A Lifting Approach to Models of Opinion Dynamics with Antagonisms

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***Abstract*— Different recent works have studied polarized versions of models of opinion dynamics, in which an agent opinion can be attracted by the opinions of some agents and by the opinions opposite to those of some others, representing a form of antagonism.**

We show that these systems correspond to the projection of specific trajectories of classical opinion dynamics systems involving twice as many agents, to which a large number of existing results apply. We take advantage of this to prove several convergence results for models with antagonisms, extending those previously available. Our approach can be applied in both discrete and continuous time.

I. INTRODUCTION

We propose a lifting-based approach to analyze models of opinion dynamics with antagonisms

$$\dot{x}_i = \sum_{j=1}^{n} |a_{ij}(t)| \left(\text{sign}\left(a_{ij}(t)\right) x_j - x_i\right) \qquad (1)$$

where $x_i(t) \in \Re$ represents the opinion of an agent $i \in \{1, \ldots, n\}$ at time t, and the a_{ij} are interaction coefficients. Specifically, $a_{ij}(t) > 0$ means that the opinion of agent i is attracted by that of agent j, while $a_{ji}(t) < 0$ mans that it is attracted by the opinion *opposite* to that of agent j.

Over the last decade, different authors have studied models describing the dynamics of opinions of the form

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j - x_i),$$

(or their discrete time counterpart) where the dependence of the $a_{ij} \geq 0$ on time, on x, or on some pre-specified social network depends on the specific models, see for example [3]–[5], [7], [8], [13].

Although these models are based on the attraction of opinions, they also allow for persistent disagreement. They typically lead to the emergence of multiple clusters of agents, where all the agents of a cluster eventually share a same opinion, different from that in the other clusters. These disagreements and clustering phenomena do however almost always result from the lack of interactions between agents or the inhibition of their mutual attractions; agents end up disagreeing because they do not influence each other sufficiently. In [4], [7], [8] for example the agents only influence each other if their opinions are sufficiently similar. As a result, when groups of agents develop opinions that are too different, they do not influence each other anymore and keep disagreeing, leading to different clusters.

Real-life phenomena can be much more complex, and one observes that people can often continuously influence each other while persistently disagreeing, or even be negatively influenced by each other. In [1], [2], Altafini has introduced and studied a model of opinion dynamics with antagonisms of the form (1). For constant interactions weights a_{ij}, he has shown that such models could lead either to a consensus at 0 or to a *bipartite consensus* in which one part of the agents agree on a value $x^* \in \Re$ while the other part of the agents agree on the opposite value $-x^*$. Moreover, he provided a combinatorial condition that determines the regime of convergence based on the signed graph G associated to the interaction weights, where (j, i) is a positive edge if $a_{ij} > 0$ and a negative edge if $a_{ij} < 0$: Assuming that it is connected, if G contains a cycle with an odd number of negative edges, all agents' opinions converge to 0. Otherwise, it can converge to a bipartite consensus. The addition of one single negative edge can thus dramatically affect the asymptotic behavior of the system. He partially extended his results to systems where the coefficients $a_{ij}(t)$ switch in a finite set, provided that the associated graph remains connected at all time. He also considered certain systems with state-dependent coefficients.

The model (1) with constant coefficients was further studied, for example by Hu and Zheng [10], who established convergence to 0 or to bipartite consensus for different classes of (constant) interactions, using signed and signless Laplacian matrices.

Similar results were obtained by Xia and Cao [20] for an analogous discrete time model with switching interaction weights. This discrete time model is further studied in a recent preprint by Meng et al. [14] who show that the absolute values $|x_i|$ of the opinions converge to a common value under some repeated connectivity assumptions.

Contribution and outline

In this work, we show that the analysis of systems such as (1) can be made easier by considering an associated classical consensus system on $2n$ agents. More specifically, we show in Section II that the trajectories of models of opinion dynamics with antagonisms can be seen as the projection of those of a lifted classical model of opinion dynamics on $2n$ agents, to which results on classical consensus can directly be applied. We take advantage of this representation to derive general convergence results for the case of reciprocal interactions in Section III, and for non-reciprocal interactions

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in Section IV, extending several of those presented in [2], [10]. We show how similar results can be obtained for discrete time systems in Section V, and conclude by a discussion in Section VI.

Note that there exist other models of negative interactions. In the context of multi-agent control, the authors of [18] consider for example systems where one faulty agent is linearly repulsed by the other agent opinions ($\dot{x}_i = \sum_j (x_i - x_j)$).

II. Model Description and Lifting

We consider the continuous-time system

$$\dot{x}_i(t) = \sum_{j=1}^{n} |a_{ij}(t)| \left(\text{sign}\left(a_{ij}(t)\right) x_j(t) - x_i(t)\right), \qquad (2)$$

where the $a_{ij}(t)$ are arbitrary functions measurable on any bounded interval. We remind that when $a_{ij}(t)$ is positive the opinion of agent i is attracted by that of agent j, and when $a_{ij}(t)$ is negative it is attracted by the opinion opposite to that of agent j. System (2) generalizes that initially considered by Altafini [2] by allowing for arbitrary time-dependent interaction coefficients $a_{ij}(t)$.

Since the interaction coefficients $a_{ij}(t)$ can be discontinuous, we will consider Caratheodory solutions to (2). A Caratheodory solution is a locally absolutely continuous function $x : \Re^+ \to \Re^n : t \to x(t)$ satisfying (2) at almost all times t. It can be equivalently defined as the solution to the integral version of (2). It follows from Theorem 54 and Proposition C.3.8 in [17, pages 473-482] that there exists a unique Caratheodory solution to (2). In the sequel, we only consider Caratheodory solutions to the systems that we introduce, but do not mention it explicitly. We also assume that all the time-varying coefficients are measurable on bounded intervals.

We now show that every solution of (2) is the projection on its first n coordinates of the solution of a usual consensus system with only positive interactions and involving $2n$ agents. Observe first that system (2) is a particular case of the following system

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}(t) \left(x_j(t) - x_i(t)\right) + \sum_{j=1}^{n} r_{ij}(t) \left(-x_j(t) - x_i(t)\right), \qquad (3)$$

where the functions $p_{ij}(t)$ and $r_{ij}(t)$ are nonnegative and represent respectively the positive attractions and the reverse attractions. Indeed, one can obtain system (2) by taking $p_{ij}(t) = \max(0, a_{ij}(t))$ and $r_{ij}(t) = \max(0, -a_{ij}(t))$. Multiplying (3) by -1, one gets

$$\frac{d}{dt}(-x_i) = \sum_{j=1}^{n} p_{ij} \left((-x_j) - (-x_i)\right) + \sum_{j=1}^{n} r_{ij} \left(x_j - (-x_i)\right), \qquad (4)$$

where we omit the dependence of t for the sake of concision. Together with (3), this implies that the vector $(x^T, -x^T)^T$ is a solution of the system

$$\dot{z}_i = \sum_{j=1}^{n} p_{ij} \left(z_j - z_i\right) + \sum_{j=1}^{n} r_{ij} \left(z_{j+n} - z_i\right),$$
$$\dot{z}_{i+n} = \sum_{j=1}^{n} p_{ij} \left(z_{j+n} - z_{i+n}\right) + \sum_{j=1}^{n} r_{ij} \left(z_j - z_{i+n}\right).$$

Taking $z_i = x_i$ and $z_{i+n} = -x_i$ for $i = 1, \dots n$, one reobtains indeed (3) and (4). The equivalence that we have shown is summarized in the following proposition.

Proposition 1: $x : \Re^+ \to \Re^n$ is a solution of (2) for certain functions $a_{ij}(t)$ if and only if $z = (x^T, -x^T)^T$ is a solution of the "classical" consensus system

$$\dot{z}_i(t) = \sum_{j=1}^{2n} \tilde{a}_{ij}(t) \left(z_j(t) - z_i(t)\right), \qquad (5)$$

with $\tilde{a}_{ij}(t) = \tilde{a}_{i+n,j+n}(t) = \max(0, a_{ij}(t)) \geq 0$ and $\tilde{a}_{i+n,j}(t) = \tilde{a}_{i,j+n}(t) = \max(0, -a_{ij}(t)) \geq 0$.

In the following sections, we use Proposition 1 to translate results on classical consensus systems available in the literature into results on systems with antagonisms as (2).

Remark 1: The approach behind Proposition 1 can be seen as a generalization of the gauge transformation used for some results in [2]. The latter corresponds indeed to building a classical consensus system on n agents by picking for each i either z_i (corresponding to x_i) or z_{i+n} (corresponding to $-x_i$).

III. Reciprocal Interactions

The main results of this section rely on the following theorem from [9].

Theorem 1: Let $x : \Re^+ \to \Re^n : t \to x(t)$ be a solution of

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(t) \left(x_j(t) - x_i(t)\right),$$

where the $a_{ij}(t) \geq 0$, and let $G_\infty(\{1, \dots, N\}, E)$ be the undirected graph of *persistent interactions* defined by letting $(j, i) \in E$ if $\int_{t=0}^{\infty} a_{ij}(t) dt$ is infinite.
If there exists a $K > 0$ such that

$$a_{ij}(t) \leq K a_{ji}(t) \qquad (6)$$

holds for all $t \in R^+$ and i, j, then

(a) The system converges: $x^* = \lim_{t\to\infty} x(t)$ exists.
(b) If two agents i, j belong to the same connected component of G_∞, that is, are connected by a path in G_∞, there holds $x_i^* = x_j^*$.
(c) If i and j do not belong to the same connected component of G_∞, then $x_i^* \neq x_j^*$ unless the initial condition $x(0)$ belongs to a specific vectorial subspace of $\Re^n$ of dimension $n - 1$ or less.

Part (b) of Theorem 1 means thus that a local consensus is attained in each connected component of the graph G_∞ of persistent interactions, and part (c) states that the consensus values of the different connected components are generically different. Note that we restrict here our attention to bounded ratio symmetric interactions ($a_{ij} \leq K a_{ji}$) for the sake of simplicity, but Theorem 1 is proved in [9] under the weaker assumption that the interactions are cut-balance. We now show how it can be used to treat systems with antagonisms.

Proposition 2: Suppose that the coefficients $a_{ij}(t)$ in system (2) have symmetric signs $\mathrm{sign}\,(a_{ij}(t)) = \mathrm{sign}\,(a_{ji}(t))$ and satisfy the bounded ratio symmetry condition $|a_{ij}(t)| \leq K\,|a_{ji}(t)|$ for some constant $K > 0$ and all i, j and t. Then $x_i^* = \lim_{t\to\infty} x_i(t)$ exists for every i, and

(a) $x_i^* = x_j^*$ if $\int_0^\infty \max(0, a_{ij}(t))dt$ is infinite,
(b) $x_i^* = -x_j^*$ if $\int_0^\infty \max(0, -a_{ij}(t))dt$ is infinite.

Proof: It follows from Proposition 1 that $(x^T, -x^T)$ is a solution of the system (5). We first show that the coefficients $\tilde{a}_{ij}(t)$ of that system satisfy the bounded ratio symmetry condition (6) with the same K. Indeed, suppose that $a_{ij}(t) \geq 0$ and therefore that $a_{ji}(t) \geq 0$ since they are assumed to have the same sign. It follows from the definition of the $\tilde{a}_{ij}$ that $\tilde{a}_{ij}(t) = \tilde{a}_{i+n,j+n}(t) = a_{ij}(t) \leq K a_{ji}(t) = K\tilde{a}_{ji}(t) = K\tilde{a}_{j+n,i+n}(t)$, and $\tilde{a}_{i+n,j}(t) = \tilde{a}_{i,j+n}(t) = 0 \leq K.0 = \tilde{a}_{j,i+n}(t) = \tilde{a}_{j+n,i}(t)$, so that the bounded ratio symmetry condition 6 holds. A symmetric argument applies to the case where $a_{ij}(t) < 0$.

It follows then from an application of Theorem 1 to system (5) that $z = (x^T, -x^T)^T$ converges to some $z^* = (x^{*T}, -x^{*T})^T$, and that $z_i^* = z_j^*$ if $\int_0^\infty \tilde{a}_{ij}(t)dt$ is infinite. Consider now $i, j \in \{1, \ldots, n\}$ and remember that $\tilde{a}_{ij}(t) = \max(0, a_{ij}(t))$. Therefore, if $\int_{t=0}^\infty \max(0, a_{ij}(t))dt$ is infinite, there holds $z_i^* = z_j^*$ and thus $x_i^* = x_j^*$, which establishes condition (a). On the other hand, we have $a_{i,j+n}(t) = a_{i+n,j}(t) = \max(0, -a_{ij}(t))$. So if $\int_{t=0}^\infty \max(0, -a_{ij}(t))dt$ is infinite, there holds $z_i^* = z_{j+n}^*$ and therefore $x_i^* = -x_j^*$, which establishes condition (b). ■

Proposition 2 provides a set of equality constraints on the different limiting values x_i^*. To analyze their consequences, it is convenient to represent them by a graph with two sorts of edges. We define the *graph of persistent interactions* $G_\infty(\{1, \ldots, n\}, P, R)$ containing positive edges P and negative (or repulsive) edges R by letting $(j, i) \in P$ if $\int_0^\infty \max(0, a_{ij}(t))dt$ is infinite, and $(j, i) \in R$ if $\int_0^\infty \max(0, -a_{ij}(t))dt$ is infinite. Note that a pair of nodes can simultaneously belong to P and R. Under the assumptions of Proposition 2, this graph is symmetric, in the sense that $(i, j) \in P$ (resp. R) if and only if $(j, i) \in P$ (resp. R). In the sequel, we consider paths (and cycles) in G_∞ that can contain both sorts of edges. In particular, *a connected component* C of G_∞ is a subset of nodes such that (i) any two of them are connected by a path of edges of $P \cup R$, and (ii) no edge of P or R connects any node of C to a node out of C.

Theorem 2: Under the assumptions of Proposition 2 (symmetric signs and bounded ratio symmetry), $x_i^* = \lim_{t\to\infty} x_i(t)$ exists for every i, and

(a) If $(i, j) \in P$ then $x_i^* = x_j^*$. If $(i, j) \in R$, then $x_i^* = -x_j^*$. As a result, If i and j belong to the same connected component of G_∞ there holds $|x_i^*| = |x_j^*|$.
(b) If a connected component C contains a cycle with an odd number of edges in R, then $x_i^* = 0$ for every $i \in C$.
(c) Conversely, if a connected component C contains no cycle with odd number of edges in R, then $|x_i^*| > 0$ for all $i \in C$, except if the initial condition $x(0)$ belongs to a certain vectorial subspace of $\Re^n$ of dimension $n-1$ or less.

Note that the situation described in part (c) is often called a "bipartite consensus".

Proof: Part (a) follows directly from Proposition 2 and the definition of G_∞. For part (b), consider a cycle Q consisting of edges $\{(q_0, q_1), (q_1, q_2), \ldots (q_{|Q|-1}, q_{|Q|})\} \in P \cup R$ with $q_{|Q|} = q_0$, and let r_Q be the number of edges of Q belonging to R. It follows from the application of part (a) to every edge of Q that $x_{q_0}^* = x_{q_{|Q|}}^* = x_{q_0}(-1)^{r_Q}$. Therefore, if r_Q is odd, we have $x_{q_0}^* = 0$, and as a result $x_i^* = 0$ for every node i of connected component to which q_0 belongs.

To prove part (c), suppose now that the connected component C contains no cycle with an odd number of edges in R (and in particular, no edge that belongs to both P and R). In that case, it follows from a classical result of graph theory (see for example [6]) that C can be partitioned into two disjoints subsets C^+, C^- such that no negative edge connects two nodes of the same subset, and no positive edge connects two nodes of different subsets, that is, $i \in C^+, j \in C^-$ implies that $(i, j) \notin P$ and $i \in C^+, j \in C^+$ or $i \in C^-, j \in C^-$ implies that $(i, j) \notin R$.

We claim that there exists an initial condition $x(0)$ for which $\lim_{t\to\infty} x_i(t) = x_C$ if $i \in C^+$ and $\lim_{t\to\infty} x_i(t) = -x_C$ if $i \in C^-$, for some nonzero $x_C \in \Re$. If the interactions were constant or restricted to their long term behavior (nonnegative values a_{ij} for pairs $(j, i) \in P$, nonpositive values for pairs $(j, i) \in R$, and $a_{ij} = 0$ for other pairs) this could directly be established by selecting an initial condition where $x_i(0) = x_C > 0$ if $i \in C^+$ and $x_i = -x_C < 0$ if $i \in C^-$, and observing that $x_i(t)$ would then remain constant for every $i \in C$. The difficulty here comes from the need to take into account the "non-persitent" interactions whose integral is bounded, and that do not correspond to edges in P or R. The proof of our claim in the general case involves several technical aspects and is deferred to the Appendix.

Fix now an index $i \in C$ and let e_i be the i^{th} unit vector. To conclude the proof of (c), we note that the function that sends $x(0)$ onto the limiting value is linear and can be represented by a matrix L. Since there is an initial condition $x(0)$ for which $x_i^* \neq 0$, we have $e_i^T L x(0) \neq 0$. Therefore $e_i^T L \neq 0$, and the set of initial conditions $x(0)$ for which $e_i^T L x(0) = 0$ is a vectorial subspace of $\Re^n$ of dimension $n-1$ or less. For any $x(0)$ out of that set, we have $x_i^* \neq 0$, and $x_j^* = \pm x_i^* \neq 0$ holds thus for all $j \in C$ by part (a) and (b), which concludes the proof of part (c) of this theorem. ■

Using an argument similar to that of part (c), one can show that when the opinions of the agents in a connected component of G_∞ do not converge to 0, their limiting value is generically different from that of the other connected components. Note also that part (c) of Theorem 1 cannot directly be applied to our systems with antagonisms because all the initial conditions $z(0)$ of (5) are of the form $z^T = (x^T, -x^T)$, and belong thus to a specific common subspace.

IV. Non-Reciprocal Interactions

To demonstrate the application of our approach to systems where interactions are not necessarily reciprocal, we use the following convergence result which is a particular case of Theorem 1 in [15] from Moreau.

Theorem 3: Suppose that the $a_{ij}(t)$ are piecewise continuous with respect to t and are uniformly bounded. If there exist a $T > 0$ and a $\delta > 0$ such that the directed graph $\tilde{G}_t^{\delta,T}$ defined by connecting (j,i) if $\int_t^{t+T} a_{ij}(s)ds \geq \delta$ is strongly connected for every t, then all $x_i(t)$ converge to a common value, $\lim_{t\to\infty} x_i(t) = x^*$ for some x^*.

We remind the reader that a directed graph is strongly connected if every node is connected to every other node by a directed path. Note that the connectivity condition in Theorem 3 does not require the interactions to form a strongly connected graph at every time, but only on average over every interval of length T.

To apply Theorem 3 to our system (2), we define for all $\delta, T > 0$ the graph $G_t^{\delta,T}(\{1,\ldots,n\}, P, R)$ with positive and negative (repulsive) edges by letting $(j,i) \in P$ if $\int_t^{t+T} \max(0, a_{ij}(t))ds \geq \delta$ and $(j,i) \in R$ if $\int_t^{t+T} \max(0, -a_{ij}(t))ds \geq \delta$.

Consider now the coefficients $\tilde{a}_{ij}(t)$ defined in Proposition 1, and define the graph $\tilde{G}_t^{\delta,T}(\{1,\ldots,2n\}, \tilde{E})$ by letting $(j,i) \in \tilde{E}$ if $\int_t^{t+T} a_{ij}(s)ds \geq \delta$. One can verify that $\tilde{G}_t^{\delta,T}$ is strongly connected if and only if $G_t^{\delta,T}$ is strongly connected and contains at least one directed cycle with an odd number of edges of R. Combining this observation with Theorem 3 and Proposition 1 leads to the following result.

Theorem 4: Consider the system with antagonisms (2) and suppose that the $a_{ij}(t)$ are uniformly bounded and piecewise continuous. If there exist $\delta, T > 0$ such that for every t, the graph $G_t^{\delta,T}$ is strongly connected and contains a cycle with an odd number of edges of R, then $\lim_{t\to\infty} x_i(t) = 0$ for every i.

We can also treat the case of convergence to a "bipartite consensus", although the assumptions that need to be made are somewhat stronger. The following theorem is proved by observing that under its assumptions, the system defined in Proposition 1 can be decomposed into two strictly independent subsystems to each of which one can apply Theorem 3.

Theorem 5: Consider the system with antagonisms (2) and suppose that the $a_{ij}(t)$ are uniformly bounded and piecewise continuous.

Suppose in addition that the agents can be partitioned in two groups V_1, V_2 such that $a_{ij} \geq 0$ if i,j belong to the same group and $a_{ij} \leq 0$ if they belong to different groups. If there exist $\delta, T > 0$ such that for every t, the graph $G_t^{\delta,T}$ is strongly connected, then there exists $x_C^* \in \Re$ such that $\lim_{t\to\infty} x_i(t) = x_C^*$ for every $i \in V_1$ and $\lim_{t\to\infty} x_i(t) = -x_C^*$ exists for every $i \in V_2$. Moreover, $x_C^* \neq 0$ except if the initial condition $x(0)$ belongs to a certain specific linear subspace of $\Re^n$ of dimension $n-1$ or less.

Theorem 4 and 5 extend Theorem 2 and Remark 4 in [2] that apply respectively to constant interactions and to switching interactions satisfying a permanent strong connectivity condition.

V. Discrete Time Systems

Our approach also applies to discrete time models. Following Xia and Cao [20], we consider the system

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t) x_j(t) \tag{7}$$

with $\sum_{j=1}^n |a_{ij}(t)| = 1$. This means that the new opinion of agent i is a weighted average of the opinions of some agents (when $a_{ij}(t) > 0$) and of the opinions opposite to those of some others (when $a_{ij}(t) < 0$).

System (7) is a particular case of

$$x_i(t+1) = \sum_{j=1}^{n} p_{ij}(t) x_j(t) + \sum_{j=1}^{n} r_{ij}(t)(-x_j(t)), \tag{8}$$

where $\sum_{j=1}^n (p_{ij} + r_{ij}) = 1$, and $p_{ij}, r_{ij} \geq 0$. One reobtains indeed (7) by taking $p_{ij} = \max(0, a_{ij})$ and $r_{ij} = \max(0, -a_{ij})$. By considering (8) together with its opposite describing the evolution of $-x_i(t)$ as in Section II, we obtain the following result analogous to Proposition 1.

Proposition 3: $x : \mathbb{N} \to \Re^n$ is a solution of (7) for certain functions $a_{ij}(t)$ if and only if $z = (x^T, -x^T)^T$ is a solution of the "classical" discrete time consensus system

$$z_i(t) = \sum_{j=1}^{2n} \tilde{a}_{ij}(t) z_j(t), \tag{9}$$

with $\tilde{a}_{ij}(t) = \tilde{a}_{i+n,j+n}(t) = \max(0, a_{ij}(t)) \geq 0$ and $\tilde{a}_{i+n,j}(t) = \tilde{a}_{i,j+n}(t) = \max(0, -a_{ij}(t)) \geq 0$.

Based on this equivalence, convergence results for discrete time systems can be derived exactly as in Sections III and IV. We can for example apply the following result on consensus systems with so-called "type-symmetric" interactions, proved in [12].

Theorem 6: Suppose that $x : \mathbb{N} \to \Re^n$ satisfies $x_i(t+1) = \sum_{j=1} a_{ij}(t)$, and that the coefficients $a_{ij}(t) \geq 0$ satisfy the three following conditions

(i) Lower bound on positive coefficients: There exists an $\alpha > 0$ such that if $a_{ij}(t) > 0$, then $a_{ij}(t) \geq \alpha$.
(ii) Positive diagonal coefficients: $a_{ii}(t) \geq \alpha$.
(iii) Type symmetry: $a_{ij}(t) > 0 \Leftrightarrow a_{ji}(t) > 0$.

Then $x_i^* = \lim_{t\to\infty} x_i(t)$ exists for every i, and $x_i^* = x_j^*$ if the set of times at which $a_{ij}(t) > 0$ is infinite.

We define here the (undirectd) graph of persistent interactions $G_\infty(\{1,\ldots,n\},P,R)$ of the system (8) by letting $(j,i)\in P$ if $a_{ij}(t)>0$ infinitely often and $(j,i)\in R$ if $a_{ij}(t)<0$ infinitely often. A reasoning parallel to that of Section III (to the exception of part (c) of Theorem 2) leads to the following convergence result.

Theorem 7: Suppose that the coefficients $a_{ij}(t)$ in the system (7) satisfy the three following conditions.

(i) Lower bound on nonzero coefficients: There exists an $\alpha>0$ such that if $a_{ij}(t)=0$ then $|a_{ij}(t)|\geq\alpha$.

(ii) Positive diagonal coefficients: $a_{ii}(t)\geq\alpha>0$.

(iii) "Type-symmetry": $a_{ij}(t)\neq 0 \Leftrightarrow a_{ji}(t)\neq 0$, and in that case $\mathrm{sign}\,(a_{ij}(t))=\mathrm{sign}\,(a_{ji}(t))$.

Then every trajectory of the system converges: $x_i^*=\lim_{t\to\infty}x_i(t)$ exists for every i and every initial condition. Moreover,

(a) If $(i,j)\in P$ then $x_i^*=x_j^*$. If $(i,j)\in R$, then $x_i^*=-x_j^*$. As a result, If two nodes i,j belong to the same connected component of G_∞ then $|x_i|=|x_j|$.

(b) If the connected component C contains a cycle with an odd number of edges in R, then $x_i^*=0$ for every $i\in C$.

The following example shows that part (c) of Theorem 2, stating that the opinions generically do not converge to 0 in the absence of cycles with an odd number of edges in R, does not extend to discrete time systems.

Example 1: Consider a system with two agents where the coefficients $a_{ij}(0),a_{ij}(1)$ are summarized in these matrices

$$A(0)=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},\qquad A(1)=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

There holds $\sum_j|a_{ij}(t)|=1$ for $t=0,1$ and the assumptions of Theorem 7 are satisfied. Nevertheless, we have $x_1(2)=x_2(2)=0$ and thus $x_1^*=x_2^*=0$ for all initial conditions, irrespectively of the further interactions, and thus of the structure of G^* and the presence or absence of cycles with an odd number of edges in R.

The mathematical reason for which part (c) of Theorem 2 cannot be extended is that the linear function that sends the initial condition $x(0)$ onto $x(t)$ is not necessarily invertible in discrete time, unlike in continuous-time. As a result, one cannot apply the argument developed in the Appendix to prove part (c) of Theorem 2.

Our approach can also be applied to discrete time systems without reciprocity conditions such as type-symmetry. Results analogous to those in Section IV and similar to those in [10] can for example be obtained by combining Proposition 3 with consensus results available in the literature such as those in [11], [16], [19].

Finally, we note that a result related to Theorem 7 for a single connected component was proposed in a recent preprint by Meng et al. [14]. By analyzing the evolution of the absolute values $|x_i(t)|$ of the opinions, they show that these all converge to a same value (asymptotic modulus consensus) when the graph of persistent interactions G is connected. Their result is valid under the same assumptions as Theorem 7 except that condition (iii) is relaxed: $a_{ij}(t)$ and $a_{ji}(t)$ are not required to have the same sign when they are nonzero. They also provide a convergence result for non-reciprocal systems, guaranteeing the convergence of all $|x_i(t)|$ to a same value under some (uniform) repeated connectivity assumption that makes no distinction between positive and negative interactions.

VI. Discussion

We have shown that the trajectories of systems of opinion dynamics with antagonisms such as those studied in [1], [2], [10], [20] can be seen as the first n components of the trajectories of an associated classical model of opinion dynamics on $2n$ agents with a specific structure.

By applying results on classical consensus systems available in the literature to that associated system, we were able to obtain different general convergence results for the initial model, extending several of those previously available.

On the one hand, the relative simplicity of our proofs and the large number of results on consensus available in the literature suggest that our approach is a very convenient way of analyzing opinion dynamics models with antagonism.

On the other hand, separating the positive and negative interactions in the associated system could be a drawback in certain situations. It might for example prove harder to treat systems where the presence of interactions is symmetric but theirs signs are not (as in [14]), that is, $a_{ij}\neq 0\Leftrightarrow a_{ji}\neq 0$ but a_{ij} and a_{ji} may have different signs. This symmetry would indeed not imply a symmetry of the associated system on $2n$ agents.

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APPENDIX

We prove that under the assumptions of Theorem 2, there exists an initial condition $x(0)$ such that the opinion of at least one agent in the connected component C (and in fact all of them) does not converge to 0.

We begin by selecting a time after which the integral of the "non-persistent" interactions is sufficiently small. Let

$$\alpha(t) = \sum_{i,j:(j,i)\notin P} \max(0, a_{ij}(t)) + \sum_{i,j:(j,i)\notin R} \max(0, -a_{ij}(t)). \tag{10}$$

It follows from the definition of P and R in Section III that $\int_{t=0}^{\infty} \alpha(t)dt < \infty$. We can thus select a $T_{1/4}$ such that $\int_{t=T_{1/4}}^{\infty} \alpha(t)dt < 1/4$.

Now, since $\int_0^t |a_{ij}(t)|\, dt < \infty$ for every t and i,j, it can be proved that the state transition (or fundamental) matrix, which maps the initial conditions $x(0)$ to $x(t)$ has full rank for any finite t, see [17] (specifically, Theorem 54, Proposition C3.8, appendix C3, and appendix C4). Therefore, we can chose $x(0)$ in such a way that $x_i(T_{1/4}) = 1$ for every $i \in C^+$, -1 for every $i \in C^-$ and 0 for every $i \notin C$, where C^+ and C^- have been defined in the main part of the proof of Theorem 2. As a consequence, $x_i(t) \in [-1, 1]$ for all $t \geq T_{1/4}$. We show that $|x_i(t)| \geq 1/2$ holds for all $t > T_{1/4}$ and $i \in C$.

Consider a $t^* > T_{1/4}$ such that for every $t \in [T_{1/4}, t^*]$, there holds $x_i(t) > 0$ if $i \in C^+$ and $x_i(t) < 0$ if $i \in C^-$. The existence of such t^* follows from the continuity of x. For every $t \in [T_{1/4}, t^*]$, let $m(t) = \arg\min_{i\in C} |x_i(t)|$ be the index of the agent in C with the smallest opinion in absolute value (using the order of the indices to break ties). The pattern of change of $m(t)$ can be very complex. However, it follows from Proposition 2 in the extended version of [9] that

$$\left|x_{m(t)}(t)\right| = \left|x_{m(T_{1/4})}(T_{1/4})\right| + \int_{s=T_{1/4}}^{t} \frac{dx_{m(s)}(\tau)}{d\tau}\Big|_{\tau=s} ds, \tag{11}$$

that is, the smallest opinion (in absolute value) in C at time t can be obtained by integrating over $s \in [T_{1/4}, t]$ the derivative $\frac{dx_{m(s)}(\tau)}{d\tau}|_{\tau=s}$ of the opinion of the agent $m(s)$ that happens to have the smallest opinion at that time s.

Let us now fix a s and assume without loss of generality that $m(s) \in C^+$ so that $x_{m(s)}(s) > 0$. By definition of the system (2), this derivative $\frac{dx_{m(s)}(\tau)}{d\tau}|_{\tau=s}$ can be written as (omitting s for brevity)

$$\begin{aligned}
&\sum_{j=1}^{n} (\max(0, a_{mj})(x_j - x_m) + \max(0, -a_{mj})(-x_j - x_m)) \\
&= \sum_{j:(j,m)\in P} \max(0, a_{mj})(x_j - x_m) && (12)\\
&+ \sum_{j:(j,m)\in R} \max(0, -a_{mj})(-x_j - x_m) && (13)\\
&+ \sum_{j:(j,m)\notin P} \max(0, a_{mj})(x_j - x_m) && (14)\\
&+ \sum_{j:(j,m)\notin R} \max(0, -a_{mj})(-x_j - x_m). && (15)
\end{aligned}$$

Remember that $|x_m| = \arg\min_{i\in C} |x_i|$, and that we have assumed that $m \in C^+$ so that $x_m > 0$. Therefore, there holds $x_j - x_m \geq 0$ for every $j \in C^+$ and $-x_j - x_m \geq 0$ for every $j \in C^-$. Moreover, by definition of C^+ and C^- and since $m \in C^+$, $(j,m) \in P$ implies that $j \in C^+$, and $(j,m) \in R$ implies that $j \in C^-$. The terms in the sums (12) and (13) are thus all nonnegative. Consider now the terms of (14) and (15). Since $x_j(t) \in [-1, 1]$, we have $(\pm x_j - x_m) \geq -2$. It follows then from the definition (10) of $\alpha(s)$ that the sum of all the terms in (14) and (15) is lower bounded by $-2\alpha(s)$. Combining all these bounds, we obtain $\frac{d|x_{m(s)}(\tau)|}{d\tau}|_{\tau=s} = \frac{dx_{m(s)}(\tau)}{d\tau}|_{\tau=s} \geq -2\alpha(s)$ if $m \in C^+$. A similar argument applies if $m \in C^-$. Therefore, it follows from (11) that

$$\begin{aligned}
\left|x_{m(t)}\right|(t) &\geq \left|x_{m(T_{1/4})}(T_{1/4})\right| - 2\int_{s=T_{1/4}}^{t} \alpha(s)ds \\
&\geq 1 - 2\frac{1}{4} = \frac{1}{2},
\end{aligned}$$

holds for all $t \in [T_{1/4}, t^*]$, where we have used the definition of $T_{1/4}$ and the fact that $\left|x_{m(T_{1/4})}(T_{1/4})\right| = 1$ by construction. By definition of $m(t)$, this implies that $|x_i(t)| \geq 1/2$ for all $t \in [T_{1/4}, t^*]$ for every $i \in C$. It follows then from the continuity of x that t^* can be taken arbitrarily large, and that $|x_i(t)| \geq 1/2$ holds for every $i \in C$ and $t \geq T_{1/4}$. In particular, $|x_i^*| = |\lim_{t\to\infty} x_i(t)| \geq 1/2$ for every $i \in C$.