Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard*

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Abstract

The lower and average spectral radii measure the minimal and average growth rates, respectively, of long products of matrices taken from a finite set. The logarithm of the average spectral radius is traditionally called the Lyapunov exponent. When one forms these products in the max-algebra, we obtain quantities that measure the performance of Discrete Event Systems. We show that approximating the lower and average max-algebraic spectral radii is NP-hard.

1 Introduction

For all positive real numbers p, the semiring $\mathbb{R}_p$ is the set of real nonnegative numbers, $\mathbb{R}^+$, equipped with the addition

$$a +_p b \stackrel{\text{def}}{=} (a^p + b^p)^{\frac{1}{p}} \quad , \tag{1}$$

together with the usual multiplication. This family of semirings was introduced independently by Maslov and Pap (see e.g. [17, 19] and the references therein). It has the following remarkable property: all the semirings $\mathbb{R}_p$ are isomorphic to the ordinary semiring $\mathbb{R}_1$ of real nonnegative numbers equipped with the usual operations. Letting p tend to ∞ in (1), we obtain:

$$a +_\infty b = \max(a, b) \quad .$$

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The corresponding semiring $\mathbb{R}_\infty$ (the set $\mathbb{R}^+$, equipped with $+_\infty$ and the usual multiplication) is the max-times semiring or "max-algebra", whose role in dynamic programming, discrete event system theory, optimal control, and asymptotic analysis is well known (see e.g. [1, 18, 17, 13, 16]). In contrast to the semirings $\mathbb{R}_p$ for finite p, this semiring is not isomorphic to $\mathbb{R}_1$. In discrete event systems applications, the max-algebra more frequently appears in an isomorphic additive form, the semiring $\mathbb{R}_{\max}$, which is the set $\mathbb{R}\cup\{-\infty\}$, equipped with max as addition, and $+$ as multiplication. The isomorphism is given by $x \mapsto \log x : \mathbb{R}_\infty \to \mathbb{R}_{\max}$. To emphasize the parallel with existing results, we will state all our results in terms of $\mathbb{R}_\infty$ (see Table 1).

In the sequel, we will use the familiar algebraic notation in the context of the semiring $\mathbb{R}_p$, without further comments: e.g., if $A \in \mathbb{R}_p^{r\times s}$ and $B \in \mathbb{R}_p^{s\times t}$, AB is the $r \times t$ matrix with entries $A_{ij} = A_{i1}B_{1j} +_p \cdots +_p A_{is}B_{sj}$. Let $\|\cdot\|$ denote a (conventional) norm on $\mathbb{R}^{r\times r}$. To a finite set of matrices $\{A_1,\dots,A_l\} \subset \mathbb{R}_p^{r\times r}$, we associate:

$$\rho_{\max}(A_1,\dots,A_l) \stackrel{\text{def}}{=} \lim_{k\to\infty} \max_{i_1,\dots,i_k\in\{1,\dots,l\}} \|A_{i_1}\cdots A_{i_k}\|^{\frac{1}{k}} \ , \tag{2a}$$

$$\rho_{\min}(A_1,\dots,A_l) \stackrel{\text{def}}{=} \lim_{k\to\infty} \min_{i_1,\dots,i_k\in\{1,\dots,l\}} \|A_{i_1}\cdots A_{i_k}\|^{\frac{1}{k}} \ , \tag{2b}$$

$$\rho_{\mathbb{E}}(A_1,\dots,A_l) \stackrel{\text{def}}{=} \text{a.s.} \lim_{k\to\infty} \|A_{i_1}\cdots A_{i_k}\|^{\frac{1}{k}} \ , \tag{2c}$$

where in (2c), $i_1, i_2, \dots$ is a random i.i.d. sequence of elements of $\{1,\dots,l\}$ drawn with the uniform distribution. The existence and values of all the limits in (2) are clearly independent of the choice of the norm. In particular, we may take the norm $\|A\| = \max_{1\le i\le r} |A_{i1}| +_p |A_{i2}| +_p \cdots +_p |A_{ir}$ which satisfies $\|AB\| \le \|A\|\|B\|$. Then, by a classical argument, the existence of the limit (2a) follows easily from the fact that the sequence $w_k = \max_{i_1,\dots,i_k} \|A_{i_1}\cdots A_{i_k}\|$ is submultiplicative, i.e. $w_{k+r} \le w_k w_r$. The existence of $\rho_{\min}$ is proved by the same argument. As shown in [7] and [1, Chap. 7], the existence of $\rho_{\mathbb{E}}$ follows from Kingman's subadditive ergodic theorem. We will call $\rho_{\max}$, $\rho_{\min}$ and $\rho_{\mathbb{E}}$ the *upper*, *lower*, and *average spectral radius* of $\{A_1,\dots,A_l\}$, respectively. The logarithm of $\rho_{\mathbb{E}}$ is traditionally called the *Lyapunov exponent* or *Lyapunov indicator*. We note that, trivially,

$$\rho_{\min} \le \rho_{\mathbb{E}} \le \rho_{\max} \ . \tag{3}$$

When $p = 1$, both the upper and average spectral radius are much studied quantities which are notoriously difficult to compute or approximate in practice. In [4, Th. 1 and 2], it was shown that even in the case of two matrices A_0, A_1 with entries in $\{0,1\}$, approximating $\rho_{\max}$, $\rho_{\min}$ and $\rho_{\mathbb{E}}$ is NP-hard. Using the fact that all the semirings $\mathbb{R}_p$ with finite p are isomorphic, it follows that analogous results hold for all semirings $\mathbb{R}_p$. In [4, Th. 1 and 2] it was also shown that, if we allow the entries of A_0, A_1 to be in $\mathbb{Z}$, there is no algorithm that can distinguish between instances with $\rho_{\min} = 0$ from instances with $\rho_{\min} = 1$, and similarly for $\rho_{\mathbb{E}}$. In particular, $\rho_{\min}$ and $\rho_{\mathbb{E}}$ cannot be approximated algorithmically, and the problem of deciding whether they are zero is undecidable. The situation for $\rho_{\max}$ is different. Using the inequalities derived in [8], it is immediate to see that $\rho_{\max}$ can be approximated algorithmically to arbitrary precision. One such algorithm is given in [15].

When $p = \infty$, the quantities $\rho_{\max}$, $\rho_{\min}$, and $\rho_{\mathbb{E}}$ have been much studied by the Discrete Event Systems community. As shown in [1], the Lyapunov exponent $\log \rho_{\mathbb{E}}$ measures the *cycle time* (inverse of the throughput) of random max-plus linear discrete event systems. Its study (structural properties, bounds, etc.) is one of the central themes of [1]. The logarithm of $\rho_{\max}$ was called worst case Lyapunov exponent in [10], for it measures the worst case cycle time of certain max-plus linear discrete event systems. For a dual reason, the logarithm of $\rho_{\min}$ was called optimal case Lyapunov exponent. Since the maximization operation which is involved in the definition of $\rho_{\max}$ is somehow compatible with the structure laws of the max-algebra, $\rho_{\max}$ can be computed quite easily: as shown in [10], it coincides with the spectral radius of the single matrix $A = A_1 +_\infty \cdots +_\infty A_l$, which can be computed in polynomial time. So far, the basic general technique to compute $\rho_{\min}$ and $\rho_{\mathbb{E}}$ consists of using an "induced Markov chain" construction in the max-algebraic projective space [1, §8.4],[10, §VII]: when this chain is finite, both $\rho_{\min}$ and $\rho_{\mathbb{E}}$ can be computed with a number of arithmetic operations which is polynomial in the number of states of the chain. In some other special cases, $\rho_{\mathbb{E}}$ can also be computed via generating series techniques [14], or, as illustrated in [6], by finding a closed form expression for the invariant measure of the above mentioned Markov chain, which is denumerable, in general. A different approach was used in [2]: we can define more generally $\rho_{\mathbb{E}}$ in (2c) by drawing $i_1, \ldots, i_k$ with a i.i.d. but non uniform distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ where π_j is the probability of $\{i_1 = j\}$. Under some technical restrictions, $\rho_{\mathbb{E}}$ is an analytic function of $\pi_1, \ldots, \pi_l$ near $\pi = (1, 0, \ldots, 0)$, and the coefficients of its power series expansion can be effectively computed. When this series is still convergent at $\pi = (1/l, \ldots, 1/l)$, this gives a way of approximating the average spectral radius.

The purpose of this paper is to analyze the complexity of computing $\rho_{\min}$ and $\rho_{\mathbb{E}}$ when $p = \infty$.

In a first section (§2 below) we show that, when $p = \infty$, approximating $\rho_{\min}$ or $\rho_{\mathbb{E}}$ is NP-hard. Our proof of this result is based on a reworking of the argument of [4, Proof of Th. 1], which, in loose terms, builds an automaton whose number of accepting paths measures the number of satisfied clauses in a given instance of the satisfiability problem SAT. This argument does not work when $p = \infty$: since $+_\infty$ is idempotent (i.e. $a +_\infty a = a$), several paths count as one. However, a variant of the reduction of [5, Proof of Th. 2] can be used to prove that approximating $\rho_{\min}$ and $\rho_{\mathbb{E}}$ is NP-hard.

In a second section (§3 below) we give a simple, independent, geometrical argument that shows that computing $\rho_{\min}$ is NP-hard. The argument is based on an intuitive interpretation of products of matrices in terms of the height of a heap of pieces. In [11, 12], it was shown that the total height of a Tetris-like heap of k pieces is equal to $\log \|A_{i_1} \ldots A_{i_k}\|$, where $A_{i_1}, \ldots, A_{i_k}$ are matrices associated to the pieces, and $\|A\| = \max_{ij} A_{ij}$. When all the pieces are of height

	$\rho_{\max}$	$\rho_{\min}$ and $\rho_{\mathbb{E}}$
$(\mathbb{R}, +, \times)$	Approximation algorithm [8]	No approximation algorithm [4]
$\mathbb{R}_p = (\mathbb{R}^+, +_p, \times)$ (finite p)	Approximation is NP-hard [4]	Approximation is NP-hard [4]
$\mathbb{R}_\infty = (\mathbb{R}^+, +_\infty, \times)$	Exact polynomial time algorithm [10]	Approximation is NP-hard [this paper]

Table 1: Summary of complexity results available for $\rho_{\max}, \rho_{\min}, \rho_{\mathbb{E}}$.

1, $\log \rho_{\min}$ coincides with the inverse of the maximal cardinality of the sets of mutually disjoint pieces, hence, the NP-hardness of computing $\rho_{\min}$ follows from the NP-hardness of Set Packing [9].

2 Reduction from SAT

In the remaining part of the paper, we will assume that $p = \infty$ and we will use the matrix norm $\|A\| = \max_{ij} A_{ij}$.

Let $\Sigma \mapsto \rho(\Sigma)$ be a non-negative function that we wish to compute. We say that ρ is *polynomial-time approximable* if there exists an algorithm which, for every rational numbers $\epsilon, \epsilon' > 0$ and every Σ, returns an approximation $\rho^*(\Sigma, \epsilon, \epsilon')$ such that $|\rho^* - \rho| \leq \epsilon\rho + \epsilon'$, in time polynomial in the description size of ϵ, ϵ' and Σ. This allows for both an absolute and a relative error.

Theorem 1. *Unless P=NP, the lower and average spectral radii of pairs of matrices with entries in* $\{0, 1\}$ *are not polynomial-time approximable.*

Proof. Let A_1, A_2 be square matrices with entries in $\{0, 1\}$. We claim that

$$\rho_{\min}(A_1, A_2) = \rho_{\mathbb{E}}(A_1, A_2) \in \{0, 1\} \ . \tag{4}$$

Indeed, in the max-algebra, any product of matrices with entries in $\{0, 1\}$ gives a matrix with entries in $\{0, 1\}$. A fortiori, $\|A_{i_1} \cdots A_{i_k}\| \in \{0, 1\}$ for all $i_1, \ldots, i_k$. Hence, if none of the products $A_{i_1} \cdots A_{i_k}$ is 0, $\rho_{\min}(A_1, A_2) = \rho_{\mathbb{E}}(A_1, A_2) = 1$. But if one of these products is 0, then $\rho_{\min}(A_1, A_2) = 0$ and the product that gives 0 will appear almost surely as a factor of any infinite, uniform, i.i.d., random product $A_{j_1} A_{j_2} \ldots$, which implies that $\rho_{\mathbb{E}}(A_1, A_2) = 0$.

Due to (4), it suffices to establish the theorem for $\rho_{\min}$. Any polynomial time approximation algorithm for $\rho_{\min}$ gives a polynomial time algorithm for distinguishing the cases $\rho_{\min} = 0$ and $\rho_{\min} = 1$. Thus, in order to establish the theorem, it suffices to show that the problem of determining whether $\rho_{\min}(A_1, A_2) = 0$ is NP-hard, even for the case of binary matrices. The proof is by reduction from SAT and is inspired by [5, Proof. of Th. 2].

Consider an instance of SAT [9], with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. We can write each clause C_i as $C_i = C_{i,1}$ `or` $\cdots$ `or` $C_{i,n}$, where $C_{i,j}$ is either x_j, or `not`(x_j), or the Boolean constant `false`.

Let $C = C_1$ `and` $\cdots$ `and` C_m. For any $y \in \{$`true`, `false`$\}$ and $k \in \{1, \ldots, n\}$, let $M_k(y)$ denote the diagonal $m \times m$ Boolean matrix with diagonal entries:

$$(M_k(y))_{i,i} = \begin{cases} 1 & \text{if } C_{i,k}(y) = \texttt{false}, \\ 0 & \text{if } C_{i,k}(y) = \texttt{true}. \end{cases}$$

Then, for all $x \in \{\texttt{true}, \texttt{false}\}^n$,

$$M_1(x_1) \cdots M_n(x_n) = 0 \text{ if and only if } x \text{ satisfies } C. \tag{5}$$

Let U denote the $m \times m$ matrix whose entries are all equal to one. We now consider the $nm \times nm$ matrices $A_1 = \mathcal{M}(\texttt{false})$, $A_2 = \mathcal{M}(\texttt{true})$, where

$$\mathcal{M}(y) = \begin{pmatrix} & UM_1(y) & & & \\ & & M_2(y) & & \\ & & & \ddots & \\ & & & & M_{n-1}(y) \\ M_n(y)U & & & & \end{pmatrix}$$

(the blocks which are not shown are zero). We claim that

$$\begin{aligned} C \text{ is not satisfiable} &\implies \rho_{\min}(A_1, A_2) = 1 \ , \\ C \text{ is satisfiable} &\implies \rho_{\min}(A_1, A_2) = 0 \ . \end{aligned}$$

In order to establish our claim, note first that for all k, and for all Boolean sequences y of length kn, $\mathcal{M}(y_1) \cdots \mathcal{M}(y_{kn})$ is a block diagonal matrix with diagonal blocks:

$$\begin{aligned} B_{1,k} &= UM_1(y_1) \cdots M_n(y_n) UUM_1(y_{n+1}) \cdots M_n(y_{kn})U \ , && \text{(6a)} \\ B_{2,k} &= M_2(y_1) \cdots UM_1(y_n) M_2(y_{n+1}) \cdots UM_1(y_{kn}) \ , && \text{(6b)} \\ &\vdots \\ B_{n,k} &= M_n(y_1)U \cdots M_{n-1}(y_n) M_n(y_{n+1}) \cdots M_{n-1}(y_{kn}) \ . && \text{(6c)} \end{aligned}$$

Assume that C is not satisfiable. Using (5), we get that $B_{1,k} = 1$ for all possible Boolean sequences y of length kn. This implies that $\rho_{\min}(A_1, A_2) = 1$.

Next, assume that C is satisfied by the Boolean sequence $x_1 \dots x_n$, and consider the infinite sequence of period $n+1$:

$$y = x_1 \dots x_n \sharp x_1 \dots x_n \sharp x_1 \dots x_n \sharp \dots \ ,$$

where $\sharp$ can take an arbitrary Boolean value. For $k = n+1$, each of the n products that give $B_{1,k}, \dots, B_{n,k}$ in (6) contains a factor of the form $M_1(x_1) \dots M_n(x_n)$. Since $M_1(x_1) \cdots M_n(x_n) = 0$ we conclude that $\rho_{\min}(A_1, A_2) = 0$. □

Remark. It is not known whether the statement of the theorem remains valid if we require that the matrices have positive entries, or have a fixed, large enough, dimension.

3 Reduction from SET PACKING

In Discrete Event Systems applications, the quantity of interest is the logarithm of $\rho_{\min}$, rather than $\rho_{\min}$. In this section we show that the following problem is NP-hard.

Problem (COMPUTING $\rho_{\min}$).
Instance: *Matrices* $A_1, \dots, A_l \in \{0,1,2\}^{n\times n}$, *a rational number* q.
Question: *Does* $\log_2\ \rho_{\min}(A_1, \dots, A_l) < q$?

Theorem 2. COMPUTING $\rho_{\min}$ *is NP-hard.*

Proof. The proof is based on a simple geometrical argument that involves a Tetris-like heap of pieces.

Consider a horizontal axis with $n \geq 1$ slots labelled $\{1, 2, \dots, n\}$. A piece is a solid, possibly disconnected, block of height one occupying some of the slots. Consider now a set of pieces $A = \{a_1, \dots, a_l\}$ each piece a_i being defined by the subset $R(a_i) \subseteq \{1, 2, \dots, n\}$ of slots it occupies. To an ordered sequence of pieces $w = a_{i_1} \dots a_{i_k}$ we associate a heap by piling up the pieces in the given order on a horizontal ground. Pieces are only subject to vertical translations and occupy the lowest possible position that is above the ground and above the pieces previously piled up. The height of a heap w on slot i is denoted by $h_i(w)$. The height $h(w)$ of a heap $w = a_{i_1} \dots a_{i_k}$ is the largest of the heights on all slots. For instance, when $n = 3$, $A = \{a_1, a_2, a_3\}$, $R(a_1) = \{1, 2\}, R(a_2) = \{3\}, R(a_3) = \{1, 3\}$, and $w = a_1a_3a_2a_1a_3$, we obtain the heap with height $h(w) = 4$ depicted at the right of Fig. 1.

Figure 1: A sequence of pieces and its associated heap.

To $k \geq 1$, we associate the lowest possible height of a heap of k pieces taken from A

$$\lambda_k = \min\{h(a_{i_1} \dots a_{i_k}) \mid a_{i_1}, \dots, a_{i_k} \in A\} \ .$$

We claim that the limit

$$\lambda = \lim_{k\to+\infty} \frac{\lambda_k}{k}$$

is equal to $1/M$, where M is equal to the maximal number of pieces in a heap of height one. Indeed, a heap w with k pieces has at most M pieces per height level, and thus $k \leq h(w) \times M$. This implies that $\lambda_k/k \geq 1/M$. Moreover, if z is a heap of height one with M pieces, the heap w obtained by repeating k-times z satisfies $h(w) = k$ and contains kM pieces. Thus, $\lambda_{kM}/kM \leq 1/M$, and the claim is established.

For instance, for the pieces a_1, a_2, and a_3 depicted in Fig. 1, the set of heaps of height one is $\{a_1, a_2, a_3, a_1a_2\}$, and $\lambda = 1/2$.

Consider now the following NP-hard problem (see [9]):

Problem (Set Packing).
Instance: a collection C of finite sets, a positive integer $K \leq |C|$.
Question: does C contain at least K mutually disjoint sets?

Modulo some changes of notation, this result can be rephrased in our framework by saying that, for a given set of pieces $A = \{a_1, \ldots, a_l\}$ and a positive integer $K \leq |A| = l$, the problem of determining if K is larger than the maximal number of pieces in a heap of height one, is NP-hard. Since $\lambda = 1/M$ an analogous statement is possible for λ.

To conclude, we describe a connection between heights of heaps of pieces and norms of products of matrices in the semiring $\mathbb{R}_\infty$. To a piece $a_k \in A$ occupying the slots $R(a_k) \subseteq \{1, 2, \ldots, n\}$ we associate the matrix $A_k \in \{0, 1, 2\}^{n\times n}$ whose entries are identical to the entries of the identity matrix, with the exception of the elements $(A_k)_{ij}$ which are set to 2 whenever $i, j \in R(a_k)$. It is shown in [12] that the height $h(w)$ of the heap $w = a_{i_1} \ldots a_{i_k}$ is given by

$$h(w) = \log_2 \|A_{i_1} \cdots A_{i_k}\|.$$

From this it follows that

$$\lambda = \lim_{k\to\infty} \min_{i_1,\ldots,i_k\in\{1,\ldots,l\}} \log_2 \|A_{i_1} \cdots A_{i_k}\| = \log_2 \rho_{\min}(A_1, \ldots, A_l).$$

Since the instance of Computing $\rho_{\min}$ is constructed from the instance of Set Packing in polynomial time, it follows that Computing $\rho_{\min}$ is NP-hard. □

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