

Numerical Algorithms as Discrete-Time Control Systems

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Contents:



Numerical algorithms as discrete-time control systems

Contents:

- Introduction



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Contents:

- Introduction
- Examples



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Contents:

- Introduction
- Examples
- Reachable Sets



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- Introduction
- Examples
- Reachable Sets
- Controllability



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- Introduction
- Examples
- Reachable Sets
- Controllability
- Current and Future Work



Introduction

Iterative Algorithms

$\Phi : \mathcal{M} \rightarrow \mathcal{M}$ iteration map

$$x_t \in \mathcal{M}, \quad x_t \longrightarrow x_{t+1} \in \mathcal{M}$$



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Shifted Iterative Algorithms

$\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ iteration map, \mathcal{U} set of shift parameters

$$x_t \in \mathcal{M}, u_t \in \mathcal{U} \quad (x_t, u_t) \longrightarrow x_{t+1} \in \mathcal{M}$$



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Example: Shifted Inverse Iteration on \mathbb{S}^{n-1} .

$A \in \mathbb{R}^{n \times n}$, $\mathcal{M} = \mathbb{S}^{n-1}$, $\mathcal{U} = \mathbb{R} \setminus \sigma(A)$.

Iteration Step

$$\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}, \quad \Phi(x, u) = \frac{(A - uI)^{-1}x}{\|(A - uI)^{-1}x\|}$$

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Shifted Iterative Algorithms

$\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ iteration map, \mathcal{U} set of shift parameters

$$x_t \in \mathcal{M}, u_t \in \mathcal{U} \quad (x_t, u_t) \longrightarrow x_{t+1} \in \mathcal{M}$$

Observation: $(\mathcal{M}, \mathcal{U}, \Phi)$ describes a discrete-time control System:

$$\begin{aligned} x_0 &\in \mathcal{M} \\ x_{t+1} &= \Phi(x_t, u_t) \end{aligned}$$

Introduction

Motivation:



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- Control theoretical description of numerical algorithms



Introduction

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- Better understanding of algorithms



Introduction

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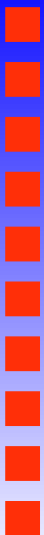
- Control theoretical description of numerical algorithms
- Better understanding of algorithms
- Optimization of shift strategies using control theoretical tools



Introduction

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- Optimization of shift strategies using control theoretical tools
- Justify existing shift strategies



Introduction

Motivation:

- Control theoretical description of numerical algorithms
- Better understanding of algorithms
- Optimization of shift strategies using control theoretical tools
- Justify existing shift strategies
- Creation of new algorithms



Examples

Example I: Shifted Inverse Iteration on Projective Spaces

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}\mathbb{P}^{n-1} = \{\text{Set of one dimensional subspaces of } \mathbb{F}^n\}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by

$$\Phi(\mathcal{X}, u) = (A - uI)^{-1}\mathcal{X}$$

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Questions and remarks:

- Can we find feedback controls to reach eigenvectors by arbitrary initial points?
- Can we find feedback controls to reach specific eigenvectors by arbitrary initial points?

Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

- $A \in \mathbb{R}^{n \times n}$
- $\mathcal{M} = \text{Grass}(p, n) = \{\text{Set of } p\text{-dimensional subspaces of } \mathbb{R}^n\}$
- \mathcal{U} set of feedback maps $F : \text{ST}(p, n) \rightarrow \mathbb{R}^{p \times p}$

$$\mathcal{U} = \{\forall X \in \text{ST}(p, n), \forall M \in \text{GL}_p(\mathbb{R}) : F(XM) = M^{-1}F(X)M\}$$

- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by procedure

- 1) Choose $X \in \text{ST}(p, n)$ such that $\langle X \rangle = \mathcal{X}$
- 2) Solve $AX^+ - X^+F(X) = X$
- 3) $\Phi(\mathcal{X}, F) := \mathcal{X}^+ := \langle X^+ \rangle$

Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

Questions and remarks

- The choice $F = R$ with $R(X) = (X^T X)^{-1} X^T A X$ leads to Grassmann Rayleigh Quotient Iteration (Absil, Mahony, Sepulchre, Van Dooren, 2002).
- Does the Grassmann Rayleigh Quotient Iteration has global convergence properties?
- Can we find feedback laws for global convergence?
- Can we find feedback laws for global convergence to a specific eigenspace?

Examples

Example III: Shifted Inverse Iteration on Flag Manifolds

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \text{Flag}(\mathbb{F}^n) = \{\mathcal{V} = (V_1, \dots, V_n) \mid V_i \subseteq V_{i+1}, \dim_{\mathbb{F}} V_i = i\}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by

$$\Phi(\mathcal{V}, u) = ((A - uI)^{-1}V_1, \dots, (A - uI)^{-1}V_n)$$



Examples

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Questions and Remarks

- Algorithm is closely related to the QR algorithm (on isospectral manifolds).
- Can we steer to arbitrary eigenflags?

Examples

Example IV: Shifted QR Algorithm on Isospectral Manifolds

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \{Q^* A Q \mid Q \in U_n(\mathbb{C})\}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by

$$\Phi(X, u) = (X - uI)_{U_n(\mathbb{C})}^* (X - uI) (X - uI)_{U_n(\mathbb{C})}$$

where $(X - uI)_{U_n(\mathbb{C})}$ is the unitary factor of the QR decomposition of $(X - uI)$.

Examples

Example V: Shifted Inverse Iteration on Hessenberg Manifolds

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n) = \{\mathcal{V} = (V_1, \dots, V_n) \in \text{Flag}(\mathbb{F}^n) \mid AV_i \subseteq V_{i+1}\}$
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Questions and Remarks

- Algorithm is closely related to the QR algorithm on Hessenberg Matrices.
- Can we steer to arbitrary Hessenberg eigenflags?

Examples

Example VI: Restart-Shifts of Krylov Methods

- $\mathcal{M} = \text{Grass}(p, n)$
- $\mathcal{U} = \mathbb{R}^k[t]$
- $x_0 \in \mathbb{R}^n \setminus \{0\}$, $K(x_0) = \langle x_0, Ax_0, \dots, A^{p-1}x_0 \rangle \in \mathcal{M}$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by

$$\Phi(K(x), u) = u(A)K(x) = K(u(A)x)$$

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Questions and Remarks

- Find shifts to approximate specific eigenspaces (Beattie, Embree, Sorensen, Rossi).

Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi); x_0$.

Definition: **Reachable set** of $x_0 \in \mathcal{M}$

$R(x_0) := \{x \text{ which can be reached from } x_0 \text{ in finite many steps}\}$



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Definition: $k \in \mathbb{N}$, $\Phi_k : \mathcal{M} \times \mathcal{U}^k \rightarrow \mathcal{M}$

$$\Phi_1(x, u) = \Phi(x, u)$$

$$\Phi_k(x, u_1, \dots, u_k) = \Phi(\Phi(x, u_1, \dots, u_{k-1}), u_k)$$



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Proposition:

$$R(x_0) := \{x \in \mathcal{M} \mid \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : x = \Phi_N(x_0, u)\}$$

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Definition: System Semigroup

$$\Gamma_{\Sigma} := \{\Phi : \mathcal{M} \rightarrow \mathcal{M} \mid \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : \Phi = \Phi_N(\cdot, u)\}$$



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Proposition: The reachable set of $x_0 \in \mathcal{M}$ is always an orbit of the semigroup action $\alpha : \Gamma_\Sigma \times \mathcal{M} \rightarrow \mathcal{M}$, $\alpha(\Phi, x) = \Phi(x)$. I.e

$$R(x_0) = \alpha(\Gamma_\Sigma, x_0)$$

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$$R(x_0) = \alpha(\Gamma_\Sigma, x_0)$$

Corollary: If Γ_Σ is a group the reachable sets form a partition on \mathcal{M} .

Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- $A \in \mathbb{F}^{n \times n}$ regular, $\mathcal{M} = \mathbb{F}\mathbb{P}^{n-1}, \text{Flag}(\mathbb{F}^n), \text{Hess}_A(\mathbb{F}^n)$,
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Proposition: For all Scalar Shifted Inverse Iterations

$$\Gamma_{\Sigma} := \left\{ \prod_{t=1}^N (A - u_t I)^{-1} \mid N \in \mathbb{N}, u_t \in \mathbb{F} \setminus \sigma(A) \right\}$$

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Theorem: (Helmke, J 2002) For $\mathbb{F} = \mathbb{C}$, Γ_{Σ} is a Group. If A is diagonalizable then Γ_{Σ} is homeomorphic to $(\mathbb{C}^*)^k$ whereas k is the number of different eigenvalues of A .

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Theorem: (J 2003) For $\mathbb{F} = \mathbb{R}$ there exists an open set of Matrices

$\mathcal{S} \subset \mathbb{R}^{n \times n}$ such that Γ_{Σ} is not a Group.

Reachable Sets

Example I, Shifted Inverse Iteration on $\mathbb{C}\mathbb{P}^{n-1}$

- $A \in \mathbb{C}^{n \times n}$, cyclic (i.e: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \dots, A^{n-1}x \rangle = \mathbb{C}^n$).
- $\mathcal{M} = \mathbb{C}\mathbb{P}^{n-1}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

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Theorem: (Helmke, Fuhrmann 2000) Let $\mathbb{F} = \mathbb{C}$ and A be cyclic. There is a bijective correspondence between the closures of the reachable sets $\overline{R(x)}$ and the A -invariant subspaces of \mathbb{C}^n .

Controllability

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

Definition: System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is said to be **controllable** if there exist $x_0 \in \mathcal{M}$ such that every point in \mathcal{M} can be reached from x_0 at least arbitrarily close. (i.e.

$$\exists x_0 \in \mathcal{M} : \overline{R(x_0)} = \mathcal{M}$$



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Remark: If $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is controllable and Γ_Σ is a group, then every neighbourhood can be reached from every neighbourhood.

Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$

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Theorem:(Helmke, Fuhrmann 2000)

Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathbb{C}\mathbb{P}^{n-1}$ is controllable if and only if A is cyclic (i.e.: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \dots, A^{n-1}x \rangle = \mathbb{C}^n$).

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Corollary: For $\mathbb{F} = \mathbb{C}$, A cyclic. One can steer nearly every initial point to every specific target point.

Controllability

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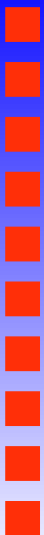
Corollary: For $\mathbb{F} = \mathbb{C}$ controllability is a generic property of the Shifted Inverse Iteration. I.e.: It holds true for an open and dense set of matrices

$\mathcal{S} \subset \mathbb{C}^{n \times n}$.

Controllability

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Remark:

Let $\mathbb{F} = \mathbb{R}$. If A is not cyclic, the Shifted Inverse Iteration on $\mathbb{R}\mathbb{P}^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $\mathbb{R}\mathbb{P}^{n-1}$ is not controllable.

Controllability

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Remark:

For $\mathbb{F} = \mathbb{R}$ it is unknown if controllability is a generic property of the Shifted Inverse Iteration.

Controllability

Example I $_{\frac{1}{2}}$, Polynomial-Shift Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$, $\mathcal{M} = \mathbb{F}\mathbb{P}^{n-1}$
- $p_{\alpha}(u) = A - uI$, $p_{\beta}(v, w) = A^2 + vA + wI$
- $\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 \mid p_{\alpha}(u), p_{\beta}(v, w) \in \text{GL}_n(\mathbb{F})\}$
- $\Phi(x, u) = p_{\pi(t)}^{-1}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

Controllability

Example I $_{\frac{1}{2}}$, Polynomial-Shift Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$

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Theorem:(J 2003)

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Polynomial-Shift Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$ is controllable if and only if A is cyclic.

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Corollary: Controllability is a generic property of the Polynomial-Shift Inverse Iteration.

Controllability

Example III, Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \text{Flag}(\mathbb{C}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$



Controllability

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Remark:

Let $n > 2$. The Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$ is not controllable.

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Corollary: Let $n > 2$. The Shifted Inverse Iteration on the isospectral manifold $M_A = \{Q^* A Q \mid Q \in U_n(\mathbb{C})\}$ is not controllable.

Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
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Theorem:(Helmke, J 2002)

Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$ is controllable (for A) if and only if
Shifted Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$ is controllable (for A).

Controllability

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Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

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Corollary: Let $\mathbb{F} = \mathbb{C}$. Controllability is a generic property of the Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{C}^n)$.

Current and Future Work

- Is shifted Inverse Iteration on \mathbb{RP}^{n-1} resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?



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- Is shifted Inverse Iteration on \mathbb{RP}^{n-1} resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?
- Characterizations of system semigroups



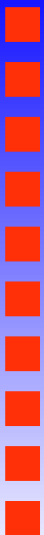
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- Characterizations of system semigroups
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- Characterizations of system semigroups
- Criteria for controllability
- Adherence structure of reachable sets
- Constructive controllability



Thank you for your attention

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