CORRELATION-BASED TUNING OF MULTIVARIABLE CONTROLLERS: VARIANCE ANALYSIS

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Abstract: The iterative Correlation-based Tuning (CbT) has been proposed recently to tune multivariable linear time-invariant controllers. A key feature is the fact that the controller parameters are updated by performing a single experiment per iteration. In this contribution, the accuracy properties of the estimated parameters for 2×2 systems are compared for two cases of excitation, either a single reference or the two references simultaneously. The performed analysis reveals that the presence of both reference signals does not improves the accuracy of the estimated controller parameters compared to the case with a single reference excitation. A heuristic solution to this problem has been proposed. The results presented here show that there is a trade-off between the experimental cost and the accuracy of the estimated parameters when performing a single experiment per iteration.

Keywords: Data-driven control, iterative tuning, variance analysis

1. INTRODUCTION

Data-driven methods have drawn wide attention in the control community in the last ten years. Several methods have appeared such as controller unfalsification (Safonov and Tsao, 1997), simultaneous perturbation stochastic approximation control (Spall and Cristion, 1998), iterative feedback tuning (Hjalmarsson *et al.*, 1998) and virtual reference feedback tuning (Campi *et al.*, 2002). A key question that arises in this research area is how to cope with the noise that necessarily corrupts the measurements and therefore also affects the closed-loop performance.

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Recently, another approach to data-driven controller tuning appeared in (Karimi *et al.*, 2003). The underlying idea of this method, labelled Correlation-based Tuning (CbT), is inspired from the well-known correlation approach in system identification (Söderström and Stoica, 1983). The controller parameters are tuned to decorrelate the closed-loop output error between the designed and achieved closed-loop systems with the external reference signal. Ideally, the closed-loop output error only contains the contribution of the noise. Moreover, the calculated controller parameters are asymptotically insensitive to measurement noise. A theoretical survey of this method can be found in (Karimi *et al.*, 2004).

In (Mišković *et al.*, 2005), the tuning of LTI multivariable controllers using the CbT approach is proposed. Assuming that the same number of inputs and outputs, the off-diagonal elements of the controller transfer function matrix are tuned to diagonalize the closed-loop system, while the elements on the main diagonal are tuned to provide the desired closed-loop performance. The aspect that makes this approach particularly appealing for the tuning of MIMO controllers is that a single experiment per iteration is sufficient for tuning all controllers and decouplers regardless of the number of inputs and outputs. This is a considerable advantage compared to some other methods where the number of experiments needed to estimate the gradient increases with the number of plant inputs n_u and outputs n_y. For example, the IFT approach requires $n_y n_u + 1$ experiments per iteration to tune the controller transfer function (Hjalmarsson, 1999). Now, the following question arises in this context: Is the simultaneous excitation of all reference signals advantageous or detrimental to the accuracy of the simulated controller parameters?

To answer this question, a multivariable controller is tuned for an 2×2 system. The variance of the estimated controller parameters is analyzed for two cases of excitation, i.e. when a single reference signal is used or when both reference signals are applied simultaneously. The result of this analysis reveals that the addition of the second reference signal impairs the variance of the estimated controller parameters. An ad hoc solution to cope with this increase in the variance is proposed.

The paper is organized as follows. Some notations and the idea of the multivariable CbT approach are given in Section 2. Section 3 deals with tuning of LTI multivariable controllers by the decorrelation procedure. The variance analysis is presented in Section 4. Finally, some concluding remarks are given in Section 5.

2. PRELIMINARIES

2.1 Notations

Consider the block diagram of the model-following problem presented in Fig. 1. The upper part shows the achieved closed-loop system with the unknown 2×2 true plant whose outputs can be described by the following LTI multivariable discrete-time model:

$$\mathbf{y}(t) = \mathbf{G}(q^{-1})\mathbf{u}(t) + \mathbf{v}(t) \tag{1}$$

where $\mathbf{y}(t) \in \mathcal{R}^2$ denotes the outputs of the true plant at time t, $\mathbf{u}(t) \in \mathcal{R}^2$ the control signals, $\mathbf{v}(t) \in \mathcal{R}^2$ disturbances on the outputs and $\mathbf{G}(q^{-1}) \in \mathcal{R}^{2\times 2}$ a transfer function matrix with q^{-1} being the backward-shift operator. It is assumed that $\mathbf{v}(t)$ is a zero-mean quasi-stationary stochastic process.

The controller transfer function matrix $\mathbf{K}(\boldsymbol{\rho}) \in \mathcal{R}^{2\times 2}$ is parameterized by some parameter vector $\boldsymbol{\rho} \in \mathcal{R}^{n_\rho}$, and $\mathbf{r}(t) \in \mathcal{R}^2$ represents external reference signals.

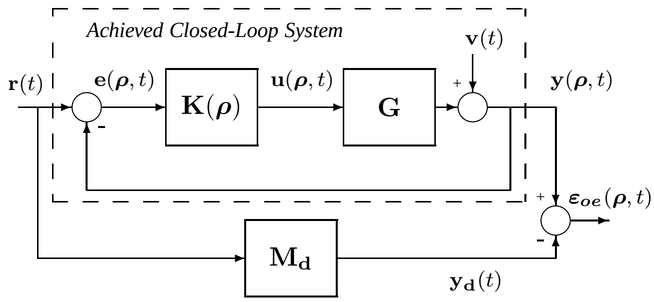

Fig. 1. Achieved multivariable closed-loop system and its reference model

The reference signals $\mathbf{r}(t)$ are assumed to be uncorrelated with the disturbances $\mathbf{v}(t)$. Furthermore, the elements of the reference signal vector $\mathbf{r}(t)$ are assumed to be mutually independent.

The (i, j) element of the controller transfer function matrix is:

$$K^{(ij)}(q^{-1}, \boldsymbol{\rho}) = \frac{S^{(ij)}(q^{-1}, \boldsymbol{\rho})}{R^{(ij)}(q^{-1}, \boldsymbol{\rho})} \tag{2}$$

where

$$R^{(ij)}(q^{-1}, \boldsymbol{\rho}) = 1 + r_1^{(i,j)} q^{-1} + \cdots + r_{n_r}^{(ij)} q^{-n_r}$$
$$S^{(ij)}(q^{-1}, \boldsymbol{\rho}) = s_0^{(ij)} + s_1^{(ij)} q^{-1} + \cdots + s_{n_s-1}^{(ij)} q^{-n_s+1}$$

It is assumed that all controllers $K^{(ij)}(q^{-1}, \boldsymbol{\rho})$, $i = 1, 2,\ j = 1, 2$ have the same order and have no common parameters. Hence, the controller parameter vector $\boldsymbol{\rho}$ can be written as follows:

$$\boldsymbol{\rho}^T = [\boldsymbol{\rho}_{11}^T, \boldsymbol{\rho}_{12}^T, \boldsymbol{\rho}_{21}^T, \boldsymbol{\rho}_{22}^T] \tag{3}$$

where

$$\boldsymbol{\rho}_{ij}^T = [r_1^{(ij)}, r_2^{(ij)}, \ldots, r_{n_r}^{(ij)}, s_0^{(ij)}, s_1^{(ij)}, \ldots, s_{n_s-1}^{(ij)}]$$

Note that $n_\rho = 4(n_r + n_s)$.

The lower part in Fig. 1 shows the reference model $\mathbf{M_d}$ defining the desired behavior of the closed-loop outputs $\mathbf{y_d}(t)$. The reference model can be constructed, for example, as the closed-loop system containing a model of the plant $\mathbf{G_0}$ and the controller $\mathbf{K_0}$:

$$\mathbf{M_d} = (I + \mathbf{G_0}\mathbf{K_0})^{-1}\mathbf{G_0}\mathbf{K_0}. \tag{4}$$

It is assumed that the reference model $\mathbf{M_d}$ has a diagonal structure. In this way, the controller $\mathbf{K_0}$ meets the control and decoupling specifications with respect to $\mathbf{G_0}$.

The closed-loop response can be written as:

$$\mathbf{y}(\boldsymbol{\rho}, t) = \boldsymbol{\mathcal{T}}\mathbf{r}(t) + \boldsymbol{\mathcal{S}}\mathbf{v}(t), \tag{5}$$

and the control error is:

$$\mathbf{e}(\boldsymbol{\rho}, t) = \mathbf{r}(t) - \mathbf{y}(\boldsymbol{\rho}, t) = \boldsymbol{\mathcal{S}}\left(\mathbf{r}(t) - \mathbf{v}(t)\right), \tag{6}$$

where $\boldsymbol{\mathcal{S}}$ denotes the output sensitivity function:

$$\boldsymbol{\mathcal{S}} = (I + \mathbf{G}\mathbf{K})^{-1} \tag{7}$$

and $\boldsymbol{\mathcal{T}}$ the complementary sensitivity function:

$$\boldsymbol{\mathcal{T}} = (I + \mathbf{GK})^{-1}\mathbf{GK} \quad (8)$$

with $I \in \mathcal{R}^{2\times 2}$ being the identity matrix.

The closed-loop output error is defined as:

$$\boldsymbol{\varepsilon_{oe}}(\boldsymbol{\rho}, t) = \mathbf{y}(\boldsymbol{\rho}, t) - \mathbf{y_d}(t). \quad (9)$$

A few words regarding the notations: The signals collected under closed-loop operation using the controller $\mathbf{K}(\boldsymbol{\rho})$ will carry the argument $\boldsymbol{\rho}$. The elements of vector signals and transfer function matrices will carry the position as a superscript in the parenthesis and will not be in bold. For example, $y^{(i)}(\boldsymbol{\rho}, t)$ will denote the i^{th} component of the output vector $\mathbf{y}(\boldsymbol{\rho}, t)$. Furthermore, the backward-shift operator q^{-1} will be omitted whenever appropriate.

2.2 Idea of Multivariable Correlation-based Tuning

For the controller structure presented in Fig. 2, consider the following design specification : controllers $K^{(21)}(\boldsymbol{\rho})$ and $K^{(12)}(\boldsymbol{\rho})$ are to decouple the outputs $y^{(2)}(\boldsymbol{\rho}, t)$ and $y^{(1)}(\boldsymbol{\rho}, t)$ from $r^{(1)}(t)$ and $r^{(2)}(t)$, respectively; controllers $K^{(11)}(\boldsymbol{\rho})$ and $K^{(22)}(\boldsymbol{\rho})$ are to provide satisfactory tracking of $y_d^{(1)}(t)$ by $y^{(1)}(\boldsymbol{\rho}, t)$ and $y_d^{(2)}(t)$ by $y^{(2)}(\boldsymbol{\rho}, t)$, respectively. In other words, the desired output $\boldsymbol{\mathcal{S}_d}$ and complementary $\boldsymbol{\mathcal{T}_d}$ sensitivity functions are in a block-diagonal form.

Consider first the tuning of the decoupler $K^{(12)}(\rho)$. When applying the controller $\mathbf{K_0}$ to the true plant excited by the reference signal $\mathbf{r}(t)$, the output $y^{(1)}(\boldsymbol{\rho}, t)$ contains the contributions due to the disturbance $\mathbf{v}(t)$ and the reference signals $r^{(1)}(t)$ and $r^{(2)}(t)$. The reference signals $r^{(1)}(t)$ and $r^{(2)}(t)$ are mutually independent and uncorrelated with $\mathbf{v}(t)$. Hence, the idea of adjusting the parameters of $K^{(12)}(\boldsymbol{\rho})$ is to make the output $y^{(1)}(\boldsymbol{\rho}, t)$ uncorrelated with the reference signal $r^{(2)}(t)$. The resulting decoupler provides $y^{(1)}(\boldsymbol{\rho}, t)$ that contains only the contributions due to $v^{(1)}(t)$ and $r^{(1)}(t)$, i.e. the influence of $v^{(2)}(t)$ and $r^{(2)}(t)$ on $y^{(1)}(\boldsymbol{\rho}, t)$ is eliminated.

Now consider the tuning of $K^{(11)}(\rho)$. Again, with $\mathbf{K_0}$ operating in the loop, the observed closed-loop output error $\varepsilon_{oe}^{(1)}(\boldsymbol{\rho}, t)$ contains a contribution due to the disturbance $\mathbf{v}(t)$ and another contribution stemming from the difference between $\mathbf{G}$ and $\mathbf{G_0}$ that, in turn, has two parts originating from $r^{(1)}(t)$ and $r^{(2)}(t)$. The idea is to adjust the parameters of $K^{(11)}(\boldsymbol{\rho})$ so as to make $\varepsilon_{oe}^{(1)}(\boldsymbol{\rho}, t)$ uncorrelated with $r^{(1)}(t)$. The controller updated in such a way compensates the effect of modeling errors to the extent that the closed-loop error $\varepsilon_{oe}^{(1)}(\boldsymbol{\rho}, t)$ contains only the disturbance filtered by the closed-loop system. This way, the output $y^{(1)}(\rho, t)$ will achieve the desired output $y_d^{(1)}(t)$. Note that the effect of modeling errors due to $r^{(2)}(t)$ is eliminated by the decoupler $K^{(12)}(\boldsymbol{\rho})$.

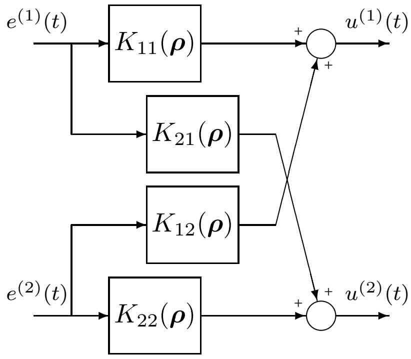

Fig. 2. Multivariable 2×2 controller

A similar reasoning follows for $K^{(21)}(\boldsymbol{\rho})$ and $K^{(22)}(\boldsymbol{\rho})$ that are related to the output $y^{(2)}(\boldsymbol{\rho}, t)$.

3. DECORRELATION PROCEDURE

In (Mišković *et al.*, 2005) the controller transfer function parameters are computed by iterative minimization of the two-norm of the cross-correlation function between an instrumental variable matrix and a vector consisting of the closed-loop output errors and outputs. Here, the parameters of the controller are calculated as solutions of the following system of correlation equations:

$$F(\boldsymbol{\rho}) \triangleq E\left\{\bar{F}(\boldsymbol{\rho})\right\} = 0 \quad (10)$$

where $E\{\cdot\}$ is the mathematical expectation, and the vector $\bar{F}(\boldsymbol{\rho}) \in \mathcal{R}^{n_\rho \times 1}$ reads:

$$\bar{F}^T(\boldsymbol{\rho}) = \left[\bar{f}_{11}^T(\boldsymbol{\rho}), \bar{f}_{12}^T(\boldsymbol{\rho}), \bar{f}_{21}^T(\boldsymbol{\rho}), \bar{f}_{22}^T(\boldsymbol{\rho})\right] \quad (11)$$

with

$$\bar{f}_{ij}(\boldsymbol{\rho}) = \frac{1}{N}\sum_{t=1}^{N} \zeta_{ij}(\boldsymbol{\rho}, t)\eta_{ij}(\boldsymbol{\rho}, t) \quad i, j = 1, 2 \quad (12)$$

where N is the number of data and $\zeta_{ij}(\boldsymbol{\rho}, t) \in \mathcal{R}^{n_\zeta}$ the vector of instrumental variables. Note that $n_\zeta = n_r + n_s$, and $n_\rho = 4n_\zeta$. According to the discussion in Section 2.2, the variable $\eta_{ij}(\boldsymbol{\rho}, t) \in \mathcal{R}$ is constructed in the following way:

$$\eta_{ij}(\boldsymbol{\rho}, t) = \begin{cases} \varepsilon_{oe}^{(i)}(\boldsymbol{\rho}, t), & i = j \\ y^{(i)}(\boldsymbol{\rho}, t), & i \neq j \end{cases} \quad (13)$$

and the vectors of instrumental variables $\zeta_{ij}(\boldsymbol{\rho}, t)$ read:

$$\zeta_{ij}(\boldsymbol{\rho}, t) = \begin{cases} \zeta_{ii}(r^{(i)}, \boldsymbol{\rho}, t), & i = j \\ \zeta_{ij}(r^{(j)}, \boldsymbol{\rho}, t), & i \neq j. \end{cases} \quad (14)$$

The rationale behind (13) and (14) is as follows. First, note that the component $\bar{f}_{ij}(\boldsymbol{\rho}) \in \mathcal{R}^{n_\zeta}$ of $\bar{F}(\boldsymbol{\rho})$ is used to tune the controller $K^{(ij)}(\boldsymbol{\rho})$. Then, to tune the diagonal elements of the controller transfer function, the vector of instrumental variables $\zeta_{ii}(\boldsymbol{\rho}, t)$ is chosen to be correlated with $r^{(i)}(t)$ and independent of $v^{(i)}(t)$. Note that, when all design specifications are met, $\varepsilon_{oe}^{(i)}(\boldsymbol{\rho}, t)$ contains only the contribution of $v^{(i)}(t)$. On the other hand, to tune the decouplers it is sufficient the decorrelate $r^{(j)}(t)$ and $y^{(i)}(t)$, i.e. $\zeta_{ij}(\boldsymbol{\rho}, t)$, $i \neq j$ should be correlated with $r^{(j)}(t)$.

A solution of equations (10) can be found using one iterative stochastic approximation procedure, for example the Robbins-Monro algorithm (Robbins and Monro, 1951):

$$\boldsymbol{\rho}_{i+1} = \boldsymbol{\rho}_i - \gamma_i \bar{F}(\boldsymbol{\rho}_i) \tag{15}$$

where γ_i is a scalar step size.

In (Karimi *et al.*, 2004), it is shown that under the assumptions:

(i) Boundedness of the signals in the loop (the calculated controllers stabilize the closed-loop system at each iteration).
(ii) The step size tends to zero appropriately fast.
(iii) The decorrelating controller K^* exists and belongs to the set of parameterized controllers (the corresponding parameters will be denoted as $\boldsymbol{\rho}^*$).
(iv) $F(\boldsymbol{\rho})$ possesses continuous partial derivatives of first and second order with respect to $\boldsymbol{\rho}$.

this scheme converges to a solution of the correlation equations (10), provided that:

$$Q(\boldsymbol{\rho}) = E\left\{\frac{\partial \bar{F}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\right\} > 0 \tag{16}$$

In this work we are interested to investigate the accuracy properties of the parameter estimates in the vicinity of the solution. Therefore, in the sequel it will be assumed that this matrix is positive definite.

For this analysis, the structure of this matrix will be studied. It follows from (3), (11), (12) and (16) that Q($\boldsymbol{\rho}$) $\in \mathcal{R}^{n_\rho \times n_\rho}$ can be expressed as:

$$Q(\boldsymbol{\rho}) = \begin{pmatrix} Q_{11}^{11} & Q_{11}^{12} & Q_{11}^{21} & Q_{11}^{22} \\ Q_{12}^{11} & Q_{12}^{12} & Q_{12}^{21} & Q_{12}^{22} \\ Q_{21}^{11} & Q_{21}^{12} & Q_{21}^{21} & Q_{21}^{22} \\ Q_{22}^{11} & Q_{22}^{12} & Q_{22}^{21} & Q_{22}^{22} \end{pmatrix} \tag{17}$$

with Q_{kl}^{ij} being the derivative of $E\{\bar{f}_{ij}(\boldsymbol{\rho})\}$ with respect to $\boldsymbol{\rho}_{kl}$:

$$Q_{kl}^{ij} = E\left\{\frac{1}{N}\sum_{i=1}^{N} \frac{\partial \zeta_{ij}(\boldsymbol{\rho}, t)}{\partial \boldsymbol{\rho}_{kl}} \eta_{ij}(\boldsymbol{\rho}, t) + \frac{\partial \eta_{ij}(\boldsymbol{\rho}, t)}{\partial \boldsymbol{\rho}_{kl}} \zeta_{ij}^T(\boldsymbol{\rho}, t)\right\} \tag{18}$$

In the vicinity of the solution, the first term in (18) vanishes since the derivative of the instrumental variable vector $\zeta_{ij}(\boldsymbol{\rho}, t)$ is not correlated with $\eta_{ij}(\boldsymbol{\rho}, t)$. Note that $\partial \eta_{ij}(\boldsymbol{\rho}, t)/\partial \boldsymbol{\rho}_{kl} = \partial y^i(\boldsymbol{\rho}, t)/\partial \boldsymbol{\rho}_{kl}$.

At the solution $\boldsymbol{\rho}^*$, using (5)-(8) one gets

$$\left.\frac{\partial \mathbf{y}(\boldsymbol{\rho}, t)}{\partial \rho_{kl}^{(m)}}\right|_{\rho_{kl}^{*\,(m)}} = \boldsymbol{\mathcal{S}}(\boldsymbol{\rho}^*)\mathbf{G} \left.\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_{kl}^{(m)}}\right|_{\rho_{kl}^{*\,(m)}} \mathbf{e}(\boldsymbol{\rho}^*, t) \tag{19}$$

where $m = 1, n_r + n_s$.

Now, considering that the subvectors $\boldsymbol{\rho}_{kl}$ are independent, and $\mathcal{S}$ is block-diagonal at $\boldsymbol{\rho}^*$, it follows from (19) and the second equality in (6)

$$\left.\frac{\partial \mathbf{y}(\boldsymbol{\rho}, t)}{\partial \rho_{kl}^{(m)}}\right|_{\rho_{kl}^{*\,(m)}} \sim e^{(l)}(\boldsymbol{\rho}^*, t) \sim r^{(l)}(t),\ v^{(l)}(t) \tag{20}$$

where $\sim$ denotes that the signal on the left side of this operator is a function of the right-side signal. Using this relationship, the expressions (14) and (18), and the fact that $r^{(1)}(t)$, $r^{(2)}(t)$, $v^{(1)}(t)$ and $v^{(2)}(t)$ are not correlated, it can be deduced that:

$$Q_{kl}^{ij} = 0, \quad j \neq l \tag{21}$$

i.e. the matrix $Q(\boldsymbol{\rho}^*)$ takes the following form:

$$Q(\boldsymbol{\rho}^*) = \begin{pmatrix} Q_{11}^{11} & 0 & Q_{11}^{21} & 0 \\ 0 & Q_{12}^{12} & 0 & Q_{12}^{22} \\ Q_{21}^{11} & 0 & Q_{21}^{21} & 0 \\ 0 & Q_{22}^{12} & 0 & Q_{22}^{22} \end{pmatrix} \tag{22}$$

where

$$Q_{kj}^{ij} \sim r^{(j)}(t), v^{(j)}(t). \tag{23}$$

Now, assuming the conditions for the algorithm (15) to converge to $\boldsymbol{\rho}^*$ hold, it is of interest to investigate the accuracy properties of the parameter estimates around this solution as functions of the external reference signals $r^{(1)}(t)$ and $r^{(2)}(t)$. This is the topic of the next section.

4. VARIANCE ANALYSIS

To perform the analysis in this section, one will make use of the following theorem (Karimi *et al.*, 2004):

Theorem 4.1. Assume that

(1) The iterative algorithm (15) converges to $\boldsymbol{\rho}^*$ almost surely as $i \to \infty$.
(2) The step size $\gamma_i = \frac{\alpha}{i}$ where α is a positive constant.
(3) The matrix $D = I/2 - \alpha Q(\boldsymbol{\rho}^*)$ has all eigenvalues with negative real part.

Then, the sequence $\sqrt{i}(\boldsymbol{\rho}_i - \boldsymbol{\rho}^*)$ converges asymptotically in distribution to a zero-mean normal distribution with covariance V

$$V = \alpha^2 \int_0^\infty e^{Dx}\, P\, e^{D^T x} dx \tag{24}$$

where

$$P = \lim_{i\to\infty} E\left\{\bar{F}(\boldsymbol{\rho}^*)\bar{F}^T(\boldsymbol{\rho}^*)\right\} \tag{25}$$

Before proceeding to the main result of this work, let's consider the forms of matrices P, D and V. Note that (13) at $\boldsymbol{\rho} = \boldsymbol{\rho}^*$ reduces to:

$$\eta_{ij}(\boldsymbol{\rho}, t) = \begin{cases} \mathcal{S}^{(ii)}(\boldsymbol{\rho}^*) v^{(i)}(t), & i = j \\ \left\{\mathcal{T}^{(ii)}(\boldsymbol{\rho}^*) r^{(i)}(t) + \mathcal{S}^{(ii)}(\boldsymbol{\rho}^*) v^{(i)}(t)\right\}, & i \neq j \end{cases} \tag{26}$$

Then, taking into account that $r^{(1)}(t)$, $r^{(2)}(t)$, $v^{(1)}(t)$ and $v^{(2)}(t)$ are independent and using (11), (12), (25) and (26), one gets , after straightforward but tedious calculations, for the covariance matrix$P$:

$$P(\boldsymbol{\rho^*}) = \begin{pmatrix} P_{11}^{11} & 0 & 0 & 0 \\ 0 & P_{12}^{12} & P_{12}^{21} & 0 \\ 0 & P_{21}^{12} & P_{21}^{21} & 0 \\ 0 & 0 & 0 & P_{22}^{22} \end{pmatrix} \tag{27}$$

where

$$P_{ij}^{kl} = E\left\{ \frac{1}{N^2} \sum_{t=1}^{N} \zeta_{ij}(\boldsymbol{\rho},t)\eta_{ij}(\boldsymbol{\rho},t) \right. \\ \left. \times \sum_{s=1}^{N} \zeta_{kl}^T(\boldsymbol{\rho},s)\eta_{kl}(\boldsymbol{\rho},s) \right\}. \tag{28}$$

Observe also that the matrix $D = 1/2I - Q(\boldsymbol{\rho}^*)$ has the same structure as $Q(\boldsymbol{\rho}^*)$ in (22), and that its elements, due to (23), satisfy:

$$D_{kj}^{ij} \sim r^{(j)}(t), v^{(j)}(t). \tag{29}$$

Finally, note that the covariance matrix V can be partitioned as:

$$V = \begin{pmatrix} V_{11}^{11} & V_{11}^{12} & V_{11}^{21} & V_{11}^{22} \\ V_{12}^{11} & V_{12}^{12} & V_{12}^{21} & V_{12}^{22} \\ V_{21}^{11} & V_{21}^{12} & V_{21}^{21} & V_{21}^{22} \\ V_{22}^{11} & V_{22}^{12} & V_{22}^{21} & V_{22}^{22} \end{pmatrix}. \tag{30}$$

Next, two cases will be considered:

a) when the closed-loop system is excited by a single reference signal, say $r^{(1)}(t)$; the corresponding matrices and its elements will carry the subscript "a", for example $V_{a,ij}^{kl},\ i,j,k,l = 1,2$, or D_a.
b) when the closed-loop system is excited by both components of $\mathbf{r}(t)$; the corresponding matrices and its elements will carry the subscript "b".

Note that, when only $r^{(1)}(t)$ is excited it is obvious from (12) and (14) that one can tune only the parameters of the controllers $K^{(11)}(\boldsymbol{\rho})$ and $K^{(21)}(\boldsymbol{\rho})$. Hence, only the variances V_{11}^{11} and V_{21}^{21} can be compared for the two excitation cases. In order to provide a fair comparison, it will be assumed when $r^{(1)}(t)$ alone is excited, that the controllers $K^{(12)}(\boldsymbol{\rho})$ and $K^{(22)}(\boldsymbol{\rho})$ are kept fix to their optimal values $K^{(12)}(\boldsymbol{\rho^*})$ and $K^{(22)}(\boldsymbol{\rho^*})$, respectively.

Now, the following result can be established.

Theorem 4.2. Consider the tuning of the parameters $\boldsymbol{\rho}_{11}$ and $\boldsymbol{\rho}_{21}$, related to the controllers $K^{(11)}(\boldsymbol{\rho})$ and $K^{(21)}(\boldsymbol{\rho})$, respectively. Assume that the iterative algorithm (15) converges to $\boldsymbol{\rho}^*$. Let the components $r^{(1)}(t)$ and $r^{(2)}(t)$ be independent and persistently exciting of sufficient order. Then, the covariance matrices of the parameter estimates $\hat{\boldsymbol{\rho}}_{11}$ and $\hat{\boldsymbol{\rho}}_{21}$ cannot decrease by addition of the second excitation $r^{(2)}(t)$, i.e.

$$V_{b,11}^{11} \geq V_{a,11}^{11} \quad \text{and} \quad V_{b,21}^{21} \geq V_{a,21}^{21} \tag{31}$$

Proof. Observe that the matrices V, D and P are related via the following Lyapunov equation (Horn and Johnson, 1990):

$$P + DV + VD^T = 0. \tag{32}$$

A straightforward computation of this expression, due to the specific form of D and P, gives the following relation that includes the variances V_{11}^{11} and V_{21}^{21}:

$$\bar{P} + \bar{D}\bar{V} + \bar{V}\bar{D}^T = 0, \tag{33}$$

where

$$\bar{P} = \begin{pmatrix} P_{11}^{11} & 0 \\ 0 & P_{21}^{21} \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} D_{11}^{11} & D_{11}^{21} \\ D_{21}^{11} & D_{21}^{21} \end{pmatrix} \tag{34}$$

and

$$\bar{V} = \begin{pmatrix} V_{11}^{11} & V_{11}^{21} \\ V_{21}^{11} & V_{21}^{21} \end{pmatrix}. \tag{35}$$

Now, from (29), observe that $\bar{D}$ depends on $r^{(1)}(t)$ and not on $r^{(2)}(t)$. Therefore, $\bar{D}$ is identical for both cases of excitation, i.e. $\bar{D}_a = \bar{D}_b$. Furthermore, note that at the solution the closed-loop system is perfectly decoupled. Then, from (26) and (28), one can conclude that P_{11}^{11} is also identical for both cases of excitation. Let's consider P_{21}^{21}. It is obvious from (14), (26) and (28) that the contribution of $r^{(2)}(t)$ to P_{21}^{21} is positive definite. This contribution will be denoted as ΔP_{21}^{21}. Therefore, one can write:

$$\bar{P}_b = \bar{P}_a + \begin{pmatrix} 0 & 0 \\ 0 & \Delta P_{21}^{21} \end{pmatrix} \triangleq \bar{P}_a + \Delta\bar{P} \tag{36}$$

where $\Delta\bar{P} \geq 0$. For the covariance matrices $\bar{V}_a$ and $\bar{V}_b$, it can be written $\bar{V}_b = \bar{V}_a + \Delta\bar{V}$. Consequently, it follows that:

$$\bar{P}_b + \bar{D}_b\bar{V}_b + \bar{V}_b\bar{D}_b^T = \\ \left(\bar{P}_a + \Delta\bar{P}\right) + \bar{D}_b\left(\bar{V}_a + \Delta\bar{V}\right) + \left(\bar{V}_a + \Delta\bar{V}\right)\bar{D}_b^T = \\ \Delta\bar{P} + \bar{D}_b\Delta\bar{V} + \Delta\bar{V}\bar{D}_b^T = 0 \tag{37}$$

The last equality can be written more illustratively as:

$$\Delta\bar{V} = \int_0^\infty e^{D_b x}\,\Delta\bar{P}\,e^{D_b^T x}dx. \tag{38}$$

It is standard result in the literature that if $\Delta\bar{P} \geq 0$ then $\Delta\bar{V} \geq 0$ (Zhou and Doyle, 1998). The inequalities (31) follows from the fact that any principal submatrix of a positive semi-definite matrix is positive semi-definite. □

Theorem 4.2 states that the presence of the component $r^{(2)}(t)$ does not improves at all the accuracy of the parameters related to the controllers $K^{(11)}(\boldsymbol{\rho})$ and $K^{(21)}(\boldsymbol{\rho})$. In fact, the accuracy is impaired in most cases. This result is rather interesting taking into account the work of the same authors where, in the case of direct closed-loop identification using prediction error methods, it is shown that the addition of $r^{(2)}(t)$ almost always improves the variance of the estimated parameters (Mišković *et al.*, 2006).

The instrumental-variable method in the field of system identification brings about two opposite effects of the excitation on the variance of the parameter estimates: 1) An increase in the variance of the excitation signal induces an increase in the variance of the criterion, which in turn increases the variance of the parameter estimates; 2) An increase in the variance of the excitation signal induces an increase in the derivative of the predictor of the output. This derivative enters inversely in the expression for the variance of the parameter estimates. In general, if one chooses the instruments as noise-free estimates of this derivative, then the overall effect is that the variance of the parameter estimates decrease as the variance of the excitation signal increases. For more details, the reader is referred to Section 9.5 in (Ljung, 1999).

Here, because $\bar{D}$ is insensible to the changes in $r^{(2)}(t)$, only the first effect is present. A remedy to this would be to do the following. Since the variance of the criterion increases due to the increase of the variance of the instrumental variables, see (14) and (28), the instruments can be multiplied by an small positive constant k_ζ. This way, the positive definiteness of $Q(\boldsymbol{\rho}^*)$ is not compromised and, at the same time, the variance of the parameter estimates can be made small. However, observe that k_ζ cannot be made arbitrarily small since, in the limiting case, the criterion becomes zero.

5. CONCLUSIONS

This contribution has presented a variance analysis for the estimated parameters of a linear time-invariant multivariable controller for two cases of excitation. It has been shown that by simultaneous excitation of both reference signals the variance of the estimated controller parameters is larger than or equal to that of the case with a single reference excitation. A heuristic solution to this problem has been proposed.

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