

4.1 Experiment design for two-input two-ouput systems

Let us assume a situation where the system is to be identified in closed loop and some accuracy level is imposed on the parameter covariance matrix. Furthermore, let us assume that it is not possible (or reasonable) to perform this task without external excitation. This situation may happen when the controller is not complex enough with respect to the chosen model structure, or when the data length required to satisfy the prescribed level of accuracy is unreasonably large. We explore whether it is possible to attain the desired accuracy by exciting one of the reference signals only. There are many industrial examples where it is not of interest to excite all reference signals. A typical example is when the product quality and the production rate are the reference signals or directly determined by them. It might then be reasonable to perform the identification experiments with an additional excitation on the production rate while not interfering with the product quality.

A generic experiment design problem that tackles the above mentioned question can be formulated as follows. Let us consider that the chosen model structure is identifiable at θ_0 and that the data length is fixed to a given N. The

objective is to construct a closed-loop experiment informative enough subject to constraints on: (i) model quality - here we shall require $trP_\theta \leq \gamma$; (ii) reference signals - it may be desirable to limit some reference signals or not to excite some of them at all; we will impose that one of the reference signals, say $r_1(t)$, is equal to zero; (iii) other signals in the loop - typical examples are the constraints on the input and/or the output signals.

This problem can be expressed as an LMI optimization problem. Let us consider the parametrization of the reference signal spectra [5],

$$\Phi_{r_i}(\omega) = R_{r_i}(0) + 2\sum_{k=1}^{m} R_{r_i}(k)cos(k\omega) \geq 0 \quad \forall\omega \tag{45}$$

where $i = 1, 2$, and m is a positive integer. To ensure that the constraints in (45) are satisfied, the KYP lemma can be used [26, 23]. The quality constraint $trP_\theta \leq \gamma$ may be expressed more conveniently as

$$\begin{array}{rcl} \gamma - TrZ & \geq & 0 \\ \left[\begin{array}{cc} Z & I \\ I & P_\theta^{-1} \end{array}\right] & \geq & 0 \end{array} \tag{46}$$

with Z being the slack variable. When the parametrization (45) is used, the inverse covariance matrix (44) is given by

$$\begin{aligned} P_\theta^{-1}(\theta_0) &= M_p^{e_1}(\theta_0) + M_p^{e_2}(\theta_0) + M_p^{r_1}(\theta_0, 0) + M_p^{r_2}(\theta_0, 0) \\ &+ \sum_{k=1}^{m} R_{r_1}(k)M_p^{r_1}(\theta_0, k) + \sum_{k=1}^{m} R_{r_2}(k)M_p^{r_2}(\theta_0, k) \end{aligned} \tag{47}$$

where

$$M_p^{e_i}(\theta_0) = \lambda_i \int_{-\pi}^{\pi} \frac{\Pi_{1,e_i}\Pi_{1,e_i}^H}{2\pi\lambda_1} + \frac{\Pi_{2,e_i}\Pi_{2,e_i}^H}{2\pi\lambda_2} d\omega,$$

$$M_p^{r_i}(\theta_0, 0) = R_{r_i}(0) \int_{-\pi}^{\pi} \left(\frac{\Pi_{1,r_i}\Pi_{1,r_i}^H}{2\pi\lambda_1} + \frac{\Pi_{2,r_i}\Pi_{2,r_i}^H}{2\pi\lambda_2} \right) d\omega,$$

and

$$M_p^{r_i}(\theta_0, k) = \int_{-\pi}^{\pi} \left(\frac{\Pi_{1,r_i}\Pi_{1,r_i}^H}{\pi\lambda_1} + \frac{\Pi_{2,r_i}\Pi_{2,r_i}^H}{\pi\lambda_2} \right) cos(k\omega) d\omega,$$

with $i = 1, 2$. Observe that $P_\theta^{-1}(\theta_0)$ in (47) is linear in $R_{r_1}(k)$ and $R_{r_2}(k)$. Since the constraint (46) is convex in P_θ^{-1} and the spectra Φ_{r_1} and Φ_{r_2} are finitely parametrized, it is straightforward to incorporate this constraint into the convex optimization procedure.

Due to actuator limitations it is common to include energy constraints on the input signals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{u_i}(\omega) d\omega \leq E_{u_i}, \; i = 1, 2. \tag{48}$$

These can be expressed linearly in $R_{r_1}(k)$ and $R_{r_2}(k)$ as follows. From (41) and (45) one has:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{u_i}(\omega)d\omega = M_{u_i}^{e}(\theta_0) + M_{u_i}^{r_1}(\theta_0,0) + M_{u_i}^{r_2}(\theta_0,0) + \sum_{k=1}^{m} R_{r_1}(k)M_{u_i}^{r_1}(\theta_0,k) + \sum_{k=1}^{m} R_{r_2}(k)M_{u_i}^{r_2}(\theta_0,k) \tag{49}$$

where, for $i = 1, 2$:

$$M_{u_i}^{e} = \int_{-\pi}^{\pi}\frac{\lambda_1}{2\pi}|U_{i1}|^2|H_1|^2 + \frac{\lambda_2}{2\pi}|U_{i2}|^2|H_2|^2 d\omega,$$

$$M_{u_i}^{r_j}(\theta_0,0) = R_{r_j}(0)\int_{-\pi}^{\pi}\frac{\lambda_1}{2\pi}|U_{ij}|^2 d\omega,$$

and

$$M_{u_i}^{r_j}(\theta_0,k) = \int_{-\pi}^{\pi}\frac{\lambda_1}{\pi}|U_{ij}|^2 cos(k\omega)d\omega.$$

4.2 Case study

The following ARX structure is considered:

$$\begin{aligned} A_1 y_1(t) &= B_{11}u_1(t) + B_{12}u_2(t) + e_1(t) \\ A_2 y_2(t) &= B_{21}u_1(t) + B_{22}u_2(t) + e_2(t) \end{aligned}$$

with $A_1 = 1 - 0.9535q^{-1}$, $B_{11} = 0.744q^{-1}$, $B_{12} = -0.8789q^{-1}$, $A_2 = 1 - 0.9329q^{-1}$, $B_{21} = 0.5786q^{-1}$, $B_{22} = -1.302q^{-1}$; $e_1(t)$ and $e_2(t)$ are mutually independent zero-mean white Gaussian noises with $\lambda_1 = \lambda_2 = 0.05$. We consider the parameter vector $\theta = (b_{11}, b_{12}, b_{21}, b_{22}, a_1, a_2)^T$. The plant is controlled by a 2×2 PI controller which stabilizes the plant without other performance considerations:

$$K(q^{-1}) = \begin{pmatrix} \frac{0.55-0.45q^{-1}}{(1-q^{-1})} & \frac{0.15-0.08q^{-1}}{(1-q^{-1})} \\ \frac{0.24-0.18q^{-1}}{(1-q^{-1})} & \frac{-0.35+0.33q^{-1}}{(1-q^{-1})} \end{pmatrix}. \tag{50}$$

We want to attain the desired accuracy by exciting only one of the reference signals. Thus, we parametrize the reference signal spectra as in (45) with $i = 1, 2$, and $m = 50$, and we consider the following experiment design problem [13]:

$$\begin{array}{ll} \underset{\Phi_{r_1},\Phi_{r_2}}{\text{minimize}} & E_{r_2} \\ \text{subject to} & trP_\theta \le \gamma \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{u_i}(\omega)d\omega \le E_{u_i},\ i = 1,2 \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{r_2}(\omega)d\omega \le E_{r_2} \\ & \Phi_{r_1}(\omega) = 0,\ \Phi_{r_2}(\omega) \ge 0\ \ \forall\omega \end{array} \tag{51}$$

where $\gamma = 1$, $E_{u_1} = E_{u_2} = 0.4$. The solution $r_2(t)$ with the minimal energy required to satisfy the constraints yields $E_{r_2} = 1.777$. A Monte-Carlo simulation is performed to verify the results of the experiment design: $r_1(t)$ is kept equal to zero, while $r_2(t)$ is generated according to the solution of the optimization problem (51). The data length is $N = 1000$. The trace of P_θ computed by 500 Monte-Carlo runs is $trP_\theta = 1.047$. This slightly exceeds the bound $\gamma = 1$; this is due to the fact that the optimization problem is solved using the asymptotic covariance expression (see (21)-(22)), while the Monte-Carlo computations are based on estimates obtained from 1000 data. The simulation confirms that it is not necessary to excite both reference signals to attain a given accuracy level.

5 Conclusions

The origin of this work was to take a critical look at a statement made to us: *"For the identification of a MIMO system based on closed-loop data collected with a time-invariant controller, it is necessary to excite all reference inputs."* We have shown that this is not the case. In analyzing this question, we have made a clear distinction between identifiability of a model structure, and selection of experimental conditions that make the data informative with respect to that structure. We have shown that, once a model structure is selected, the user can always chose a controller of sufficient complexity that will make the data informative with respect to that model structure; we have quantifed this statement, by providing a result which gives the required controller order. We have also shown that the addition of reference signal excitation may serve to improve the quality of the parameter estimates, but that several excitation scenarios can typically be considered.

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