Numerical Algorithms as Discrete-Time Control Systems

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Numerical algorithms as discrete-time control systems

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Iterative Algorithms

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$$x_t \in \mathcal{M}, \qquad x_t \longrightarrow x_{t+1} \in \mathcal{M}$$

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Shifted Iterative Algorithms

$\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ iteration map, $\mathcal{U}$ set of shift parameters

$$x_t \in \mathcal{M}, u_t \in \mathcal{U} \qquad (x_t, u_t) \longrightarrow x_{t+1} \in \mathcal{M}$$

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Example: Shifted Inverse Iteration on $\mathbb{S}^{n-1}$.

$A \in \mathbb{R}^{n\times n}$, $\mathcal{M} = \mathbb{S}^{n-1}$, $\mathcal{U} = \mathbb{R} \setminus \sigma(A)$.

Iteration Step

$$\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}, \qquad \Phi(x,u) = \frac{(A-uI)^{-1}x}{\|(A-uI)^{-1}x\|}$$

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Shifted Iterative Algorithms

$\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ iteration map, $\mathcal{U}$ set of shift parameters

$$x_t \in \mathcal{M}, u_t \in \mathcal{U} \qquad (x_t, u_t) \longrightarrow x_{t+1} \in \mathcal{M}$$

Observation: $(\mathcal{M}, \mathcal{U}, \Phi)$ describes a discrete-time control System:

$$\begin{aligned} x_0 &\in \mathcal{M} \\ x_{t+1} &= \Phi(x_t, u_t) \end{aligned}$$

Introduction

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- Control theoretical description of numerical algorithms
- Better understanding of algorithms
- Optimization of shift strategies using control theoretical tools
- Justify existing shift strategies
- Creation of new algorithms

Examples

Example I: Shifted Inverse Iteration on Projective Spaces

- $A \in \mathbb{F}^{n\times n}$
- $\mathcal{M} = \mathbb{FP}^{n-1} = \{$Set of one dimensional subspaces of $\mathbb{F}^n\}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$\Phi(\mathcal{X}, u) = (A - uI)^{-1}\mathcal{X}$$

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Questions and remarks:

- Can we find feedback controls to reach eigenvectors by arbitrary initial points?
- Can we find feedback controls to reach specific eigenvectors by arbitrary initial points?

Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

- $A \in \mathbb{R}^{n \times n}$
- $\mathcal{M} = \mathrm{Grass}(p, n) = \{$Set of p-dimensional subspaces of $\mathbb{R}^n\}$
- $\mathcal{U}$ set of feedback maps $F : \mathrm{ST}(p, n) \to \mathbb{R}^{p \times p}$

$$\mathcal{U} = \{\forall X \in \mathrm{ST}(p, n), \forall M \in \mathrm{GL}_p(\mathbb{R}) : F(XM) = M^{-1} F(X) M\}$$

- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by procedure

1) Choose $X \in \mathrm{ST}(p, n)$ such that $\langle X \rangle = \mathcal{X}$
2) Solve $AX^+ - X^+ F(X) = X$
3) $\Phi(\mathcal{X}, F) := \mathcal{X}^+ := \langle X^+ \rangle$

Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

Questions and remarks

- The choice $F = R$ with $R(X) = (X^T X)^{-1} X^T A X$ leads to Grassmann Rayleigh Quotient Iteration (Absil, Mahony, Sepulchre, Van Dooren, 2002).
- Does the Grassmann Rayleigh Quotient Iteration has global convergence properties?
- Can we find feedback laws for global convergence?
- Can we find feedback laws for global convergence to a specific eigenspace?

Examples

Example III: Shifted Inverse Iteration on Flag Manifolds

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathrm{Flag}(\mathbb{F}^n) = \{\mathcal{V} = (V_1, \ldots, V_n) \,|\, V_i \subseteq V_{i+1}, \dim_{\mathbb{F}} V_i = i\}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$\Phi(\mathcal{V}, u) = \left((A - uI)^{-1} V_1, \ldots, (A - uI)^{-1} V_n\right)$$

Examples

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Questions and Remarks

- Algorithm is closely related to the QR algorithm (on isospectral manifolds).
- Can we steer to arbitrary eigenflags?

Examples

Example IV: Shifted QR Algorithm on Isospectral Manifolds

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \{Q^* A Q \,|\, Q \in \mathrm{U}_n(\mathbb{C})\}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$\Phi(X, u) = (X - uI)^*_{\mathrm{U}_n(\mathbb{C})}(X - uI)(X - uI)_{\mathrm{U}_n(\mathbb{C})}$$

where $(X - uI)_{\mathrm{U}_n(\mathbb{C})}$ is the unitary factor of the QR decomposition of $(X - uI)$.

Dynamics & Control

Examples

Example V: Shifted Inverse Iteration on Hessenberg Manifolds

- $A \in \mathbb{F}^{n\times n}$ regular
- $\mathcal{M} = \mathrm{Hess}_A(\mathbb{F}^n) = \{\mathcal{V} = (V_1, \ldots, V_n) \in \mathrm{Flag}(\mathbb{F}^n) \mid AV_i \subseteq V_{i+1}\}$
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Questions and Remarks

- Algorithm is closely related to the QR algorithm on Hessenberg Matrices.
- Can we steer to arbitrary Hessenberg eigenflags?

Examples

Example VI: Restart-Shifts of Krylov Methods

- $\mathcal{M} = \mathrm{Grass}(p, n)$
- $\mathcal{U} = \mathbb{R}^k[t]$
- $x_0 \in \mathbb{R}^n \setminus \{0\}$, $K(x_0) = \langle x_0, Ax_0, \ldots, A^{p-1}x_0 \rangle \in \mathcal{M}$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$\Phi(K(x), u) = u(A)K(x) = K(u(A)x)$$

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Questions and Remarks

- Find shifts to approximate specific eigenspaces (Beattie, Embree, Sorensen, Rossi).

Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$; x_0.

Definition: Reachable set of $x_0 \in \mathcal{M}$

$$R(x_0) := \{x \text{ wich can be reached from } x_0 \text{ in finite many steps}\}$$

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Definition: $k \in \mathbb{N}$, $\Phi_k : \mathcal{M} \times \mathcal{U}^k \to \mathcal{M}$

$$\begin{aligned} \Phi_1(x, u) &= \Phi(x, u) \\ \Phi_k(x, u_1, \ldots, u_k) &= \Phi(\Phi(x, u_1, \ldots, u_{k-1}), u_k) \end{aligned}$$

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Proposition:

$$R(x_0) := \{x \in \mathcal{M} \,|\, \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : x = \Phi_N(x_0, u)\}$$

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Definition: System Semigroup

$$\Gamma_\Sigma := \{\Phi : \mathcal{M} \to \mathcal{M} \,|\, \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : \Phi = \Phi_N(\cdot, u)\}$$

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Proposition: The reachable set of $x_0 \in \mathcal{M}$ is always an orbit of the semigroup action $\alpha : \Gamma_\Sigma \times \mathcal{M} \to \mathcal{M}$, $\alpha(\Phi, x) = \Phi(x)$. I.e

$$R(x_0) = \alpha(\Gamma_\Sigma, x_0)$$

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$$R(x_0) = \alpha(\Gamma_\Sigma, x_0)$$

Corollary: If Γ_Σ is a group the reachable sets form a partition on $\mathcal{M}$.

Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- $A \in \mathbb{F}^{n\times n}$ regular, $\mathcal{M} = \mathbb{FP}^{n-1}, \mathrm{Flag}(\mathbb{F}^n), \mathrm{Hess}_A(\mathbb{F}^n)$, $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
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Proposition: For all Scalar Shifted Inverse Iterations

$$\Gamma_\Sigma := \{\prod_{t=1}^{N} (A - u_t I)^{-1} \,|\, N \in \mathbb{N}, u_t \in \mathbb{F} \setminus \sigma(A)\}$$

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Theorem: (Helmke, J 2002) For $\mathbb{F} = \mathbb{C}$, Γ_Σ is a Group. If A is diagonalizable then Γ_Σ is homeomorphic to $(\mathbb{C}^*)^k$ whereas k is the number of different eigenvalues of A.

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Theorem: (J 2003) For $\mathbb{F} = \mathbb{R}$ there exists an open set of Matrices

$\mathcal{S} \subset \mathbb{R}^{n\times n}$ such that Γ_Σ is not a Group.

Reachable Sets

Example I, Shifted Inverse Iteration on $\mathbb{CP}^{n-1}$

- $A \in \mathbb{C}^{n\times n}$, cyclic (i.e: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \ldots A^{n-1}x\rangle = \mathbb{C}^n$).
- $\mathcal{M} = \mathbb{CP}^{n-1}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

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Theorem: (Helmke, Fuhrmann 2000) Let $\mathbb{F} = \mathbb{C}$ and A be cyclic. There is a bijective correspondence between the closures of the reachable sets $\overline{R(x)}$ and the A-invariant subspaces of $\mathbb{C}^n$.

Controllability

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

Definition: System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is said to be controllable if there exist $x_0 \in \mathcal{M}$ such that every point in $\mathcal{M}$ can be reached from x_0 at least arbitrarily close. (I.e.

$$\exists x_0 \in \mathcal{M} : \quad \overline{R(x_0)} = \mathcal{M}$$

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$$\exists x_0 \in \mathcal{M} : \quad \overline{R(x_0)} = \mathcal{M}$$

Remark: If $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is controllable and Γ_Σ is a group, then every neighbourhood can be reached from every neighbourhood.

Controllability

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Theorem:(Helmke, Fuhrmann 2000)
Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathbb{CP}^{n-1}$ is controllable if and only if A is cyclic (i.e.: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \ldots A^{n-1}x \rangle = \mathbb{C}^n$).

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Corollary: For $\mathbb{F} = \mathbb{C}$, A cyclic. One can steer nearly every initial point to every specific target point.

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Corollary: For $\mathbb{F} = \mathbb{C}$, A cyclic. One can steer nearly every initial point to every specific target point.

Corollary: For $\mathbb{F} = \mathbb{C}$ controllability is a generic property of the Shifted Inverse Iteration. I.e.: It holds true for an open and dense set of matrices $\mathcal{S} \subset \mathbb{C}^{n\times n}$.

Controllability

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Remark:

Let $\mathbb{F} = \mathbb{R}$. If A is not cyclic, the Shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ is not controllable.

Controllability

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Let $\mathbb{F} = \mathbb{R}$. If A is not cyclic, the Shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ is not controllable.

Remark:

For $\mathbb{F} = \mathbb{R}$ it is unknown if controllability is a generic property of the Shifted Inverse Iteration.

Controllability

Example I$\frac{1}{2}$, Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$

- $A \in \mathbb{F}^{n\times n}$, $\mathcal{M} = \mathbb{FP}^{n-1}$
- $p_\alpha(u) = A - uI$, $p_\beta(v,w) = A^2 + vA + wI$
- $\mathcal{U} := \{u \in \mathbb{F}, (v,w) \in \mathbb{F}^2 \,|\, p_\alpha(u), p_\beta(v,w) \in \mathrm{GL}_n(\mathbb{F})\}$
- $\Phi(x,u) = p_{\pi(t)}^{-1}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

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- $\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 \,|\, p_\alpha(u), p_\beta(v, w) \in \mathrm{GL}_n(\mathbb{F})\}$
- $\Phi(x, u) = p_{\pi(t)}^{-1}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

Theorem:(J 2003)
Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$ is controllable if and only if A is cyclic.

Controllability

Example I$\frac{1}{2}$, Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$, $\mathcal{M} = \mathbb{FP}^{n-1}$
- $p_\alpha(u) = A - uI$, $p_\beta(v, w) = A^2 + vA + wI$
- $\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 \,|\, p_\alpha(u), p_\beta(v, w) \in \mathrm{GL}_n(\mathbb{F})\}$
- $\Phi(x, u) = p^{-1}_{\pi(t)}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

Theorem:(J 2003)
Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$ is controllable if and only if A is cyclic.

Corollary: Controllability is a generic property of the Polynomial-Shift Inverse Iteration.

Controllability

Example III, Shifted Inverse Iteration on $\mathrm{Flag}(\mathbb{C}^n)$

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \mathrm{Flag}(\mathbb{C}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

Controllability

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Remark:

Let $n > 2$. The Shifted Inverse Iteration on $\mathrm{Flag}(\mathbb{C}^n)$ is not controllable.

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Remark:
Let $n > 2$. The Shifted Inverse Iteration on $\mathrm{Flag}(\mathbb{C}^n)$ is not controllable.

Corollary: Let $n > 2$. The Shifted Inverse Iteration on the isospectral manifold $M_A = \{Q^* A Q \,|\, Q \in \mathrm{U}_n(\mathbb{C})\}$ is not controllable.

Controllability

Example III, Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
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Theorem:(Helmke, J 2002)
Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{F}^n)$ is controllable (for A) if and only if Shifted Invesre Iteration on $\mathbb{FP}^{n-1}$ is controllable (for A).

Controllability

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Theorem:(Helmke, J 2002)
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Corollary: Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{C}^n)$ is controllable if and only if A is cyclic.

Controllability

Example III, Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{F}^n)$)

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Corollary: Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{C}^n)$ is controllable if and only if A is cyclic.

Corollary: Let $\mathbb{F} = \mathbb{C}$. Controllability is a generic property of the Shifted Inverse Iteration on $\mathrm{Hess}_A(\mathbb{C}^n)$.

Current and Future Work

- Is shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ resp. $\mathrm{Hess}_A(\mathbb{R}^n)$ generic?

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- Characterizations of system semigroups
- Criteria for controllability
- Adherence structure of reachable sets
- Constructive controllability

Thank you for your attention

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