DIRECT CLOSED-LOOP IDENTIFICATION OF 2×2 SYSTEMS: VARIANCE ANALYSIS

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Abstract: An analysis of the variance for the parameters of a 2×2 plant estimated in closed-loop operation is performed. Two cases of excitation are considered, i.e. via either a single reference signal or both references simultaneously. The resulting expressions are valid for all conventional Prediction Error Models (PEM). It is shown that, regardless of the parametrization, the presence of the second reference signal never impairs and, in most cases, improves the accuracy of the parameter estimates. The analytical results are illustrated by two simulation examples. The results presented here are straightforward to extend to the case of direct closed-loop identification of an arbitrary multi-input system.

Keywords: Direct closed-loop identification, variance analysis, multi-input systems.

1. INTRODUCTION

In (Gevers *et al.*, 2005), the authors consider the open-loop identification of multi-input systems. The effect of an additional input signal on the variance of the polynomial coefficients in the case of FIR, ARX, ARMAX, OE and BJ models is investigated. Necessary and sufficient conditions on the parametrization of MISO models under which the addition of an input decreases the covariance of the parameter estimates are provided. It is shown that, for SISO model structures that have common parameters in the plant and noise model, any additional input reduces the covariance of all parameters, including the noise model parameters and the parameters associated with the other input. It is also shown that for systems with several inputs, the accuracy improvement resulting from an additional input extends beyond the case of common parameters in all transfer functions. The reader is referred to (Gevers *et al.*, 2005) for details.

The present contribution is a continuation of that work but in the context of *direct closed-loop* identification. A Linear Time-Invariant (LTI) system with two inputs and two outputs is to be identified using data collected in closed-loop operation. Assume that there are no common parameters between the models associated with each of the outputs. Assume furthermore that the disturbances acting on the outputs are not correlated. For systems fulfilling these two assumptions, a MISO model can be first identified for each output separately and then the resulting individual models combined into a final MIMO model (Dayal and MacGregor, 1997). Now, the following questions arise: Do the conditions on the parametrization of the MISO structures that apply to the open-loop identification also hold for the case of direct closed-loop identifica-

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tion (with the difference that in closed-loop operation the external reference signals are excited instead of the inputs)? In what way does the correlation of the input signals, due to the presence of the feedback, affects the accuracy of the parameter estimates?

To answer these questions, a general model structure is introduced that encompasses all commonly-used parametric model structures. It is assumed that the system (including the noise model) is in the model set. An analysis, asymptotic in data length but not in model order, of the variance of the estimated parameters in this structure is performed for two cases of excitation: (i) a single reference signal is used to excite the closed-loop system; (ii) both references are applied simultaneously. A similar asymptotic analysis is performed in (Bombois *et al.*, 2005), where the variances in both open and closed loop for the BJ models are compared for SISO systems.

The result of this analysis is that in the case of closed-loop identification the following two situations can be distinguished:

(i) If all parameters of the noise model are present in the plant model, or if there is no noise model at all, then the accuracy of all parameter estimates is always improved by applying both references simultaneously. For the FIR and OE structures, this result is in contrast to the open-loop case where existence of common parameters between the plant and noise models is required to improve the accuracy of all parameter estimates.
(ii) If the noise model contains some parameters that are independent of the plant model, then the simultaneous excitation of both reference signals cannot worsen the quality of the parameter estimates.

The paper is organized as follows. Preliminaries concerning prediction error identification are given in Section 2. In Section 3, the expression describing the influence of the reference signals on the information matrix is derived. This expression is used for the computation of the variance of the parameter and transfer function estimates in Section 4. Section 5 illustrates the analytical result via two simulation examples. Finally, the conclusions are given in Section 6.

2. PRELIMINARIES

Consider the unknown LTI 2×2 "true" plant:

$$\begin{aligned}\mathcal{S} : y(t) &= G(q^{-1})u(t) + H(q^{-1})\eta(t) \\ &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} u(t) + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \eta(t) \quad (1)\end{aligned}$$

where G_{11}, G_{12}, G_{21} and G_{22} are strictly causal, finite-order, rational transfer functions not necessarily analytic outside the unit circle, and H_1 and H_2 are stable and inversely stable transfer functions. The

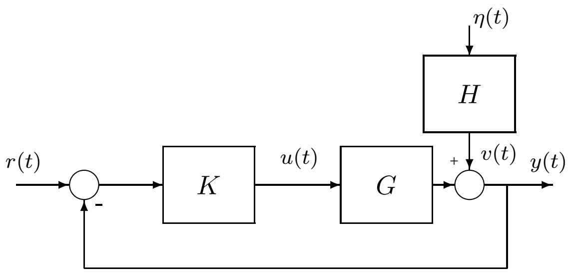

Fig. 1. Closed-loop configuration

backward-shift operator q^{-1} will be omitted in the sequel whenever appropriate. The signal $y(t) \in \mathcal{R}^2$ is the output of the true plant, $u(t) \in \mathcal{R}^2$ the control signal, $r(t) \in \mathcal{R}^2$ an external reference signal and $\eta(t) \in \mathcal{R}^2$ white noise input with variance $\sigma_\eta^2 = \mathrm{diag}(\sigma_{\eta_1}^2, \sigma_{\eta_2}^2)$. The system $\mathcal{S}$ is controlled by the stabilizing controller $K \in \mathcal{R}^{2\times 2}$ as depicted in Fig. 1. The control signal $u(t)$ can be expressed as a function of $r(t)$ as follows:

$$\begin{aligned}u(t) &= U\left(r(t) - H\eta(t)\right) \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} (r(t) - H\eta(t)) \quad (2)\end{aligned}$$

where the input sensitivity function U is $U = KS$, with $S = (I + GK)^{-1}$ the output sensitivity function.

Consider now the direct closed-loop identification of the subsystem $\mathcal{S}_1$ of the system $\mathcal{S}$:

$$\mathcal{S}_1 : y_1(t) = G_{11}u_1(t) + G_{12}u_2(t) + H_1\eta_1(t) \quad (3)$$

using the following model structure:

$$\begin{aligned}\mathcal{M} = \{&G_{11}(\alpha), G_{12}(\alpha,\beta), H_1(\alpha,\beta,\gamma), \\ &\theta = \left(\, \alpha^T \;\; \beta^T \;\; \gamma^T \,\right)^T \in D_\theta \subset \mathcal{R}^{n_\theta} \Big\} \quad (4)\end{aligned}$$

where $G_{11}(\alpha)$, $G_{12}(\alpha,\beta)$ and $H_1(\alpha,\beta,\gamma)$ are rational transfer functions, $\theta \in \mathcal{R}^{n_\theta}$ is the vector of model parameters, and D_θ is a subset of admissible values for θ. It is assumed that the true subsystem $\mathcal{S}_1$ can be described by this model structure for some $\theta_0 = (\alpha_0^T, \beta_0^T, \gamma_0^T)^T \in D_\theta$. Note that this parametrization covers a wide range of model structures. For example, if one considers the ARMAX structure: $Ay_1(t) = B_{11}u_1(t) + B_{12}u_2(t) + C_1\eta_1(t)$ then the subvector α contains the parameters of the polynomials A and B_{11}, β contains the parameters of B_{12} and γ contains the parameters of C_1. Here $H_1 = H_1(\alpha, \gamma)$.

The direct method gives consistent estimates of the open-loop system if the data is sufficiently informative with respect to the adopted model structure and if the true system, including the noise model, can be described within the chosen parametrization (Ljung and Forssell, 1999). Here, sufficiently informative data means that the signals $u(t)$ are persistently exciting of appropriate order. In closed loop, this is ensured e.g. by a persistently exciting reference signal. Using a set of input-output data of length N acquired in closed-loop operation, the estimate $\hat{\theta}_N$ is calculated via the prediction error criterion (Ljung, 1999):

$$\hat{\theta}_N = \begin{pmatrix} \hat{\alpha}_N \\ \hat{\beta}_N \\ \hat{\gamma}_N \end{pmatrix} = \arg\min_{\theta \in D_\theta} \frac{1}{N} \sum_{t=1}^{N} [\varepsilon(t,\theta)]^2 \quad (5)$$

where the one-step ahead prediction error $\varepsilon(t,\theta)$ for (3) is defined as:

$$\begin{aligned} \varepsilon(t,\theta) &\triangleq y_1(t) - \hat{y}_1(t|t-1,\theta) \\ &= H_1(\theta)^{-1} \left(y_1(t) - G_{11}(\theta)u_1(t)\right. \\ &\quad \left. -G_{12}(\theta)u_2(t)\right) \end{aligned} \quad (6)$$

and the transfer functions are written generically as functions of the parameter vector θ.

Let us assume that the parameter estimates $\hat{\theta}_N$ converge to the true parameter vector θ_0 as N tends to infinity. Then, the parameter error converges to a Gaussian random variable:

$$\sqrt{N}\left(\hat{\theta}_N - \theta_0\right) \xrightarrow{\text{dist}} \mathcal{N}(0, P_\theta) \quad (7)$$

where the covariance matrix P_θ is given by:

$$P_\theta = \sigma^2_{\eta_1(t)} [E\psi(t,\theta_0)\psi^T(t,\theta_0)]^{-1} \quad (8)$$

with $\psi(t,\theta) \triangleq \frac{\partial \varepsilon(t,\theta)}{\partial \theta}$. Typically, to compute approximate expressions for the covariance of the parameter vector estimates, the asymptotic covariance formulas (7)-(8) are used:

$$cov(\hat{\theta}_N) \approx \frac{1}{N} P_\theta \triangleq \frac{\sigma^2_{\eta_1(t)}}{N} M^{-1}. \quad (9)$$

M is called the information matrix. In the next section, an expression for M is derived that shows the dependence of this matrix on the external excitation signals. This expression, together with (9), will help us analyze the dependence of the covariance of the parameter vector estimate $\hat{\theta}_N$ on $r_1(t)$ and $r_2(t)$.

3. EXPRESSION FOR THE INFORMATION MATRIX M

Combining (2), (3) and (6), the gradient of the prediction error with respect to the parameters at $\theta = \theta_0$ can be expressed as follows:

$$\begin{aligned} &\psi(t,\theta_0) = H_1{}^{-1} \left[\left(g^\theta_{11} U_{11} + g^\theta_{12} U_{21} \right) r_1(t) \right. \\ &+ \left(g^\theta_{11} U_{12} + g^\theta_{12} U_{22} \right) r_2(t) \\ &+ \left(h^\theta_1 - g^\theta_{11} U_{11} H_1 - g^\theta_{12} U_{21} H_1 \right) \eta_1(t) \\ &- \left. \left(g^\theta_{11} U_{12} H_2 - g^\theta_{12} U_{22} H_2 \right) \eta_2(t) \right] \\ &\triangleq \Pi_1 r_1(t) + \Pi_2 r_2(t) + \Pi_3 \eta_1(t) + \Pi_4 \eta_2(t) \end{aligned} \quad (10)$$

where

$$g^\theta_{11} = \left. \frac{\partial G_{11}(\theta)}{\partial \theta} \right|_{\theta=\theta_0} ; \; g^\theta_{12} = \left. \frac{\partial G_{12}(\theta)}{\partial \theta} \right|_{\theta=\theta_0}$$

and

$$h^\theta_1 = \left. \frac{\partial H_1(\theta)}{\partial \theta} \right|_{\theta=\theta_0}. \quad (11)$$

The quantities Π_1, Π_2, Π_3 and Π_4 are introduced in (10) for the sake of simplicity of notation.

From (8)-(11), and using Parseval's theorem and the fact that $r_1(t)$, $r_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are not correlated, the information matrix can be rewritten as:

$$\begin{aligned} M &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \Pi_1 \Pi_1^* \Phi_{r_1} + \Pi_2 \Pi_2^* \Phi_{r_2} + \right. \\ &\quad \left. + \Pi_3 \Pi_3^* \sigma_{\eta_1^2} + \Pi_4 \Pi_4^* \sigma_{\eta_2^2} \right\} d\omega \\ &\triangleq M(r_1) + M(r_2) + M(\eta_1) + M(\eta_2) \end{aligned} \quad (12)$$

where $(.)^*$ is used to denote the complex conjugate.

Consider now the partition of the parameter vector θ in (4). The sensitivities of the transfer functions G_{11}, G_{12} and H_1 with respect to θ read:

$$\begin{aligned} g^\theta_{11} &= \begin{pmatrix} g^\alpha_{11} & 0 & 0 \end{pmatrix}^T, \\ g^\theta_{12} &= \begin{pmatrix} g^\alpha_{12} & g^\beta_{12} & 0 \end{pmatrix}^T \text{ and} \\ h^\theta_1 &= \begin{pmatrix} h^\alpha_1 & h^\beta_1 & h^\gamma_1 \end{pmatrix}^T \end{aligned} \quad (13)$$

where the definition of the components of g^θ_{11}, g^θ_{12} and h^θ_1 is analogous to that in (11). It follows from (10), (11) and (13) that the quantity Π_1 reduces to:

$$\Pi_1 = H_1{}^{-1} \begin{pmatrix} g^\alpha_{11} U_{11} + g^\alpha_{12} U_{21} & g^\beta_{12} U_{21} & 0 \end{pmatrix} \quad (14)$$

Consequently, the contribution of $r_1(t)$ to M can formally be expressed as:

$$M(r_1) = \begin{pmatrix} M_{11}(r_1) & M_{12}(r_1) & 0 \\ M_{21}(r_1) & M_{22}(r_1) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

Similar calculations provide the expressions for $M(r_2)$, $M(\eta_1)$ and $M(\eta_2)$, from which one can express the information matrix M in the following form:

$$M = \begin{pmatrix} M_{11}(r,\eta) & M_{12}(r,\eta) & M_{13}(\eta_1) \\ M_{21}(r,\eta) & M_{22}(r,\eta) & M_{23}(\eta_1) \\ M_{31}(\eta_1) & M_{32}(\eta_1) & M_{33}(\eta_1) \end{pmatrix}. \quad (16)$$

If an element of M carries the argument r and/or η, this means that this particular element depends on *both* $r_1(t)$ and $r_2(t)$ and/or $\eta_1(t)$ and $\eta_2(t)$. Otherwise, the elements of M carry as argument only the particular component, for example $M_{33}(\eta_1)$ depends only on $\eta_1(t)$.

In the sequel, the effect of the presence/absence of the second external reference signal $r_2(t)$ on the variance of the elements of the parameter vector estimate is analyzed. Note that, for a given model structure, the presence/absence of a particular external reference signal does not change the structure of the information matrix M due to the fact that, in closed-loop operation, both inputs $u_1(t)$ and $u_2(t)$ are excited by both reference signals.

4. EFFECT OF THE SECOND REFERENCE SIGNAL

Consider the form of the matrix M in (16). All possible model structures that can be derived from the parametrization (4) can be divided in two groups:

$\mathcal{A}$) The model structures where the subvector γ of the vector θ is empty (there are no parameters in the noise model H_1 that are independent of the plant model), or which contain no noise model at all. Among others, this group includes the classical FIR, ARX and OE model structures.

$\mathcal{B}$) The noise model contains some (not necessarily all) parameters that are independent of the plant model. This group includes the ARMAX and BJ model structures in addition to some less conventional structures where, for example, common parameters are not shared by all transfer functions (e.g. there are common parameters between G_{12} and H_1 but not with G_{11}).

In order to study the effect of $r_1(t)$ and $r_2(t)$ on the accuracy of the parameter estimates of α, β and γ, we introduce:

$$C \triangleq M^{-1} = \begin{pmatrix} C_\alpha & C_{\alpha\beta} & C_{\alpha\gamma} \\ C_{\beta\alpha} & C_\beta & C_{\beta\gamma} \\ C_{\gamma\alpha} & C_{\gamma\beta} & C_\gamma \end{pmatrix} \tag{17}$$

Note that $cov(\hat{\alpha}_N) \approx \frac{\sigma^2_{\eta_1}}{N} C_\alpha$, $cov(\hat{\beta}_N) \approx \frac{\sigma^2_{\eta_1}}{N} C_\beta$, and $cov(\hat{\gamma}_N) \approx \frac{\sigma^2_{\eta_1}}{N} C_\gamma$. Furthermore, the variances of the identified plant models $G_{11}(\hat{\theta}_N)$ and $G_{12}(\hat{\theta}_N)$ and the identified noise model $H_1(\hat{\theta}_N)$ can be calculated using Gauss' approximation formula (Ljung, 1999). For large number of data N and by inserting (13) for g^θ_{11}, g^θ_{12} and h^θ_1, one gets:

$$\begin{aligned} var\left(G_{11}(e^{j\omega}, \hat{\theta}_N)\right) &\approx \frac{\sigma^2_{\eta_1}}{N} \left(g^\alpha_{11}\right)^* C_\alpha \, g^\alpha_{11} \\ var\left(G_{12}(e^{j\omega}, \hat{\theta}_N)\right) &\approx \frac{\sigma^2_{\eta_1}}{N} \left\{ \left(g^\alpha_{12}\right)^* C_\alpha g^\alpha_{12} \right. \\ &\qquad \left. + (g^\beta_{12})^* C_\beta \, g^\beta_{12} \right\} \\ var\left(H_1(e^{j\omega}, \hat{\theta}_N)\right) &= \frac{\sigma^2_{\eta_1}}{N} \left\{ \left(h^\alpha_1\right)^* C_\alpha h^\alpha_1 \right. \\ &\qquad \left. + (h^\beta_1)^* C_\beta \, h^\beta_1 + (h^\gamma_1)^* C_\gamma \, h^\gamma_1 \right\} . \end{aligned} \tag{18}$$

In the sequel, the analysis is performed separately for the two groups $\mathcal{A}$ and $\mathcal{B}$, and the corresponding covariance matrices C and their elements will carry the appropriate subscripts "$\mathcal{A}$" and "$\mathcal{B}$", respectively. Furthermore, the block-diagonal elements C_α, C_β, C_γ, the matrices C and M will carry the superscript "(1)" when only reference signal $r_1(t)$ is applied and "(2)" when both reference signals are applied simultaneously.

4.1 Group $\mathcal{A}$

When the vector γ is empty and both excitation signals $r_1(t)$ and $r_2(t)$ are present, the information matrix M from (16) reduces to

$$M^{(2)}_{\mathcal{A}} = \begin{pmatrix} M_{11}(r,\eta) & M_{12}(r,\eta) \\ M_{21}(r,\eta) & M_{22}(r,\eta) \end{pmatrix}. \tag{19}$$

The corresponding information matrix when exciting $r_1(t)$ alone reads:

$$M^{(1)}_{\mathcal{A}} = \begin{pmatrix} M_{11}(r_1,\eta) & M_{12}(r_1,\eta) \\ M_{21}(r_1,\eta) & M_{22}(r_1,\eta) \end{pmatrix}. \tag{20}$$

The matrix $M^{(2)}_{\mathcal{A}}$ can be written as:

$$M^{(2)}_{\mathcal{A}} = M^{(1)}_{\mathcal{A}} + \Delta\bar{M} \tag{21}$$

with

$$\Delta\bar{M} \triangleq \begin{pmatrix} M_{11}(r_2) & M_{12}(r_2) \\ M_{21}(r_2) & M_{22}(r_2) \end{pmatrix}. \tag{22}$$

The following result is an immediate consequence of the expression (21) and the fact that $\Delta\bar{M} > 0$.

Theorem 4.1. Consider the closed-loop identification of the parameter vectors α and β of the model structure $\mathcal{A} \subset \mathcal{M}$. Let the excitation signals $r_1(t)$ and $r_2(t)$ be independent and persistently exciting of sufficient order. Then, the covariance matrices of the parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ decrease by addition of the second excitation $r_2(t)$, i.e.

$$C^{(2)}_{\alpha,\mathcal{A}} < C^{(1)}_{\alpha,\mathcal{A}} \quad \text{and} \quad C^{(2)}_{\beta,\mathcal{A}} < C^{(1)}_{\beta,\mathcal{A}}. \tag{23}$$

Proof. The inequalities (23) are a direct consequence of the following expression:

$$C^{(1)}_{\mathcal{A}} - C^{(2)}_{\mathcal{A}} = C^{(2)}_{\mathcal{A}} \Delta\bar{M} C^{(1)}_{\mathcal{A}} \triangleq \Delta C_{\mathcal{A}} > 0. \tag{24}$$

□

Comments

1) For a structure from the group $\mathcal{A}$, the simultaneous excitation of $r_1(t)$ and $r_2(t)$ reduces the covariance of the estimates of the parameter vectors α and β compared to the case when $r_1(t)$ alone is excited.
2) If the variance of $r_2(t)$ tends to infinity, $\Delta\bar{M}$ and $M^{(2)}_{\mathcal{A}}$ also tend infinity and consequently $C^{(2)}_{\mathcal{A}}$ tends to zero. The intuition is that α and β become perfectly known when the power of $r_2(t)$, and therefore the power of $u_1(t)$ and $u_2(t)$, tends to infinity.
3) The presence of $r_2(t)$ reduces the variance of all transfer function estimates. If the power of $r_2(t)$ grows unbounded, the variances of $G_{11}(\hat{\theta}_N)$, $G_{12}(\hat{\theta}_N)$ and $H_1(\hat{\theta}_N)$ tend to zero.

4.2 Group $\mathcal{B}$

When only $r_1(t)$ is excited, it follows from (10)-(13) that the information matrix $M^{(1)}_{\mathcal{B}}$ has the following form:

$$M_{\mathcal{B}}^{(1)}=\begin{pmatrix} M_{11}(r_1,\eta) & M_{12}(r_1,\eta) & M_{13}(\eta_1) \\ M_{21}(r_1,\eta) & M_{22}(r_1,\eta) & M_{23}(\eta_1) \\ M_{31}(\eta_1) & M_{32}(\eta_1) & M_{33}(\eta_1) \end{pmatrix}. \quad (25)$$

When both $r_1(t)$ and $r_2(t)$ are present, the information matrix $M_{\mathcal{B}}^{(2)}$ is given by expression (16). $M_{\mathcal{B}}^{(1)}$ and $M_{\mathcal{B}}^{(2)}$ are related as follows:

$$M_{\mathcal{B}}^{(2)} = M_{\mathcal{B}}^{(1)} + \Delta M \quad (26)$$

with

$$\Delta M = \begin{pmatrix} M_{11}(r_2) & M_{12}(r_2) & 0 \\ M_{21}(r_2) & M_{22}(r_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (27)$$

Next, the following result can be established.

Theorem 4.2. Consider the closed-loop identification of the parameter vectors α, β and γ of the model structure $\mathcal{B} \subset \mathcal{M}$. Let the excitation signals $r_1(t)$ and $r_2(t)$ be independent and persistently exciting of sufficient order. Then, the covariance matrices of the parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ cannot increase by addition of the second excitation $r_2(t)$, i.e.

$$C_{\alpha,\mathcal{B}}^{(2)} \le C_{\alpha,\mathcal{B}}^{(1)} \quad \text{and} \quad C_{\beta,\mathcal{B}}^{(2)} \le C_{\beta,\mathcal{B}}^{(1)}. \quad (28)$$

In addition, the covariance matrices of $\hat{\gamma}$ are strictly smaller for $r_2(t) \neq 0$ compared to $r_2(t) = 0$, i.e.

$$C_{\gamma,\mathcal{B}}^{(2)} < C_{\gamma,\mathcal{B}}^{(1)}. \quad (29)$$

Proof. First note that the matrix ΔM is positive semi-definite (observe the non-negative contribution of $r_2(t)$ to the elements of M in (12)). Consequently,

$$\begin{aligned} C_{\mathcal{B}}^{(1)} - C_{\mathcal{B}}^{(2)} &= \left(M_{\mathcal{B}}^{(2)}\right)^{-1}\left(M_{\mathcal{B}}^{(2)} - M_{\mathcal{B}}^{(1)}\right)\left(M_{\mathcal{B}}^{(1)}\right)^{-1} \\ &= C_{\mathcal{B}}^{(2)} \Delta M C_{\mathcal{B}}^{(1)} \triangleq \Delta C_{\mathcal{B}} \ge 0. \quad (30) \end{aligned}$$

Now, the expression (28) follows from the fact that any principal submatrix of a positive semi-definite matrix is positive semi-definite. Also, it follows from (30) that $C_{\gamma,\mathcal{B}}^{(2)} \le C_{\gamma,\mathcal{B}}^{(1)}$. However, this inequality can be strengthened as follows. When $r_1(t)$ alone is present, by straightforward calculation of the inverse of the $(3,3)$ block-element of $M_{\mathcal{B}}^{(1)}$, one obtains:

$$\begin{aligned} C_{\gamma,\mathcal{B}}^{(1)} &= (M_{33}(\eta_1) - (M_{31}(\eta_1)\, M_{32}(\eta_1)) \quad (31) \\ &\times \begin{pmatrix} M_{11}(r_1,\eta) & M_{12}(r_1,\eta) \\ M_{21}(r_1,\eta) & M_{22}(r_1,\eta) \end{pmatrix}^{-1} \times \begin{pmatrix} M_{13}(\eta_1) \\ M_{23}(\eta_1) \end{pmatrix} \Bigg)^{-1}. \end{aligned}$$

Similarly, when both $r_1(t)$ and $r_2(t)$ are applied:

$$\begin{aligned} C_{\gamma,\mathcal{B}}^{(2)} &= (M_{33}(\eta_1) - (M_{31}(\eta_1)\, M_{32}(\eta_1)) \\ &\times \left(\begin{pmatrix} M_{11}(r_1,\eta) & M_{12}(r_1,\eta) \\ M_{21}(r_1,\eta) & M_{22}(r_1,\eta) \end{pmatrix} + \Delta\bar{M}\right)^{-1} \\ &\times (M_{13}(\eta_1)\, M_{23}(\eta_1))^T\Big)^{-1} \quad (32) \end{aligned}$$

where the matrix $\Delta\bar{M} > 0$ is given in (22). By comparing expressions (31) and (32), the expression (29) follows immediately. □

Comments

1) For a structure of the group $\mathcal{B}$, the presence of a second reference signal $r_2(t)$ does not increase the covariance of the estimates of the parameter vectors α, β and reduces the covariance of the estimates of γ. This statement is valid also for model structures with independent parametrization of the plant and noise models such as BJ.
2) If the energy of $r_2(t)$ grows unbounded, expressions (32) and (22) reveal that $C_{\gamma,\mathcal{B}}^{(2)}$ tends to $M_{33}^{-1}(\eta_1)$. At the same time, using (16), it is straightforward to show that $C_{\alpha,\mathcal{B}}^{(2)}$ and $C_{\beta,\mathcal{B}}^{(2)}$ tend to zero. This can be explained as follows: when $r_2(t)$ goes to infinity, $u_1(t)$ and $u_2(t)$ also go to infinity, and the parameters α and β become perfectly known; then, the estimation of γ corresponds to the identification of the unknown parameters of the Moving Average (MA) model $y(t) = H_1(q^{-1})\eta_1(t)$ (some parameters of H_1 might already be known as they are part of α and/or β).
3) The excitation $r_2(t)$ never impairs and in most cases improves the accuracy of all transfer function estimates: see (28), (29) and (18). When the power of $r_2(t)$ goes to infinity, the variances of $G_{11}(\hat{\theta}_N)$ and $G_{12}(\hat{\theta}_N)$ tend to zero.
4) Even when the plant and noise models are parameterized independently, there is a strong correlation between the parameter estimates due to closed-loop operation. A lower variance of the plant parameter estimates implies a lower variance of estimates of the parameters associated with the noise model and vice versa.

It follows from Theorems 4.2 and 4.1 that, regardless of the parametrization, the presence of the external signal $r_2(t)$ does not reduce the accuracy of the parameter vector estimates obtained via direct closed-loop identification. This conclusion holds for any controller K that guarantees informative experiments in closed loop. It follows from (12) that, for both groups $\mathcal{A}$ and $\mathcal{B}$, the contribution of the noise is never detrimental to the precision of the parameter estimates.

5. SIMULATION RESULTS

In order to illustrate the analytical results for both groups $\mathcal{A}$ and $\mathcal{B}$, two academic examples are considered. In both simulation examples, the plants are controlled by the following 2×2 controller:

$$K(q^{-1}) = \frac{0.04(1 - 0.3q^{-1})}{(1 - 0.4q^{-1})} \begin{pmatrix} 1 & 0.1 \\ -0.1 & 1 \end{pmatrix} \quad (33)$$

The controller is designed so as to stabilize both plants without any additional performance consideration.

A Monte-Carlo simulation is performed to compare the case where the reference signal $r_1(t)$ alone is excited with the case where the two reference signals are applied simultaneously. The reference signals $r_1(t)$ and $r_2(t)$ are PRBS generated by a 10-bit shift register with data length $N = 1023$ with standard deviations $\sigma_{r_1} = 1$ and $\sigma_{r_2} = 10$. The disturbance signals $\eta_1(t)$ and $\eta_2(t)$ are white noises with standard deviations $\sigma_{\eta_1} = \sigma_{\eta_2} = 4$. The signals $r_1(t)$, $r_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are mutually independent. So, the assumptions of Theorems 4.2 and 4.1 are verified.

Simulation 1: Group $\mathcal{A}$
The following FIR model is simulated:

$$y_1(t) = B_{11}u_1(t) + B_{12}u_2(t) + \eta_1(t)$$
$$y_2(t) = B_{21}u_1(t) + B_{22}u_2(t) + \eta_2(t)$$

with $B_{11} = 10q^{-1} + q^{-2}$, $B_{12} = 0.5q^{-1} + 4q^{-2}$, $B_{21} = 0.4q^{-1}+3q^{-2}$ and $B_{22} = 2q^{-1}+0.25q^{-2}$. The variance of the estimates of the following parameter vector $\theta = (b_{11}^1, b_{11}^2, b_{12}^1, b_{12}^2)$ is computed for the two cases of excitation. When $r_1(t)$ alone is excited, the asymptotic variances of the elements of θ computed by 1000 Monte-Carlo runs are:

$$var(\hat{\theta}_n^{(1)}) = (599.603 \;\; 585.486 \;\; 603.934 \;\; 612.293)$$

The asymptotic variances of θ computed when both $r_1(t)$ and $r_2(t)$ are excited simultaneously are:

$$var(\hat{\theta}_n^{(2)}) = (523.286 \;\; 461.777 \;\; 84.8456 \;\; 96.4518)$$

Once again, all variances are decreased by addition of the second excitation. This result is due to the fact that the input signals $u_1(t)$ and $u_2(t)$ are correlated, which in turn affects the correlation between the parameter estimates. Note that, in the case of open-loop identification for FIR models, the asymptotic accuracy of the estimates of the b_{11}^j coefficients is totally independent of the presence of $u_2(t)$.

Simulation 2: Group $\mathcal{B}$
The following ARMAX structure is considered

$$A_1y_1(t) = B_{11}u_1(t) + B_{12}u_2(t) + C_1\eta_1(t)$$
$$A_2y_2(t) = B_{21}u_1(t) + B_{22}u_2(t) + C_2\eta_2(t)$$

with $A_1 = 1 - 0.2q^{-1}$, $B_{11} = 10q^{-1} + q^{-2}$, $B_{12} = 0.1q^{-1}+4q^{-2}$, $C_1 = 1-1.6q^{-1}+0.64q^{-2}$, $A_2 = 1-0.25q^{-1}$, $B_{21} = 0.2q^{-1}+3q^{-2}$, $B_{22} = 2q^{-1}+0.1q^{-2}$ and $C_2 = 1 - 1.5q^{-1} + 0.5625q^{-2}$. We consider the parameter vector $\theta = (a_1, b_{11}^1, b_{11}^2, b_{12}^1, b_{12}^2, c_1^1, c_1^2)^T$ associated to the output $y_1(t)$. The Monte-Carlo simulations provide the following variances:

$$\begin{aligned} var(\hat{\theta}_n^{(1)}) = (&4.9336 \quad 675.5294 \quad 1614.4743 \\ &635.1392 \quad 620.4109 \quad 3.0875 \quad 3.1263). \\ var(\hat{\theta}_n^{(2)}) = (&1.9221 \quad 518.046 \quad 589.6793 \\ &66.99271 \quad 139.8771 \quad 1.3696 \quad 1.3517). \end{aligned}$$

Observe that the presence of $r_2(t)$ improves the precision of all estimated coefficients. The corresponding variances of the transfer function estimates $G_{11}(q^{-1}, \hat{\alpha}_n)$, $G_{12}(q^{-1}, \hat{\alpha}_n, \hat{\beta}_n,)$ and $H_1(q^{-1}, \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n)$ are computed at 500 frequency points for the two cases of excitation. The results are compared in Fig. 2. As expected, the accuracy of the three transfer function estimates is improved.

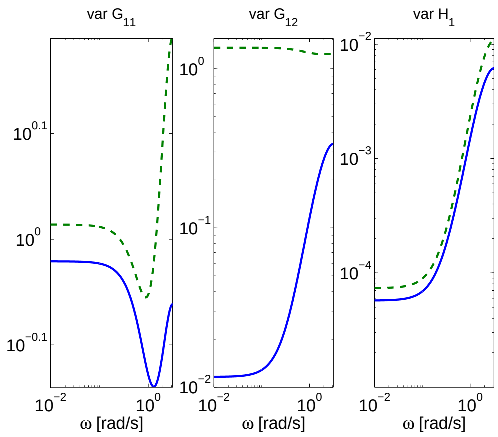

Fig. 2. Variance of the transfer function estimates: $G_{11}(q^{-1}, \hat{\alpha}_n)$ (left), $G_{12}(q^{-1}, \hat{\alpha}_n, \hat{\beta}_n,)$ (middle) and $H_1(q^{-1}, \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n)$ (right), for the ARMAX model with 2 reference inputs (solid line) and one input (dashed line).

6. CONCLUSIONS

An analysis of the variance of the estimated parameters, identified by direct closed-loop identification, is performed for two situations: (i) when only $r_1(t)$ is excited; (ii) when both $r_1(t)$ and $r_2(t)$ are excited simultaneously. It is observed that the presence of $r_2(t)$ cannot worsen the quality of the parameter estimates; in fact, the quality is improved in most cases.

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