The simultaneous stabilizability question of three linear systems is undecidable

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Abstract

Abstract. We show that the simultaneous stabilizability of three linear systems, that is the question of knowing whether three linear systems are simultaneously stabilizable, is an undecidable question. It is undecidable in the sense that it is not possible to find necessary and sufficient conditions for simultaneous stabilization of the three systems that involve only a combination of arithmetical operations (additions, substractions, multiplications and divisions), logical operations ('and' and 'or') and sign tests operations (equal to, greater than, greater than or equal to,...) on the coefficients of the three systems.

Key words: simultaneous stabilization, decidability, decidable question.

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1 Introduction

When is it possible to find a single rational controller that simultaneously stabilizes three, or more, linear systems? At present nobody is capable of giving a comprehensive answer to this question and this paper is devoted to it.

We first clearly state what we mean by 'simultaneously stabilizes'.

We restrict our attention to single-input single-output linear, time invariant systems that are rational but not necessarily causal. By 'system' we mean systems that satisfy these conditions. Each of our k systems is represented in the frequency domain by a real rational function $p_i(s) \in \mathrm{R}(s)$ $(i = 1, ..., k)$. To control our systems we allow ourselves the use of a linear, time invariant rational controller. Such a controller is also represented in the frequency domain by a real rational function $c(s) \in \mathrm{R}(s)$. By 'controller' we mean controllers that satisfy these conditions.
Finally, our goal is to achieve closed loop internal stability with the controller. That is, we require that the four closed loop transfer functions $p_i(s)c(s)(1+p_i(s)c(s))^{-1}, p_i(s)(1+p_i(s)c(s))^{-1}, c(s)(1+p_i(s)c(s))^{-1}$ and $(1+p_i(s)c(s))^{-1}$ associated to the k systems $p_i(s)$ $(i = 1, ..., k)$ have no poles in the extended right half plane. A controller that satisfies that condition is said to be a *simultaneous stabilizing controller* of $p_i(s)$ $(i = 1, ..., k)$.

Our question is:

> *Under what conditions on the systems* $p_i(s)$ $(i = 1, ..., k)$ *does there exist a simultaneous stabilizing controller?*

This problem has been formulated for some years now (see, for example, [18] or [15]) and, despite many efforts, it has remained unsolved ever since for three or more systems.

The case $k = 1$ –the stabilization of a single plant– is easily dealt with. There always exists a stabilizing controller for a single system. Moreover, once a stabilizing controller of a single system is found, it is easy to parametrize the infinite set of all stabilizing controllers of this system. This parametrization is known as the Youla-Kucera parametrization and was discovered in 1976, see [22].

By using the Youla-Kucera parametrization, it is possible to rephraze the simultaneous stabilization question of two systems into one of strong stabilization –stabilization with a stable controller– of a single system. The strong stabilization question was solved in 1974 by D. Youla *et al.* [21] and has a surprising and elegant solution: a system is stabilizable by a stable controller if and only if it has an even number of real unstable zeros between each pair of real unstable poles. What is really remarkable in this condition, known as the *parity interlacing property*, is that it involves only *real* poles and zeros and not the complex ones.
It was proved by B. Anderson [1] that the parity interlacing condition can be checked by performing only elementary arithmetic operations (additions, substractions, multiplications and divisions) on the coefficients of the system: the real poles and zeros do not have to be computed explicitly.
Thus the simultaneous stabilizability question for $k = 2$ is fully solved; we can first translate it into a strong stabilization question by using the Youla-Kucera parametrization and then check the parity interlacing property.

The picture is different for three systems. For $k = 3$ the question is open and is nowadays commonly referred to as the *simultaneous stabilization problem for three systems*. It is recognized as one of the hard open problem in linear system theory and has attracted much attention this last decade.
The presently available results are in the form of necessary conditions [8], [19], sufficient conditions [20] [5], [12] or untractable necessary and sufficient conditions [8], [7]. Despite all these efforts, *there exist at present no tractable necessary and sufficient conditions for testing the simultaneous stabilizability of three or more systems.*

Our ambition in this paper is to explain one of the reasons for the control community's failure to produce such simultaneous stabilizability conditions. We show in the central theorem of the paper –Theorem 9– that it is hopeless to search for a general tractable criterion to test the simultaneous stabilizability of three systems because such a criterion... does not exist.

The simultaneous stabilizability question for three or more linear systems is –in the sense that we give to decidability– undecidable. It is not possible to find a general criterion that involves only the coefficients of three or more linear systems, arithmetical operations (additions, substractions, multiplications and divisions), logical operations ('and' and 'or') and sign tests operations (equal to, greater than, greater than or equal to,...) and that is necessary and sufficient for simultaneous stabilizability of the systems.

Questions that can be solved by using only the above mentioned elementary operations are sometimes refered to as 'tractable'. This paper points thus to an intrinsic limitation of simultaneous stabilizability conditions: it is possible to find necessary *or* sufficient conditions for simultaneous stabilization and it is also possible to find equivalent formulations, but it is *not* possible to find *tractable* necessary and sufficient conditions.

Section 2 sets out the notations. Section 3 states the simultaneous stabilization question in the convenient and classical factorization set-up. Our notion of a decidable question is presented in Section 4. Section 5 introduces a deep result of analytic function theory that is used in our central Section 6. In that final section, we show in Theorem 9 that the simultaneous stabilization question of three or more systems is not a decidable question.

This paper contains some of the material from the first author's PhD thesis. See [6].

2 Notations

R is the set of real numbers and Q is the set of rational numbers. $\mathrm{R}[s]$ is the set of real polynomials in the variable s. $\mathrm{R}(s)$ is the set of real rational functions. $\mathrm{Q}(\beta)$ is the set of rational functions in the variable β and with coefficients in Q. C_∞ is the extended complex plane $\mathrm{C} \cup \{\infty\}$ topologized with the Riemann sphere topology and R_∞ is the extended real line, $\mathrm{R} \cup \{\infty\}$. D is the open unit disc $\{s \in \mathrm{C} : |s| < 1\}$, $\overline{D}$ is the closed unit disc $\{s \in \mathrm{C} : |s| \leq 1\}$ and $\mathrm{C}_{+\infty} = \{s \in \mathrm{C} : \Re(s) \geq 0\} \cup \{\infty\}$ is the extended closed right half plane. Assume that Ω is a subset of C_∞. A real rational function $f(s) \in \mathrm{R}(s)$ is

Ω-*stable* if it has no poles in Ω [1]. $S(\Omega)$ is the set of all Ω-stable rational functions. We use $U(\Omega)$ to denote the set of functions in $S(\Omega)$ whose inverse are in $S(\Omega)$ and we call such rational functions Ω-*bistable* rational functions. Finally, to shorten the notations, we denote $U = U(\mathsf{C}_{+\infty})$ and $S = S(\mathsf{C}_{+\infty})$.

3 Simultaneous stabilization

A rational function is stable if and only if it has no poles in the extended closed right half plane, i.e. if and only if it belongs to the set S.
Throughout this paper we consider a controller to be within a unity feedback loop with the system and we adopt the following, usual, definition of stability for this closed loop configuration.

Definition 1 *A controller $c(s) \in \mathsf{R}(s)$ is a stabilizing controller for a system $p(s) \in \mathsf{R}(s)$ if and only if the four transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}$, $c(s)(1+p(s)c(s))^{-1}$, $p(s)(1+p(s)c(s))^{-1}$ and $(1+p(s)c(s))^{-1}$ belong to S.*

We also need the definition of strong stabilization.

Definition 2 *A system $p(s) \in \mathsf{R}(s)$ is strongly stabilizable if and only if there exists a stable stabilizing controller for $p(s)$. A stable stabilizing controller is called a strong stabilizing controller.*

According to the first definition, the simultaneous stabilization question for k systems is.

Question 1 *Under what necessary and sufficient condition(s) on $\{p_i(s) : i = 1, ..., k\}$ does there exist a stabilizing controller $c(s) \in \mathsf{R}(s)$ for the k systems $p_i(s) \in \mathsf{R}(s)$?*

The question of simultaneous stabilization of $k+1$ systems encompasses that of strong simultaneous stabilization of k systems, i.e. that of simultaneous stabilization of k systems with a stable controller. Indeed, assume

[1] We draw the reader's attention on the fact that this is pure convention. Other authors define Ω-stability in exactly the opposite way.

that we have a system $p(s) = 0$, then the four transfer functions $p(s)c(s)(1 + p(s)c(s))^{-1}$, $c(s)(1 + p(s)c(s))^{-1}$, $p(s)(1 + p(s)c(s))^{-1}$ and $(1 + p(s)c(s))^{-1}$ associated to the controller $c(s) \in \mathtt{R}(s)$ and to the system $p(s) = 0$ are equal to $0, c(s), 0$ and 1. The constants 0 and 1 are, by definition, stable, and thus the controller $c(s)$ stabilizes $p(s) = 0$ if and only if it is stable.
Simultaneous stabilization of the $k+1$ systems $p_0 = 0, p_1, p_2, ..., p_k$ is therefore equivalent to simultaneous stabilization of the k systems $p_1, p_2, ..., p_k$ with a stable controller.

The condition given above for stabilization is not very convenient because it involves four transfer functions. We condense this unpractical condition by using the so-called factorization approach. We give hereafter a short introduction to this theory and refer the interested reader to [17] for more details.

It is easy to check that S is a commutative ring. The invertible elements of the ring S are the stable real rational functions whose inverse is stable. We have denoted the set of invertible elements of S by U. Two elements of S are *coprime* in S if and only if they have no common zeros in $\mathtt{C}_{+\infty}$. The next theorem says that any element of $\mathtt{R}(s)$ can be expressed as a ratio of two coprime elements of S. This factorization procedure is a generalization of the usual factorization of rational functions as ratios of polynomials.

Theorem 1 *Assume that $p(s) \in \mathtt{R}(s)$. There exists $n_p(s), d_p(s) \in S$ coprime in S such that $p(s) = \frac{n_p(s)}{d_p(s)}$. Such a fractional factorization of $p(s)$ is called a coprime fractional factorization of $p(s)$ in S.*

Note that coprime fractional factorizations are not unique.
The link between coprime fractional factorizations and stabilization is given in the next theorem[2].

Theorem 2 *Let $p, c \in \mathtt{R}(s)$ be a system and a controller and let $p = \frac{n_p}{d_p}$ and $c = \frac{n_c}{d_c}$ be any coprime fractional factorizations of p and c in S. Then c stabilizes p if and only if $n_c n_p + d_c d_p \in U$.*

[2]When clear from the context, we drop the reference to the complex variable s and write, for example, p for $p(s)$.

As a consequence of this theorem, we formulate simultaneous stabilization under the following form.

Theorem 3 *Let* $p_i \in \mathtt{R}(s)$ $(i = 1, ..., k)$ *and let* $p_i = \frac{n_i}{d_i}$ *be any coprime fractional factorizations of* p_i *in* S*. Then* p_i *are simultaneously stabilizable if and only if there exist* $n_c, d_c \in S$ *such that* $n_c n_i + d_c d_i \in U$ $(i = 1, ..., k)$*.*

It is this formulation that is used in the central section.

4 decidability and algebraic numbers

This section is in three parts. We first give our definition of decidability, thereafter we define algebraic and transcendental numbers and finally we prove a result that links decidable questions and algebraic numbers.

4.1 decidable

We give the example of polynomial stability to illustrate what we mean by a 'decidable question'. The example is then generalized to an abstract setting.

We say that the real polynomial $p(s) = a_1 + a_2 s + ... + a_{n+1} s^n$ $(a_{n+1} \neq 0)$ of degree n is *stable* if and only if it has no zeros with positive or zero real part.

For each positive integer n there exists a test on the coefficients –known as the Routh-Hurwitz criterion– that allows to check, without computing its roots, whether a polynomial of order n is stable. Up to second order polynomials the test is trivial: a polynomial of order less than or equal to two is stable if and only if all its coefficients have the same sign. The criterion becomes more interesting when $n \geq 3$. We give its formulation for third order polynomials.

The polynomial $p(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3$ $(a_4 \neq 0,\ a_i \in \mathtt{R})$ is stable if and only if either

$$a_i > 0 \ (i = 1, 2, 3, 4) \text{ and } a_2 a_3 - a_1 a_4 > 0$$

or

$$a_i < 0 \ (i = 1, 2, 3, 4) \text{ and } a_2 a_3 - a_1 a_4 < 0.$$

The remarkable feature of the Routh-Hurwitz criterion is that it involves only the *coefficients* of the polynomial and three sorts of *elementary operations*. In our criterion, we use only:

1. the coefficients a_i $(i = 1, 2, 3, 4)$,
2. substractions and multiplications,
3. the 'and' and 'or' logical operations (we denote these by $\wedge$ and $\vee$),
4. strict positivity and strict negativity tests ($>$ and $<$).

Due to this particular feature, the criterion can be rewritten under the form of a logical sentence.
The polynomial

$$p(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3$$

is stable if and only if the logical sentence

$$\begin{aligned}&(((a_1 > 0) \wedge (a_2 > 0) \wedge (a_3 > 0) \wedge (a_4 > 0) \wedge (a_2 a_3 - a_1 a_4 > 0))\\ \vee&((a_1 < 0) \wedge (a_2 < 0) \wedge (a_3 < 0) \wedge (a_4 < 0) \wedge (a_2 a_3 - a_1 a_4 < 0)))\end{aligned}$$

is true.

The binary question of deciding whether a third order polynomial is stable by using only its four coefficients is a typical example of what we mean by a *decidable* question: it can be answered by using a finite number of elementary operations.
We say that the polynomial stability question is decidable because it is decidable for each fixed polynomial degree n.

The abstract notion of decidability is only a formalization of this simple and intuitive idea. We say that a binary question Q associated to an n-uple $(a_1, ..., a_n) \in \mathrm{R}^n$ is decidable if and only if there exists a logical sentence L that involves only elementary operations on the entries of the n-uple and that is true if and only if Q is.

We formalize this concept with the next two definitions.

Definition 3 *An elementary operation is any one of*

1. *the four arithmetic operations: addition, substraction, multiplication and division. These are commonly referred to as rational operations,*
2. *the two logical operations: 'and' and 'or',*
3. *the five test operations:* $=, >, <, \geq$ *or* $\leq$.

Definition 4 *A binary question* Q *associated to an* n*-uple* $(a_1, ..., a_n) \in \mathbb{R}^n$ *is decidable if and only if there exists a meaningful logical sentence* L *of finite length that involves only elementary operations on the entries* a_i *of the* n*-uple and such that* L *is true if and only if* Q *is.*

In other words, a binary question $Q(a_1, ..., a_n)$ associated to an n-uple $(a_1, ..., a_n)$ is decidable if and only if there exists a logical sentence of finite length $L(a_1, ..., a_n)$ that is made up of elementary operations only and such that

$$\forall (a_1, ..., a_n) \in \mathbb{R}^n \ : \ (Q(a_1, ..., a_n) \text{ is true } \Leftrightarrow L(a_1, ..., a_n) \text{ is true}) \, .$$

With this definition, and for each $n \in \mathbb{N}$, the following questions are decidable: when is a real (or complex) polynomial of order n stable? when is a $n \times n$ matrix positive definite? when do two polynomials of order n have a common zero? when are two linear systems of order n simultaneously stabilizable?

Similarly to the example of polynomial stability, we say that a binary question associated to an n-uple in $\mathbb{R}^n$ –without specifying the value of n– is decidable if and only if it is decidable for each fixed value of n. Therefore, the stability of a polynomial, the positive definiteness of a matrix, the coprimeness of two polynomials and the simultaneous stabilizability of two linear systems *are all decidable questions.*

On the other hand, we show in Section 6 that the question: when are three systems simultaneously stabilizable? is *not* decidable.

4.2 Algebraic numbers

Algebraic numbers are numbers that are roots of polynomials whose coefficients are integers (see, for example, [16] or [4]).

Definition 5 *A real number is algebraic if and only it is the root of a polynomial that has integer (or rational) coefficients. A real number that is not algebraic is transcendental.*

For example, $-1, \sqrt{2}$, $i = \sqrt{-1}$ and $\frac{\sqrt{\sqrt{7}+31}}{\sqrt{13}-5}$ are algebraic numbers whereas π, e and $\Gamma(\frac{1}{4})$ are not, they are transcendental. It is in general not true that the ratio of two transcendental numbers is a transcendental number. For our simultaneous stabilization purposes we need the next non-trivial result. The proof of this theorem may safely be ignored by those readers wishing to make a fast reading. It is totally independent of the rest of the paper.

Theorem 4 *The real number* $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ *is transcendental.*

Proof
Our proof is based on a result contained in the third section of the last chapter of 'Transcendental number theory' (A. Baker, p. 158, [4]). This result states: "The transcendence degree of the field L generated by $\omega_1 = \frac{\Gamma^2(\frac{1}{4})}{\sqrt{8\pi}}$, $\omega_2 = i\omega_1$, $\eta_1 = \frac{\pi}{\omega_1}$, and $\eta_2 = -i\eta_1$ over the rationals Q is at least 2."
It is trivial to see that the field L generated by ω_1, ω_2, η_1 and η_2 can equally be written as

$$L = \mathsf{Q}(\frac{\Gamma^2(\frac{1}{4})}{\sqrt{8\pi}}, i, \pi).$$

A subfield of L is given by

$$F = \mathsf{Q}(\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, \pi).$$

Both $\frac{\Gamma^2(\frac{1}{4})}{\sqrt{8\pi}}$ and i are algebraic over F and so L is a finite extension of F.

By Steinitz Theorem (see [16], p. 140, Lemma 15.2), the combined fact that L is a finite extension of F and that L has a transcendence degree over Q of at least 2 implies that the transcendence degree of F over Q is at least 2.

Since $F = \mathbb{Q}(\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, \pi)$, this implies that the transcendence degree of F over $\mathbb{Q}$ is precisely equal to 2 and, hence, that both π and $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ are transcendental numbers. This ends the theorem. ■

4.3 decidability and algebraic numbers

In this section we establish a link between decidability and algebraic numbers. As previously, we first illustrate our point with the example of polynomial stability and then generalize the concept in an abstract setting.

Assume that $\beta \in \mathbb{R}$ and that we wish to investigate the stability of the polynomial $p(s) = 1+\beta s+\beta s^2+2s^3$. As noted above, the stability of a third order polynomial is a decidable question and, hence, using the associated logical sentence, the polynomial

$$p(s) = 1 + \beta s + \beta s^2 + 2s^3$$

is stable if and only if the logical sentence

$$\begin{aligned} &(((1 > 0) \wedge (\beta > 0) \wedge (\beta > 0) \wedge (2 > 0) \wedge (\beta^2 - 2 > 0)) \\ &\vee((1 < 0) \wedge (\beta < 0) \wedge (\beta < 0) \wedge (2 < 0) \wedge (\beta^2 - 2 < 0))) \end{aligned}$$

is true.
After some trivial simplifications it appears that this logical sentence is true if and only if

$$(\beta > 0) \wedge (\beta^2 - 2 > 0)$$

is. That is, if and only if

$$\beta \in (\sqrt{2}, \infty).$$

We thus have the chain of equivalences

$$\begin{aligned} &\text{the polynomial } 1 + \beta s + \beta s^2 + 2s^3 \text{ is stable} \\ \Leftrightarrow\ &(\beta > 0) \wedge (\beta^2 - 2 > 0) \text{ is true} \\ \Leftrightarrow\ &\beta \in (\sqrt{2}, \infty). \end{aligned}$$

In the last equivalence the stability condition is expressed by means of an open interval $(\sqrt{2}, \infty)$ whose endpoints are the point at infinity and the algebraic number $\sqrt{2}$.
A similar feature remains true in the abstract general case. Recall that $\mathsf{Q}(\beta)$ denotes the set of rational functions of β with coefficients in Q.

Theorem 5 *If $Q(a_1, ..., a_n)$ is a decidable binary question associated to an n-uple $(a_1, ..., a_n)$ and if all the entries a_i of the n-uple are in $\mathsf{Q}(\beta)$ ($a_i(\beta) \in \mathsf{Q}(\beta)$), then there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are either equal to $\pm\infty$ or to algebraic numbers, such that*

$$\begin{aligned} & Q(a_1(\beta), ..., a_n(\beta)) \textit{ is true} \\ \Leftrightarrow \quad & \beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right). \end{aligned}$$

Proof
Since the question $Q(a_1, ..., a_n)$ is decidable, there exists a meaningful logical sentence $L(a_1, ..., a_n)$ of finite length that involves only elementary operations on the entries a_i of the n-uple and such that $L(a_1, ..., a_n)$ is true if and only if $Q(a_1, ..., a_n)$ is. Thus

$$\forall \beta \in \mathsf{R} \ : \ (Q(a_1(\beta), ..., a_n(\beta)) \text{ is true } \Leftrightarrow L(a_1(\beta), ..., a_n(\beta)) \text{ is true}).$$

It remains to show that there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are either equal to $\pm\infty$ or to algebraic numbers, such that

$$\forall \beta \in \mathsf{R} \ : L(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow$$

$$\beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right).$$

To prove this we proceed by induction on the size of the logical sentence $L(a_1, ..., a_n)$.

The logical sentence $L(a_1, ..., a_n)$ is either made up of two smaller logical sentences $L_1(a_1, ..., a_n)$ and $L_2(a_1, ..., a_n)$ linked by an 'and' or an 'or' logical operation ($L(a_1, ..., a_n) = L_1(a_1, ..., a_n) \wedge L_2(a_1, ..., a_n)$ or $L(a_1, ..., a_n) = L_1(a_1, ..., a_n) \vee L_2(a_1, ..., a_n)$) or is a nucleus expression of the form $L(a_1, ..., a_n) = R_1(a_1, ..., a_n) \ R_2(a_1, ..., a_n)$ where $R_1(a_1, ..., a_n)$ and $R_2(a_1, ..., a_n)$ are rational expressions of the coefficients $a_1, ..., a_n$ ($R_i(a_1, ..., a_n) \in \mathsf{Q}(a_1, ..., a_n)$ for

$i = 1, 2)$ and is any one of the five sign test operations $<, \leq, >, \geq, =$.

We analyse these two cases successively.

First, if $L(a_1, ..., a_n)$ is a nucleus expression then $L(a_1(\beta), ..., a_n(\beta))$ is true if and only if

$$R_1(a_1(\beta), ..., a_n(\beta)) \ \ R_2(a_1(\beta), ..., a_n(\beta))$$

for some $\ \in \{<, \leq, >, \geq, =\}$. By hypothesis $a_i(\beta)$ are rational expressions of β ($a_i(\beta) \in \mathtt{Q}(\beta)$ for $i = 1, 2, ..., n$) and $R_j(a_1, ..., a_n)$ are rational expressions of $a_1, ..., a_n$ ($R_j(a_1, ..., a_n) \in \mathtt{Q}(a_1, a_2, ..., a_n)$ for $j = 1, 2$). Hence, $R'_j(\beta)$, $R_j(a_1(\beta), ..., a_n(\beta))$ are also rational expressions of β. The condition $R_1(a_1(\beta), ..., a_n(\beta)) \ \ R_2(a_1(\beta), ..., a_n(\beta))$ is satisfied if and only if $R'_1(\beta) \ \ R'_2(\beta)$ is, and this last condition is equivalent to

$$\beta \in \left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are equal to $\pm\infty$ or to algebraic numbers. Thus the theorem is proved in the case of a nucleus expression.

Secondly, suppose that $L(a_1, ..., a_n)$ is made up of two logical sentences $L_1(a_1, ..., a_n)$ and $L_2(a_1, ..., a_n)$ linked by an 'and' or an 'or' logical operation. By induction hypothesis assume that the values $\overline{\sigma}^1_{k,j}$ and $\underline{\sigma}^1_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m^1_k$) and $\overline{\sigma}^2_{k,j}$ and $\underline{\sigma}^2_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m^2_k$) are equal to $\pm\infty$ or to algebraic numbers and are such that

$$L_1(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\cup_{j=1}^{m_1}(\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_1}[\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)$$

and

$$L_2(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\cup_{j=1}^{m_1}(\underline{\sigma}^2_{1,j}, \overline{\sigma}^2_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2}[\underline{\sigma}^2_{2,j}, \overline{\sigma}^2_{2,j})\right).$$

Then, if $L(a_1, ..., a_n) = L_1(a_1, ..., a_n) \wedge L_2(a_1, ..., a_n)$ we have

$$L(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\left(\cup_{j=1}^{m_1}(\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2}[\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)\right)$$

$$\cap\left(\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j}^2,\overline{\sigma}_{1,j}^2]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j}^2,\overline{\sigma}_{2,j}^2)\right)\right)$$

whereas, if $L(a_1,...,a_n) = L_1(a_1,...,a_n) \vee L_2(a_1,...,a_n)$ we have

$$L(a_1(\beta),...,a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j}^1,\overline{\sigma}_{1,j}^1]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j}^1,\overline{\sigma}_{2,j}^1)\right)\right)$$

$$\cup\left(\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j}^2,\overline{\sigma}_{1,j}^2]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j}^2,\overline{\sigma}_{2,j}^2)\right)\right).$$

It is trivial to see that, in both cases we can rewrite the unions and intersections involved under the form

$$\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j},\overline{\sigma}_{1,j}]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j},\overline{\sigma}_{2,j})\right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) equal to $\pm\infty$ or to algebraic numbers. Thus, by induction on the size of L, the theorem is proved. ■

5 Analytic functions

The results that we need are contained in two books on analytic functions (see Nehari [13] and Goluzin [10]). We pick out a theorem from each of these sources and then merge them into a single formulation that is more suitable for our subsequent treatment. In all what follows we define $A\ ,\ \frac{4\pi^2}{\Gamma^4(\frac{1}{4})} = 0.228...$.

Theorem 6 (Goluzin, [10], p.89) *Suppose that the function $F(z) = z^q + a_{q+1}z^{q+1} + a_{q+2}z^{q+2} + ...$, for $q \geq 1$, is regular in the disk $|z| < 1$. Then the image of that disk under the mapping $\xi = F(z)$ completely covers some segment of arbitrary predetermined slope that contains the point $\xi = 0$ and is of length no less than $2A = \frac{8\pi^2}{\Gamma^4(\frac{1}{4})} = 0.45...$. The number A cannot be increased without additional restrictions on $F(z)$.* ■

The proof of this theorem is not contained in the book itself but in a referenced journal [3].
The next similar result is taken from Nehari [13].

Theorem 7 (Nehari, [13], p.328) *If the odd analytic function*

$$f(z) = z + a_3 z^3 + ... + a_{2n+1} z^{2n+1} + ..., \ |z| < 1$$

is regular in $|z| < 1$*, then the values taken by* $w = f(z)$ *in* $|z| < 1$ *fully cover the circle*

$$|w| < \frac{4\pi^2}{\Gamma^4(\frac{1}{4})} = A = 0.228....$$

The statement would not be true if the constant involved was replaced by a larger value. ∎

We define, and denote the *range* of an analytic function $f(z)$ on D by $f(D)$, $\{f(z) : z \in D\}$. Theorem 7 can be seen as a consequence of Goluzin's Theorem 6. Indeed, for any odd analytic function

$$f(z) = z + a_3 z^3 + ... + a_{2n+1} z^{2n+1} + ...$$

and for any complex number a strictly less than A we know by Theorem 6 that either a or $-a$ is in the range of $f(z)$ on $|z| < 1$. But, since $f(z)$ is an odd function, this means that both a and $-a$ are in the range of $f(z)$ on $|z| < 1$. Hence the result.

Theorem 7 has the advantage for the authors that in contrast to Theorem 6, a proof of it is contained in the book [13] rather than in a 1944 Russian journal. Particularly important for us is that the proof shows that the bound A is the best achievable one by constructing an analytic function that achieves the bound.

The function $f_e(z)$ (denoted by $f(z)$ and introduced at the bottom of page 330 in [13]) is connected to the so-called elliptic modular function and is defined by the converging infinite product

$$f_e(z) \triangleq \frac{4\pi^2}{\Gamma^4(\frac{1}{4})} \left(32 e^{-\pi \frac{1+z}{1-z}} \prod_{n=1}^{\infty} \left(\frac{1 + e^{-2n\pi \frac{1+z}{1-z}}}{1 - e^{-(2n-1)\pi \frac{1+z}{1-z}}} \right) - 1 \right).$$

It is shown in [13] that $f_e(z)$ enjoys the following nice properties:

1. it is a real function: $f_e(\overline{z}) = \overline{f_e(z)}$,
2. it is analytic on $|z| < 1$,
3. it is such that $f_e(0) = 0$ and $f_e'(0) = 1$,
4. it does not take the values $\pm A$ on D, i.e. $f_e(z) \neq A$ and $f_e(z) \neq -A$, $z \in D$.

For further purposes we merge the Theorems 7 and 6 into a single one by making use of the properties of the function $f_e(z)$.

Theorem 8 *Assume that $\beta \in \mathrm{R}$. There exists an analytic function on $|z| < 1$ such that $f(\overline{z}) = \overline{f(z)}$, $f(0) = 0$, $f'(0) = 1$ that leaves out the values $\pm\beta$ if and only if $|\beta| \geq A$.*

Proof

We first prove sufficiency. Let $f_e(z)$ be the function defined above. Assume that $\beta \geq A$ and define

$$f(z) \triangleq \frac{\beta}{A} f_e(\frac{A}{\beta} z).$$

Due to the properties of $f_e(z)$ it is easy to check that $f(z)$:

1. is such that $f(\overline{z}) = \overline{f(z)}$,
2. is analytic on $|z| < 1$ (note that this fails when $|\beta| < A$),
3. is such that $f(0) = 0$ and $f'(0) = 1$,
4. leaves out the values $\pm\beta$ on D.

This ends the first part of the proof.

For necessity, assume by contradiction that $f(z)$ satifies the conditions of the theorem and that $0 < \beta < A$. By assumption, the image of the disc $|z| < 1$ under the mapping $\xi = f(z)$ contains neither the value β nor the value $-\beta$. Thus, the image does not cover any segment of the real line that contains the origin and is of length $2A$. This contradicts Theorem 6, hence the result.

■

This theorem is the crucial result that is needed for proving Theorem 9.

6 Simultaneous stabilization of three systems: an undecidable question

Simultaneous stabilization has been defined in Section 3 and decidability in Section 4. In this part we merge the results of these two sections and use our Theorem 8 on analytic functions to show that the simultaneous stabilizability question of three systems is an undecidable question.

Theorem 9 *The simultaneous stabilizability of three systems is an undecidable question.*

Proof

Assume that $\beta \in \mathtt{R}$ and consider the three systems $p_1(s) = 0$, $p_{2,\beta}(s) = \frac{(s-1)^2}{(s^2-1)-\beta(s+1)^2}$ and $p_{3,\beta}(s) = \frac{(s-1)^2}{(s^2-1)+\beta(s+1)^2}$.

We proceed in two steps.

First, we show that,

(a) when $\beta = 0$ the three systems are simultaneously stabilizable,

(b) when $|\beta| > A$ the three systems are simultaneously stabilizable,

(c) when $0 < |\beta| < A$ the three systems are *not* simultaneously stabilizable.

Note that we leave out the analysis of the case $\beta = A$.

Second, we show that the first step contradicts the fact that the simultaneous stabilizability question of three systems is a decidable question.

Step 1.

(a) This point is trivial. When $\beta = 0$ then $p_1(s) = 0$ and $p_{2,0} = p_{3,0} = \frac{s-1}{s+1}$. It is easy to check that, for example, $c(s) = 2$ is a stabilizing controller of these three systems.

In order to prove points (b) and (c) we follow the factorization approach briefly outlined in Section 3. Consider, for $\beta \neq 0$, the coprime fractional factorization in S of the three systems

$$p_1(s) = \frac{n_1(s)}{d_1(s)} = \frac{0}{1}$$

$$p_{2,\beta}(s) = \frac{n_2(s)}{d_{2,\beta}(s)} = \frac{\left(\frac{s-1}{s+1}\right)^2}{\frac{s-1}{s+1} - \beta}$$

$$p_{3,\beta}(s) = \frac{n_3(s)}{d_{3,\beta}(s)} = \frac{\left(\frac{s-1}{s+1}\right)^2}{\frac{s-1}{s+1} + \beta}.$$

Applying Theorem 3, these three systems are simultaneously stabilizable if and only if there exist two functions $n_c(s), d_c(s) \in S$ such that

$$d_c(s) \in U$$

$$\left(\frac{s-1}{s+1}\right)^2 n_c(s) + \left(\frac{s-1}{s+1} - \beta\right) d_c(s) \in U$$

$$\left(\frac{s-1}{s+1}\right)^2 n_c(s) + \left(\frac{s-1}{s+1} + \beta\right) d_c(s) \in U.$$

That is, if and only if there exist two functions $n_c(s), d_c(s) \in S$ such that

$$d_c(s)$$

$$\left(\frac{s-1}{s+1}\right)^2 n_c(s) + \left(\frac{s-1}{s+1} - \beta\right) d_c(s)$$

and

$$\left(\frac{s-1}{s+1}\right)^2 n_c(s) + \left(\frac{s-1}{s+1} + \beta\right) d_c(s)$$

have no zeros in the extended right half plane.
Using the bilinear transformation $z = \frac{s-1}{s+1}$ that maps the extended right half plane $\mathsf{C}_{+\infty}$ onto the closed unit disc $\overline{D}$, these conditions are clearly equivalent to that of the existence of two rational function $n'_c(z)$ and $d'_c(z)$ that have no poles in $\overline{D}$ such that

$$d'_c(z)$$

$$z^2 n'_c(z) + (z - \beta) d'_c(z)$$

and

$$z^2 n_c'(z) + (z + \beta) d_c'(z)$$

have no zeros in $\overline{D}$.
The first of these conditions imposes that $d_c'(z)$ has no zeros in $\overline{D}$. The rational function defined by the ratio $c'(z) \, , \frac{n_c'(z)}{d_c'(z)}$ then has no poles in $\overline{D}$ and, dividing by $d_c'(z)$, the above three conditions are equivalent to that of the existence of a rational function $c'(z)$ that has no poles in $\overline{D}$ and such that

$$z^2 c'(z) + z - \beta$$

and

$$z^2 c'(z) + z + \beta$$

have no zeros in $\overline{D}$.

It remains to show that, when $\beta < A$ such a function $c'(z)$ does not exist whereas it does exist when $\beta > A$. We prove these two points in (b) and (c), respectively.

(b) Assume, by contradiction, that $\beta < A$, that $c(z)$ has no poles in $\overline{D}$ and that

$$z^2 c(z) + z - \beta$$

and

$$z^2 c(z) + z + \beta$$

have no zeros in $\overline{D}$. Then, the function defined by

$$f(z) \, , \, z^2 c(z) + z$$

satisfies all the conditions of Theorem 8 and leaves out the values $\pm\beta$ with $\beta < A$. A contradiction is achieved and this part is proved.

(c) Assume that $\beta > A$. We construct a rational function $c(z)$ that satisfies all the requested conditions.

By Theorem 8, there exists an analytic function $f(z)$ on D such that $f(0) = 0,\ f'(0) = 1$ and that leaves out the values $\pm A$. We define the function $g(z)$ as

$$g(z) \triangleq \frac{\beta}{A} f\left(\frac{A}{\beta} z\right).$$

Due to the properties of $f(z)$, the function $g(z)$ is such that

1. $g(z)$ is such that $g(\overline{z}) = \overline{g(z)}$,
2. $g(z)$ is analytic on $|z| < \frac{\beta}{A}$ (and $1 < \frac{\beta}{A}$),
3. $g(0) = 0$ and $g'(0) = 1$,
4. $g(z)$ leaves out the values $\pm A$ on $|z| < \frac{\beta}{A}$.

In the sequel we construct, with the help of this function $g(z)$, a real *polynomial* $p(z) \in \mathbb{R}[z]$ such that $p(0) = 0,\ p'(0) = 1$ and $p(z) \neq \pm A,\ |z| < 1$.

Because of the points 2 and 4, the real number μ defined by

$$\mu \triangleq \min\{\inf_{z\in\overline{D}} |g(z) - A|, \inf_{z\in\overline{D}} |g(z) + A|\}$$

is strictly positive.

Because of the first three points, the function $h(z)$ defined by

$$h(z) \triangleq \frac{g(z) - z}{z^2}$$

is real and analytic in $\{z : |z| < \frac{\beta}{A}\}$.

By Runge's theorem (see Rudin [14]), there exists a real polynomial $q(z)$ such that

$$|h(z) - q(z)| < \mu \left(\frac{A}{\beta}\right)^2,\ z \in \overline{D}.$$

This polynomial is then also such that

$$|g(z) - z - z^2 q(z)| < \mu, \ z \in \overline{D}.$$

Our final step consists in the definition of the polynomial $p(z)$

$$p(z) \triangleq z + z^2 q(z) \in \mathbb{R}[z].$$

Clearly, $p(0) = 0$ and $p'(0) = 1$. But also, because

$$|g(z) - p(z)| < \mu, \ z \in \overline{D}$$

and

$$\mu \leq \min\{\inf_{z \in \overline{D}} |g(z) - A|, \inf_{z \in \overline{D}} |g(z) + A|\}$$

it follows that

$$|g(z) - p(z)| < |g(z) \pm A|, \ z \in \overline{D}.$$

Hence,

$$p(z) \neq \pm A, \ z \in \overline{D},$$

as requested.
A polynomial is a rational function with no poles of module less than or equal to one and, thus, point (c) of our first step is proved.

Step 2. It remains to show the second step of the proof: the fact that the first step contradicts the decidability of the simultaneous stabilizabition question of three plants.
Assume, by contradiction, that the simultaneous stabilizability of three systems is a decidable question. Then, so is the simultaneous stabilizability of our three systems $p_1(s) = 0$, $p_{2,\beta}(s) = \frac{(s-1)^2}{(s^2-1)-\beta(s+1)^2}$ and $p_{3,\beta}(s) = \frac{(s-1)^2}{(s^2-1)+\beta(s+1)^2}$.
But then, using Theorem 5, there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are either equal to $\pm\infty$ or that are algebraic numbers, such that our three systems are simultaneously stabilizable if and only if

$$\beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \ \cup \ \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right).$$

But this contradicts our first step since we know from there that the three systems are simultaneously stabilizable if and only if either

$$\beta \in (-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}) \cup [0,0] \cup (\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty)$$

or

$$\beta \in (-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}] \cup [0,0] \cup [\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty).$$

By Theorem 4, $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ is a transcendental number, a contradiction is achieved and the theorem is proved. ■

7 Conclusion

We believe that this paper closes a whole area of investigation of the simultaneous stabilization problem. There exist no criterion for simultaneous stabilizability that involve only elementary operations on the coefficients. In particular, it is not possible to find a criterion that involves only, say, solving systems of linear equations, solving a Nevanlinna type interpolation problem or evaluating a Cauchy index, because all these operations are conducted by performing elementary operations only.

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We dedicate this paper to all 1992 twins.

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