Identification of dynamical networks

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Abstract We consider the identification of networks of linear time-invariant dynamical systems whose node signals are measured and are connected by causal linear time-invariant transfer functions. The external signals at the nodes may comprise both known excitation signals and unknown stationary noise signals. The identification of such networks comprise two essentially different problems. The first is to find conditions on the external excitation signals that allow the identification of the whole network from the measured node signals and excitation signals. The second problem is the identification of a particular module (i.e. transfer function) embedded in the network. We present state of the art results for both problems.

1 Introduction

The identification of networks of dynamical systems has recently emerged as an active topic in the systems and control community. Attention has focused on networks in which the node signals are connected by scalar causal rational transfer functions. These node signals are excited through the network by a combination of known external excitation signals and unknown noise sources. The node signals and the known external excitation signals are assumed to be measured without error. The identification of such networks essentially contains two different questions.

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The first question is the identification of the whole network using all measured node signals and the known external excitation signals. This approach estimates all transfer functions of the network and, as a result, also delivers the topology of the network by detecting which of these transfer functions are zero, so that one can construct the directed graph describing its interconnection structure. As we shall show, there is a fundamental unidentifiability problem in the sense that it is impossible to reconstruct such dynamical networks from measurements of the nodes and of the known external excitation signals, unless some prior knowledge is available about the structure of the network. The question is thus to produce conditions on the network structure (in the form of prior knowledge) and on the external excitation signals that lead to a unique identification of the whole network. To illustrate how virgin this question was until recently, we quote from [8] published in 2010: "*Remarkably, while networks of dynamical systems have been deeply studied and analyzed in automatic control theory, the question of reconstructing an unknown dynamical network has not been formally investigated yet. Indeed, in most applicative scenarios the network is given or it is the very objective of design. However, there are also some interesting situations where the link structure is actually unknown and dynamic, such as in biological neural networks, biochemical metabolic pathways and financial markets with a high frequency trade.*"

The second question concerns the identification of a particular transfer function within the network, assuming that its interconnection structure is known. It involves questions such as which signals need to be measured, and which external excitation signals need to be applied in order to estimate the desired transfer function. A number of results on this topic have been obtained recently [9, 2, 3].

In this chapter, we present state of the art results on these two questions. We first consider the problem of global identification of a network of dynamical systems. An early result pointing to the unidentifiability problem mentioned above can be found in [6] where the authors showed that, for a strictly proper continuous time system with known inputs, the transformation from input-output form to a network form is non-unique. More recent research has focused on the modeling and identification of high-dimensional stochastic processes, where the focus has been on detecting the causal links between variables [1, 11, 7].

We examine under what conditions on the network structure and on the external signals such network can be uniquely identified from the measured node signals and the known external signals, for networks with both deterministic and stochastic inputs. Our results take the form of a range of sufficient conditions on the network structure and on the external signals that will guarantee that the network can be uniquely reconstructed from the measured data. They are close to those of [10], even though our approach takes a different route inspired by the deterministic approach of [6].

We adopt the network model structure studied in [10], and we first show that this network model can be transformed into an equivalent Multiple Input Multiple Output (MIMO) model with added noise, which can be identified in open loop. The identifiability conditions for open-loop MIMO systems are well established, and they lead to a unique Input Output (I/O) model and a unique noise model under the usual assumptions that the system is in the model set and that the data are informative with respect to the adopted model structure. The question of whether the network model can be identified from measured data then turns into the question of whether or not the mapping from the network model to the I/O model is injective.

We first propose a definition of *network identifiability* that relates to the objective of identifying the true network from measured data. A network model structure that is able to represent the true network will be called identifiable if no other different network model structure, that is unable to represent the true network, can produce the same I/O model. By extending the results of [6] to networks with unmeasured noise signals and with transfer functions that need not be strictly proper, we then show that, generically, there is an infinity of network models that produce the same I/O model, and we provide a parametrization of all these indistinguishable network models. These indistinguishable network models may even have different interconnection structures, i.e. the zero transfer functions are in different locations, leading to different corresponding graphs. This implies that a network model structure that is able to represent the true network will be identifiable only if some adequate prior knowledge is available about its structure. Such prior knowledge can take many different forms, such as the topology of the interconnection structure between the nodes, or the topology of the external excitation structure by the known excitation signals or by the noise signals.

We present a range of sufficient conditions on the structure of the external excitation signals - reference excitation signals and noise signals - that make the network model structure identifiable. These conditions show that the known excitation signals and the unknown noise signals play the same role in terms of their capacity to make the network structure identifiable; in other words identifiability can be achieved either by the known excitation signals, or by the noise signals, or by a combination of both.

In the second part of this chapter, we consider the problem of identitying a module (i.e. a transfer function) embedded in the network. Several contributions have recently been made for this problem [9, 2, 3]. A major open problem is that of finding conditions on the external excitation signals (known or noisy) that will lead to a consistent estimate of the desired transfer function. We illustrate this problem on a 3-dimensional network. Our contribution is twofold: show which external excitation signals are required to make the data informative, and show how adding additional excitation at other nodes affects the parameter variances of the estimated transfer function.

The outline of this chapter is as follows. The network model structure is presented in section 2. In section 3 we present a definition of identifiability of a network which relates to the objective of identifying the true network. We then show that a network is generically unidentifable and we parametrize the set of all indistinguishable network models. Using this parametrization, we present a range of sufficient conditions on the structure of the external excitation that render the network identifiable. In section 4 we illustrate the problem of obtaining an informative experiment for the identification of an embedded module. We conclude in section 5.

2 Problem statement

We consider a network made up of L nodes, with node signals denoted $\{w_1(t),\dots,w_L(t)\}$. These node signals are related to each other and to external excitation signals r_j and white noise signals e_j by the following network equations, which we call the **network model** and in which the matrix G^0 will be called the **network matrix**:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \dots & G_{1L} \\ G_{21} & 0 & \ddots & G_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1} & G_{L2} & \dots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + K^0(q) \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + H^0(q) \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix} \tag{1}$$

Or, equivalently

$$w(t) = G^0(q)w(t) + K^0(q)r(t) + H^0(q)e(t) \tag{2}$$

with the following properties:

- G_{ij} are proper but not necessarily strictly proper transfer functions. Some of them may be zero, indicating that there is no direct link from w_j to w_i.
- there is a delay in every loop going from one w_j to itself.
- the network is well-posed so that $(I-G^0)^{-1}$ is proper and stable.
- all node signals $w_j, j=1,\dots,L$ are measurable.
- r_i are external excitation signals that are available to the user in order to produce informative experiments for the identification of the G_{ij}. $K^0(q)$ reflects how the external excitation signals affect the node signals.
- $e \in \Re^L$ is a white noise vector with a positive definite covariance matrix Σ. $H(q)$ is a $L \times L$ stable rational matrix.
- the external excitation signals r_i are assumed to be uncorrelated with all noise signals $e_j, j=1,\dots,L$.
- q^{-1} is the delay operator.

The network model (2) can be rewritten in a more traditional form as follows:

$$w(t) = T^0(q)r(t) + N^0(q)e(t) \tag{3}$$

where

$$T^0(q) \triangleq (I-G^0(q))^{-1}K^0(q), \;\; N^0(q) \triangleq (I-G^0(q))^{-1}H^0(q). \tag{4}$$

The description (3) will be called the **input-output (I/O) description** of the network. A corresponding parametrized version $M_{io} = [T(q,\eta), N(q,\eta)]$ will be called the *input-output (I/O) model*.

3 Identifiability of the whole network

3.1 Definition of network identifiability

Consider now that our objective is to estimate the matrices $G^0(q), K^0(q)$ and $H^0(q)$ of the network (2) using the available measurements $w(t)$ and $r(t)$. Assuming that the network is driven by sufficiently informative excitation signals $r(t)$ and white noise signals $e(t)$, what are the conditions (in the form of required prior knowledge) on the network matrices $G^0(q), K^0(q), H^0(q)$ such that they can be uniquely identified from the known measured signals $w(t)$ and $r(t)$?

It is well known from the theory of identification of multi-input multi-output (MIMO) linear time-invariant (LTI) systems that from the signals $w(t)$ and $r(t)$ one can uniquely identify the matrices $T^0(q)$ and $N^0(q)$ of the input-output model (3) if the chosen model structure $M_{io} = [T(q,\eta), N(q,\eta)]$ is such that $[T^0(q), N^0(q)] = [T(q,\eta_0), N(q,\eta_0)]$ for some unique η_0 (this is the identifiability question), and if the signals $r(t)$ are sufficiently rich for the chosen parametrizations (this is the informativity question). The identification of (3) is an open loop identification problem.

The question of **network identifiability** then relates to the mapping from $[T^0(q), N^0(q)]$ to $[G^0(q), K^0(q), H^0(q)]$, namely *under what conditions (in the form of prior knowledge on the network matrices $G^0(q), K^0(q), H^0(q)$) can one uniquely recover the network matrices $[G^0(q), K^0(q), H^0(q)]$ from the true input-output description $[T^0(q), N^0(q)]$?* It can be formally defined as follows.

Definition 1. (*Identifiability of the true network model*): Consider the true network (2) defined by the triple $\mathscr{S} = [G^0, K^0, H^0]$ and a parametrized network model structure $\{M(\theta) = [G(\theta), K(\theta), H(\theta)], \theta \in D_\theta\}$ with the property that $M(\theta_0) = \mathscr{S} = [G^0, K^0, H^0]$ for some $\theta_0 \in D_\theta$. Let $[T^0, N^0]$ be the corresponding true I/O model defined by (4). Then $\mathscr{S}$ is network identifiable if there exists no other network model structure $\{\tilde{M}(\nu) = [\tilde{G}(\nu), \tilde{K}(\nu), \tilde{H}(\nu)], \nu \in D_\nu\}$ such that $(I - \tilde{G}(\nu_0))^{-1}\tilde{K}(\nu_0) = T^0$ and $(I - \tilde{G}(\nu_0))^{-1}\tilde{H}(\nu_0) = N^0$ for some ν_0, with $[\tilde{G}(\nu_0), \tilde{K}(\nu_0), \tilde{H}(\nu_0)] \neq [G^0, K^0, H^0]$.

We illustrate this definition with the following example studied in [10].

Example 1. Consider the following 3-node noise-free network $\mathscr{S}_1$:

$$G^0(q) = \begin{bmatrix} 0 & 0 & 0 \\ A(q) & 0 & 0 \\ 0 & B(q) & 0 \end{bmatrix}, K^0(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{5}$$

i.e. $G^0_{21}(q) = A(q), G^0_{32}(q) = B(q)$, where $A(q)$ and $B(q)$ are rational transfer functions, and all other G^0_{ij} are zero. The corresponding I/O description of the true network is given by (3) with

$$T^0(q) = \begin{bmatrix} 1 & 0 & 0 \\ A(q) & 1 & 0 \\ A(q)B(q)+1 & B(q) & 0 \end{bmatrix} \tag{6}$$

The following $\bar{G}$ and $\bar{K}$ yield a network $\mathscr{S}_2$ with the same I/O model T^0 as the "true" network (5):

$$\bar{G}(q) = \begin{bmatrix} 0 & -B(q) & 1 \\ A(q) & 0 & 0 \\ 0 & B(q) & 0 \end{bmatrix}, \bar{K}(q) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{7}$$

Thus, the network $[\bar{G}, \bar{K}]$ is indistinguishable from the true network even though it has a different topology. It means that if the same data $\{r(t)\}$ excite the two networks (5) and (7), they will generate the same data $\{w(t)\}$, despite the fact that the graphs of these two networks are different.

3.2 The set of all indistinguishable networks

We show in this section that there exists an infinite set of network models $M(\theta) = [G(q,\theta), K(q,\theta), H(q,\theta)] \in \mathscr{M}^*$ that produce the same I/O model $[T(q), N(q)]$, and we parametrize the set of these indistinguishable network models. This will allow us to derive conditions on prior knowledge of the true network model structure that will make this network identifiable in the sense of Definition 1. This parametrization is an extension to networks with noisy inputs of a result of [6] which adressed the case of a noiseless network with strictly proper transfer functions. We first introduce the notion of **admissible network matrix**.

Definition 2. (*Admissible network matrix*): A network matrix $G(q,\theta)$ is called admissible if the following conditions hold:

- the diagonal elements of $G(q,\theta)$ are zero;
- there is a delay in every loop going from one w_j to itself;
- all $G_{ij}(q,\theta)$ are proper
- $(I - G(q,\theta))^{-1}$ is stable

The following theorem describes the set of all network models that produce the same I/O model $[T \;\; N]$. For brevity of notations, we delete the (q,θ) dependence.

Theorem 1. *The set of all network models that produce an I/O model $M_{io} = [T \;\; N]$ is given by*

$$\{[\tilde{G}\ \tilde{K}\ \tilde{H}] = [\tilde{G} \;\; (I-\tilde{G})T \;\; (I-\tilde{G})N]\} \tag{8}$$

where $\tilde{G}$ is any admissible network matrix of size $L \times L$, in the sense of Definition 2.
Proof: *We first show that the set of network matrices defined in (8) produce the correct I/O model $M_{io} = [T, N]$. Indeed, the I/O transfer function matrices derived from (8) are*

$$\tilde{T} = (I-\tilde{G})^{-1}\tilde{K} = (I-\tilde{G})^{-1}(I-\tilde{G})T = T$$
$$\tilde{N} = (I-\tilde{G})^{-1}\tilde{H} = (I-\tilde{G})^{-1}(I-\tilde{G})N = N$$

Conversely, let $[\tilde{G}, \tilde{K}, \tilde{H}]$ *be any network that produces the correct* T *and* N *with* $\tilde{G}$ *admissible. Then, necessarily, we must have* $(I-\tilde{G})^{-1}\tilde{K} = T$ *and* $(I-\tilde{G})^{-1}\tilde{H} = N$. *Premultiplying both equations by* $(I-\tilde{G})$ *shows that this network has the form (8).*

This result shows that without prior knowledge about the network structure, any admissible $\tilde{G}$ can produce the true I/O model $[T^0(q), N^0(q)]$. The choice of any particular $\tilde{G}$ fixes the corresponding $\tilde{K} = (I-\tilde{G})T$ and $\tilde{H} = (I-\tilde{G})N$, and the network $[\tilde{G}, \tilde{K}, \tilde{H}]$ is then indistinguishable from the "true" $[G^0, K^0, H^0]$. This means that if they are driven by the same $\{r(t), e(t)\}$ signals, they will generate the same $\{w(t)\}$. Thus, a network is generically not identifiable from measured data $\{w(t), r(t)\}$, unless some prior information is known about $G^0(q)$ and/or $K^0(q)$ and/or $H^0(q)$.

The following corollary, which is an extension to noisy networks of Lemma 4 of [6], will help us generate constraints that make a network identifiable.

Corollary 1. *Let* $[G^0, K^0, H^0]$ *be the transfer matrices of the "true" network. Let* ΔG *be any transfer function matrix of size* $L \times L$ *such that* $\tilde{G} \triangleq G^0 + \Delta G$ *is admissible in the sense of Definition 2. Let* $\tilde{K} = K^0 + \Delta K$ *and* $\tilde{H} = H^0 + \Delta H$ *be the corresponding matrices defined by (8). Then the network* $[\tilde{G}, \tilde{K}, \tilde{H}]$ *has the same I/O model as the true network* $[G^0, K^0, H^0]$ *if and only if*

$$[\Delta G \; \Delta K \; \Delta H] \begin{bmatrix} T^0 & N^0 \\ I & 0 \\ 0 & I \end{bmatrix} = [0 \;\; 0] \tag{9}$$

Proof: *The proof follows immediately from Theorem 1 by noting that for all these* $[\tilde{G} \; \tilde{K} \; \tilde{H}]$ *we have* $(I-\tilde{G})^{-1}[\tilde{K} \;\; \tilde{H}] = [T^0 \; N^0]$.

3.3 Conditions for network identifiability

In this section we use the result of Corollary 1 to derive a range of sufficient conditions under which the true network is identifiable. These conditions take the form of prior knowledge on the structure of the excitation matrices $K^0(q)$ and $H^0(q)$. As stated in the introduction, it is a realistic situation that the way in which the external signals enter the network is known a priori. The following theorem provides a first set of sufficient conditions.

Theorem 2. *The network structure (1) is identifiable if* $L-1$ *columns of the matrix* $[K^0 \;\; H^0]$ *are known and linearly independent.*
Proof: *It follows from Corollary 1 that the network* $[G^0 \;\; K^0 \;\; H^0]$ *is identifiable if and only if there is no triple* $[\Delta G \;\; \Delta K \;\; \Delta H]$ *that satisfies*

$$\Delta G (I - G^0)^{-1} [K^0 \;\; H^0] = -[\Delta K \;\; \Delta H] \tag{10}$$

Since ΔG *has zeroes on its diagonal, it contains* $L \times (L-1)$ *unknown elements. Now let* W *denote the* $L \times (L-1)$ *submatrix of* $[K^0 \;\; H^0]$ *made up of its known and linearly* *independent columns. Then the corresponding columns of* $[\Delta K \;\; \Delta H]$ *are zero. From (10) we can thus extract the following subset of equations for* ΔG*:*

$$\Delta G(I - G^0)^{-1} W = O \tag{11}$$

where W *and* O *have size* $L \times (L-1)$*. This represents a set of* $L \times (L-1)$ *linearly independent equations for the* $L \times (L-1)$ *unknown elements of* ΔG*, from which it follows that* $\Delta G = 0$*. It then follows from (10) that* ΔK *and* ΔH *are also zero.*

By applying this theorem to Example 1 we note that if K^0 is known, then the network is identifiable. A network is also identifiable when either $K^0(q)$ or $H^0(q)$ is diagonal with nonzero diagonal elements, a situation that is not covered by Theorem 2.

Theorem 3. *Consider the network structure (1) and assume that either* $K^0(q)$ *or* $H^0(q)$ *is diagonal and of full rank. Then the network is identifiable.*
The proof can be found in [4].

Alternative sets of sufficient conditions for identifiability of the whole network have also been derived in [10].

4 Identification of an embedded module

In this section we consider the other major problem in the identification of networks, namely the identification of a single embedded module. Without loss of generality, consider that the objective is to identify the module $G_{12}(q)$ in the network (1).

Historically, this problem was addressed first in [9]. In that paper, the authors proposed several solutions to this problem, based on existing closed-loop identification methods. Indeed, if the objective is to identify $G_{12}(q)$, it is easy to show that the network model (1) can be rewritten as a Multiple Input Single Output (MISO) closed-loop system, where w_1 acts as the single output and $[w_2\; w_3 \ldots w_L]^T$ acts as the input vector of this closed-loop system. Thus, the various methods of closed-loop identification can be applied for the identification of $G_{12}(q)$.

An important problem that has not been solved so far is that of deciding which external excitation signals, measured or unmeasured, need to be applied for the identification algorithms to converge to the true G_{12}. This is the question of informativity of the identification experiment. In [9, 2] it is assumed that the vector $w(t)$ of node signals is informative, but this is an internal constraint. The difficult question is what are the requirements on the external signals, $r_i(t)$ and $e_i(t)$, that will deliver informative data for the identification of the module $G_{12}(q)$. Assuming that different choices of external signals can yield informative data, then another interesting question is how do these different choices affect the variance of the estimated $\hat{G}_{12}(q)$.

The objective of obtaining necessary and sufficient informativity conditions on the external excitation signals for the identification of a specific module, say G_{12}, is illusory, since these informativity conditions will depend on the method that is used for the identification of G_{12} and, in particular, on the signals that are used.

Thus, the aim is to find sufficient conditions for informativity. The first contribution to this informativity question for an embedded module is to be found in [3], where we have analyzed a 3-node network. We have shown that, even in such simple network, different alternatives exist for the identification of G_{12} and we have proposed a framework, based on [5], for the computation of sufficient conditions for informativity depending on the identification method used.

Here we study the *direct identification* of G_{12} using the first equation of (12), and we extend the analysis of [3] by computing not just the informativity requirements, but also the way in which informative excitation signals affect the variance of the estimated G_{12}. In order to obtain an unbiased estimate of G_{12}, the direct method requires the identification of the vector $[G_{12} \;\; G_{13}]$. To keep the analysis simple, we shall assume that $K^0 = H^0 = I$. Thus, consider the following 3-node network:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & G_{32} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{12}$$

where it is desired to identify G_{12}. For the purpose of analysing the effect of different excitation scenarios on the estimates, we adopt the following model structure for the parametrization of G_{12}, G_{13}:

$$\mathscr{M} = \left\{ G_{12}(\alpha), G_{13}(\alpha,\beta),\ \theta = \left(\alpha^T \;\; \beta^T \right)^T \in D_\theta \subset \mathscr{R}^d \right\} \tag{13}$$

where $G_{12}(\alpha)$ and $G_{13}(\alpha,\beta)$ are rational transfer functions, $\theta \in \mathscr{R}^d$ is the vector of model parameters, and D_θ is a subset of admissible values for θ. Thus, α are the possibly common parameters of $G_{12}(\theta)$ and $G_{13}(\theta)$.

We shall assume that there exists some $\theta^0 = (\alpha_0^T, \beta_0^T)^T \in D_\theta$ that represents the true G_{12}^0 and G_{13}^0. The one-step ahead prediction error for $w_1(t)$ is given by

$$\varepsilon_1(t,\theta) \stackrel{\Delta}{=} w_1(t) - \hat{w}_1(t|t-1,\theta) = [w_1(t) - G_{12}(\alpha)w_2(t) - G_{13}(\alpha,\beta)w_3(t) - r_1(t)]$$

If the model structure is identifiable and the data informative, the parameter vector estimate $\hat{\theta}^N$ converges asymptotically to the true θ^0, and the per sample asymptotic covariance matrix is given by $P_\theta = [I(\theta^0)]^{-1}$ where $I(\theta)$ is the information matrix:

$$I(\theta) \stackrel{\Delta}{=} \bar{E}[\psi(t,\theta)\psi^T(t,\theta)] \tag{14}$$

The pseudoregressor vector $\psi(t,\theta) \stackrel{\Delta}{=} \frac{\partial \varepsilon_1(t,\theta)}{\partial \theta}$ is expressed as follows as a function of the excitation signals:

$$\psi(t,\theta) = V(q,\theta) \begin{bmatrix} r_1(t) + e_1(t) \\ r_2(t) + e_2(t) \\ r_3(t) + e_3(t) \end{bmatrix} \tag{15}$$

where $V(q,\theta)$ is a $d\times 3$ matrix of transfer functions obtained as follows from the partial derivatives of $G_{12}(\theta)$ and $G_{13}(\theta)$ with respect to the unknown parameters.

$$V(q,\theta) = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}, \text{ where} \tag{16}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} T_{21}^0 & T_{31}^0 \\ T_{22}^0 & T_{32}^0 \\ T_{23}^0 & T_{33}^0 \end{bmatrix} \begin{bmatrix} \nabla_1 \\ \nabla_2 \end{bmatrix}, \text{ with } \nabla_1 = \begin{bmatrix} \frac{\partial G_{12}}{\partial \alpha} \\ 0 \end{bmatrix} \text{ and } \nabla_2 = \begin{bmatrix} \frac{\partial G_{13}}{\partial \alpha} \\ \frac{\partial G_{13}}{\partial \beta} \end{bmatrix}. \tag{17}$$

Here the T_{ij}^0 are the elements of the second and third column of the transfer matrix $T^0 \stackrel{\Delta}{=} (I-G^0)^{-1}$ of the true network (12).

A data set is informative if the information matrix that it produces is nonsingular, i.e. $I(\theta) > 0$. By the above expressions, this is equivalent with the condition that there exists no vector $\mu \in \Re^d$ with $\mu \neq \mathbf{0}$ such that

$$\mu^T V(q,\theta) = \mathbf{0}. \tag{18}$$

We now apply this informativity analysis to the identification of $G_{12} = a_1 q^{-1} + a_2 q^{-2}$ and $G_{13} = bq^{-1}$ using the direct prediction error method based on the first equation in the following 3-node example.

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & a_1 q^{-1} + a_2 q^{-2} & bq^{-1} \\ q^{-1} & 0 & 0 \\ 0 & cq^{-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{19}$$

Applying expressions (16)-(17) to this example, with $\alpha = (a_1 \;\; a_2)$ and $\beta = b$, yields:

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} q^{-2} & q^{-1} & bq^{-3} \\ q^{-3} & q^{-2} & bq^{-4} \\ cq^{-3} & cq^{-2} & q^{-1} - a_1 q^{-3} - a_2 q^{-4} \end{bmatrix} \tag{20}$$

where $\Delta = 1 - a_1 q^{-2} - (a_2 + bc) q^{-3}$. From (20) it is clear that $\mu^T [V_1 \;\; V_2] = \mathbf{0}$ for $\mu = [0 \;\; c \;\; -1]^T$, while $Ker(V^3) = \{\mathbf{0}\}$. This shows that applying either $r_3 \neq 0$ or $e_3 \neq 0$ is a necessary and sufficient condition for the generation of informative data, and thus for convergence of the parameters a_1, a_2, b to their true values. Additional signals at other nodes may reduce the variance of these estimates, and hence of $\hat{G}_{12}$, since $I(\theta)$ is given by (14), which leads to the following covariance of the estimate:

$$P_{\hat{\theta}^N} = \frac{\lambda_1}{N} [I(\theta^0)]^{-1} \qquad I(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_1^3 [V_i V_i^\star \Phi_{r_i} + V_i V_i^\star \lambda_i] \right\} d\omega \tag{21}$$

where N is the number of data used in the identification and λ_i is the variance of e_i.

We illustrate our informativity analysis and the effect of different scenarii of external excitation on parameter variance by calculating the variance from (21) using the following true parameters: $a_1 = -0.3, a_2 = 0.8, b = -0.5, c = 0.5$. We consider there different scenarii: excitation of r_3 alone, excitation of r_3 and r_1, and excitation of all inputs. In all cases, the inputs are white noise with unit variance, while a white noise with variance $\lambda_1 = 2$ is present in the first equation - the one that is used for prediction error identification. The variances of the parameter estimates are calculated for $N = 2,000$ data.

Table 1 below shows the different experimental scenarii and the corresponding values of the covariance matrix - note that the individual variances of each parameter correspond to the diagonal elements of this matrix. Recall that either $e_3 \neq 0$ or $r_3 \neq 0$ is necessary and sufficient for informativity. Sufficiency is confirmed in the first part of the Table, which gives a finite covariance for the first scenario where only node 3 is excited. Necessity is confirmed by simulations, which yield an infinite covariance matrix if identification is performed with $r_3 = e_3 = 0$. Notice in the Table how the variances are reduced as excitation in the other inputs is added, although this reduction is very slim in the variance of parameter b. Identical covariance matrices are obtained if $r_3 = 0$ while $e_3 \neq 0$ is applied with the same unit variance as that used for r_3.

Table 1 Covariance matrices using white-noise (WN) inputs and data length $N = 2,000$; all inputs have variance equal to one, and $\lambda_1 = 2$

$r_1(t) = 0, r_2(t) = 0, r_3(t) =$ WN, $e_1(t) =$ WN $(\lambda_1 = 2)$, $e_2(t) = 0$ and $e_3(t) = 0$
$P(\hat{\theta}^N) = 10^{-5} \begin{bmatrix} 4.76 & 1.09 & 1.09 \\ 1.09 & 7.35 & -5.56 \\ 1.09 & -5.56 & 11.5 \end{bmatrix}$
$r_1(t) = WN, r_2(t) = 0, r_3(t) =$ WN, $e_1(t) =$ WN $(\lambda_1 = 2)$, $e_2(t) = 0$ and $e_3(t) = 0$
$P(\hat{\theta}^N) = 10^{-5} \begin{bmatrix} 3.27 & 0.754 & 0.752 \\ 0.754 & 5.95 & -5.57 \\ 0.752 & -5.57 & 11.4 \end{bmatrix}$
$r_1(t) = WN, r_2(t) = WN, r_3(t) =$ WN, $e_1(t) =$ WN $(\lambda_1 = 2)$, $e_2(t) = 0$ and $e_3(t) = 0$
$P(\hat{\theta}^N) = 10^{-5} \begin{bmatrix} 2.49 & 0.576 & 0.573 \\ 0.576 & 5.21 & -5.58 \\ 0.573 & -5.58 & 11.4 \end{bmatrix}$

5 Conclusions

We have described two major problems of current research interest in the identification of dynamical networks: the identification of the whole network (both the topology and the transfer functions) and the identification of a particular module embedded in the network. For the first problem, we have shown that there is a fundamental identifiability problem and we have described the set of all indistinguishable networks; this parametrization has allowed us to obtain sufficient conditions for identifiability by imposing constraints in the form of prior knowledge on the excitation structure. For the second problem, a major open problem is that of finding informative excitation experiments. We have illustrated on a simple 3-node network how the experiment conditions affect informativity, as well as the variance of the estimated parameters.

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