We also point out that the Riccati approach has been extended to problems where the complete state is not available for feedback. These results are in [8]–[10] and require solution of two Riccati equations.

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Fake Algebraic Riccati Techniques and Stability

MARIE-ANTOINETTE POUBELLE, ROBERT R. BITMEAD, AND MICHEL R. GEVERS

Abstract—**Conditions, sufficient and necessary, for monotonic behavior of the solutions of the Riccati differential equation and Riccati difference equation are derived. For the optimal filtering (respectively, control) equation these results are derived without the usual requirement of detectability (respectively, stabilizability). The monotonic behavior allows us to prove stabilizing properties of the solutions subject only to requirements on the initial conditions.**

I. INTRODUCTION

We study stabilization properties of the solutions of the following equations of optimal filtering: the discrete-time Riccati difference equation (RDE)

$$P(t+1)=FP(t)F'-FP(t)H'(HP(t)H'+R)^{-1}HP(t)F'+Q$$
$$P(0)=P_0; \tag{1}$$

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M.-A. Poubelle is with the Department of Systems Engineering, Research School of Physical Sciences, Australian National University, Canberra, A.C.T. 2601 Australia, on leave from D.G.S.E.

R. R. Bitmead is with the Department of Systems Engineering, Research School of Physical Sciences, Australian National University, Canberra, A.C.T. 2601 Australia.

M. R. Gevers is with the Laboratoire d'Automatique, de Dynamique et d'Analyse des Systèmes, Université Catholique de Louvain, Louvain, la Neuve, Belgium.

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and the continuous-time Riccati differential equation (RDE)

$$\dot{P}(t)=FP(t)+P(t)F'-P(t)H'R^{-1}HP(t)+Q$$
$$P(0)=P_0, \tag{2}$$

with $R > 0$ and $Q = Q'$.

These equations are part of the Kalman filters associated with linear time-invariant systems. By duality, the same equations refer to the optimal control problem [1], [2].

Connected with these RDE's are their associated algebraic Riccati equations (ARE's):

$$P=FPF'-FPH'(HPH'+R)^{-1}HPF'+Q \tag{3}$$

and

$$0=FP+PF'-PH'R^{-1}HP+Q. \tag{4}$$

Under certain conditions the solution $P(t)$ of the RDE converges to the solution P of the associated ARE as t tends to infinity. This shall not concern us here. Rather we shall be directing our attention to the stability of the linear time-invariant "frozen" closed-loop discrete-time system

$$x(t+1)=\{F-FP(s)H'[HP(s)H'+R]^{-1}H\}x(t) \tag{5}$$

or its continuous-time counterpart

$$\dot{x}(t)=[F-P(s)H'R^{-1}H]x(t) \tag{6}$$

for fixed s.

Because stabilization properties of P, the solution of the ARE, are well known, we utilize a device pioneered in [3]–[5]—the fake algebraic Riccati equations (FARE) associated with the RDE's (1) and (2), respectively,

$$\bar{Q}(t)=P(t)-FP(t)F'+FP(t)H'(HP(t)H'+R)^{-1}HP(t)F' \tag{7}$$

indiscreet time, and

$$\bar{Q}(t)=P(t)H'R^{-1}HP(t)-FP(t)-P(t)F' \tag{8}$$

in continuous time. These FARE's are actually definitions for matrix sequences and functions $\bar{Q}(t)$. However, as explored in [3]–[5], it is the stabilizability of the pair $[F, \bar{Q}^{1/2}(s)]$ (when $\bar{Q}(s) \geqslant 0$) that determines the asymptotic stability of (7) or (8), respectively, since then $P(s)$ satisfies a legitimate ARE and standard stability results may be invoked. Specifically we recall the following.

Theorem 1: Consider the FARE (7) [respectively, (8)] and assume:

i) $[H, F]$ is a detectable pair,

ii) $\bar{Q}(s) \geqslant 0$ and $[F, \bar{Q}^{1/2}(s)]$ is a stabilizable pair.

Then the frozen closed-loop system (5) [respectively, (6)] is asymptotically stable.

Subtracting (1) from (7), and (2) from (8) yields

$$\bar{Q}(t)=Q+P(t)-P(t+1) \tag{9}$$

and

$$\bar{Q}(t)=Q-\dot{P}(t). \tag{10}$$

The connection between monotonicity of $\{P(t)\}$ and stabilizability properties of $[F, \bar{Q}^{1/2}(t)]$ then emerges; see [3].

We perform two main tasks in this note. We extend the sufficient conditions of [3], [4] for monotonicity of $P(t)$, the solution of the RDE, to necessary and sufficient conditions. This is done without the usual detestability conditions on the pair $[H, F]$. By exploiting some novel manipulation of the RDE in continuous time, we develop new stabilizability conditions applicable for $0 \leqslant P(0) \leqslant P$. Previous conditions only dealt with $0 \leqslant P \leqslant P(0)$. Here P is the solution of the ARE (3) or (4). A connection with the asymptotic stability of finite horizon optimal control strategies will be made.

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II. Discrete-Time Results

We first recall a lemma of [4], inspired by a result of Nishimura [6].

Lemma 1: Consider two discrete-time RDE's (1) with the same F, H, R matrices but possibly different Q matrices, $\hat{Q}$ and Q^*, and possibly different initial conditions. Denote the solutions $\hat{P}(t)$ and $P^*(t)$, respectively, and rewrite (1) as

$$\hat{P}(t+1)=f(\hat{P}(t), \hat{Q}), \; P^*(t+1)=f(P^*(t), Q^*). \tag{11}$$

Then

$$\hat{P}(t+1) \geqslant P^*(t+1) \text{ if } \hat{P}(t) \geqslant P^*(t) \text{ and } \hat{Q} \geqslant Q^*. \tag{12}$$

The following three results are immediate consequences.

Lemma 2: Consider the RDE (1). If for some t, $P(t) \geqslant P(t+1)$ (respectively, $P(t) \leqslant P(t+1)$), then $P(t+k) \geqslant P(t+k+1)$ (respectively, $P(t+k) \leqslant P(t+k+1)$) for all $k \geqslant 0$.

Proof: From Lemma 1 by considering $\hat{P}(t) = P(t)$, $P^*(t) = P(t+1)$, and $\hat{Q} = Q^* = Q$ (and vice versa).

Theorem 2: The sequence $\{P(t)\}$ is monotonically nonincreasing (respectively, monotonically nondecreasing) if and only if $\bar{Q}(0) \geqslant Q$ (respectively, $\bar{Q}(0) \leqslant Q$).

Proof: We prove only the nonincreasing result. Note from (9) that $P(1) - P(0) = Q - \bar{Q}(0)$. Using Lemma 2 the result follows.

The following special case was partially established by Caines and Mayne using more devious means.

Corollary 1: If $P_0 = 0$, the sequence $\{P(t)\}$ is monotonically nondecreasing if $Q \geqslant 0$, monotonically nonincreasing when $Q \leqslant 0$.

Proof: From Theorem 2 noting that $\bar{Q}(0) = 0$.

III. Continuous-Time Monotonicity Results

We derive the equivalent of the preceding two lemmas, or dilemma, but now in continuous time. First, we need a preliminary result also found in [7, p. 58].

Lemma 3: Consider the following time-varying Lyapunov equation:

$$\dot{S}(t)=A(t)S(t)+S(t)A'(t)+W(t), \; S(0)=S_0 \tag{13}$$

and denote $\Phi(t, \tau)$ the transition matrix of $A(t)$. Then the solution is

$$S(t)=\int_0^t \Phi(t, \tau)W(\tau)\Phi'(t, \tau)\, d\tau+\Phi(t, 0)S_0\Phi'(t, 0). \tag{14}$$

Proof: By differentiating $S(t)$ in (14) and observing that they are the same.

We now generalize Nishimura/Lemma 1.

Lemma 4: Consider two RDE's (2) with the same F, H, R but possibly different Q matrices, $\hat{Q}$ and Q^*, and possibly different initial conditions, $\hat{P}(0)$ and $P^*(0)$. Denote the solutions $\hat{P}(t)$ and $P^*(t)$, respectively. Then $\hat{P}(0) \geqslant P^*(0)$ and $\hat{Q} \geqslant Q^*$ implies $\hat{P}(t) \geqslant P^*(t)$ for all $t \geqslant 0$.

Proof: Denote $\tilde{P}(t) = \hat{P}(t) - P^*(t)$. Then

$$\dot{\tilde{P}}(t)=A(t)\tilde{P}(t)+\tilde{P}(t)A'(t)+W(t) \tag{15}$$

where

$$A(t)=F-\hat{P}(t)H'R^{-1}H \tag{16}$$

and $W(t) = \tilde{P}(t)H'R^{-1}H\tilde{P}(t) + \hat{Q} - Q^*$. The result follows immediately from Lemma 3.

It is well known that the continuous-time RDE (2) may be written as a Lyapunov-like equation

$$\dot{P}(t)=A(t)P(t)+P(t)A'(t)+P(t)H'R^{-1}HP(t)+Q \tag{17}$$

with $A(t) = F - P(t)H'R^{-1}H$. However, the analysis may be carried further. Differentiating the RDE (2) yields directly

$$\ddot{P}(t)=\dot{P}(t)[F-P(t)H'R^{-1}H]'+[F-P(t)H'R^{-1}H]\dot{P}(t)$$

or

$$\ddot{P}(t)=\dot{P}(t)A'(t)+A(t)\dot{P}(t). \tag{18}$$

Thus, $\dot{P}(t)$ itself satisfies a Lyapunov matrix equation and Lemma 3 yields, for $t \geqslant t_0$,

$$\dot{P}(t)=\Phi(t, t_0)\dot{P}(t_0)\Phi'(t, t_0). \tag{19}$$

This was also observed in [8]. This admits immediately the following results.

Lemma 5: Consider the RDE (2). If for some t_0, $\dot{P}(t_0) \geqslant 0$ (respectively, $\dot{P}(t_0) \leqslant 0$), then $\dot{P}(t) \geqslant 0$ (respectively, $\dot{P}(t) \leqslant 0$) for all $t \geqslant t_0$.

Theorem 3: The solution $\{P(t)\}$ of (2) is monotonically nonincreasing (respectively, monotonically nondecreasing) if and only if $\bar{Q}(0) \geqslant Q$ (respectively, $\bar{Q}(0) \leqslant Q$).

Proof: Follows by Lemma 5 upon noting (10).

IV. Continuous-Time Stability Results

The connection between asymptotic stability of (6) and monotonicity of $P(t)$ is studied in [4] by linking the conditions of Theorem 1 with monotonicity conditions of Theorem 3, noting the relationship (10) between $\bar{Q}(s)$ and $\dot{P}(s)$. In these papers monotonic nonincreasing $P(t)$, i.e., $\dot{P}(t) \leqslant 0$, were considered in the development of stability theorems. Of necessity, this treats only the case of initial conditions $P(0)$ which are greater than or equal to the steady-state solution P of the ARE (4). We shall now present results which guarantee stability of (6) for all $s \geqslant 0$ when $P(0)$ may be less than P.

The differentiation of the RDE (2) may be carried further by differentiating (18) to produce

$$\dddot{P}(t)=[F-P(t)H'R^{-1}H]\ddot{P}(t)+\ddot{P}(t)[F-P(t)H'R^{-1}H]' -2\dot{P}(t)H'R^{-1}H\dot{P}(t)$$

or

$$\dddot{P}(t)=A(t)\ddot{P}(t)+\ddot{P}(t)A'(t)-2\dot{P}(t)H'R^{-1}H\dot{P}(t). \tag{20}$$

That is, $\ddot{P}(t)$ satisfies a Lyapunov equation. Appealing to Lemma 3 for $t \geqslant t_0$

$$\ddot{P}(t)=-2\int_{t_0}^t \Phi(t, \tau)\dot{P}(\tau)H'R^{-1}H\dot{P}(\tau)\Phi'(t, \tau)\, d\tau + \Phi(t, t_0)\ddot{P}(t_0)\Phi'(t, t_0). \tag{21}$$

Thus, should $\ddot{P}(t_0)$ be nonpositive definite, then $\ddot{P}(t_0 + t)$ will remain nonpositive definite for all $t \geqslant 0$.

We may now return to the stability problem.

Theorem 4: Consider the solution $P(t)$ to the RDE (2), $t \geqslant 0$. Assume the following.

i) $[H, F]$ is a detectable pair, and $P(0) = P_0$ is such that

ii) $0 \leqslant \bar{Q}(0) = P_0H'R^{-1}HP_0 - FP_0 - P_0F'$ and $[F, \bar{Q}^{1/2}(0)]$ is a stabilizable pair,

iii) $0 \geqslant \ddot{P}(0) = (FP_0 + P_0F' - P_0H'R^{-1}HP_0 + Q)(F - P_0H'R^{-1}H)' + (F - P_0H'R^{-1}H)(FP_0 + P_0F' - P_0H'R^{-1}HP_0 + Q)$.

Then the frozen closed-loop system (6) is asymptotically stable for all $s \geqslant 0$.

Proof: From condition iii) and (21) we have $\ddot{P}(t) \leqslant 0$ for all t. Therefore, $\dot{P}(t + T) \leqslant \dot{P}(t)$ for all $t, T \geqslant 0$. From (10) this yields $\bar{Q}(t + T) \geqslant \bar{Q}(t)$ for all $t, T \geqslant 0$ and, in particular, this holds for $t = 0$ and $T = s$. Theorem 1 completes the proof. The expression for $\ddot{P}$ in iii) derives from substitution of (2) and (16) in (18).

We note here that nonpositivity of $\ddot{P}(0)$ replaces the condition of nonpositive $\dot{P}(0)$ in [4]. Indeed, from (10), there is no need to exclude initial conditions P_0 which yield $\dot{P}(0)$ [via (2)] being positive definite, as occurs, for example, if $P_0 = 0$; see Corollary 1. The condition for stability of (6) for all $s \geqslant 0$ with $P_0 = 0$ reduces to i) $[H, F]$ detectable, ii) Re $[\lambda_i(F)] < 0$, iii) $FQ + QF' \leqslant 0$. Note that ii) implies i).

For optimal control problems or, more specifically, adaptive control problems using LQG feedback control laws, e.g., [9], one is frequently concerned with the application of finite horizon LQG laws while attempting to achieve closed-loop asymptotic stability. The finite horizon is a design parameter and a meaningful question is to ask for conditions under which the closed loop can be guaranteed stable for all horizons larger than a certain value. This is directly addressed by the results of Theorem 4.

For discrete-time problems such as those in [9], a suitable version of Theorem 4 needs to be stated. This would require the development of the analog of (20). We believe that such a result is possible but have yet to master the arithmetical intricacies.

V. Conclusion

We have derived sufficient and necessary conditions for monotonic behavior of the solutions of the RDE's. This was done without an overriding requirement of detectability. New results were developed to establish stabilizability properties of these solutions.

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External Description for Multivariable Systems Sampled in an Aperiodic Way

A. FUSTER

Abstract—**An external description for nonperiodically sampled multivariable linear systems has been developed. Emphasis is on the sampling period sequence, included among the variables to be handled. The computational procedure is simple and no use of polynomial matrix theory is required. This input/output description is believed to be a basic formulation for its later application to the problem of optimal control and/or identification of linear dynamical systems.**

Introduction

There are two different ways of describing dynamical systems: i) by means of input/output relations; and ii) by means of state variables.

In the classical or frequency-domain approach, systems are described by transfer functions which reflect just the external or input/output properties of the system. However, this mode of description entails some difficulties concerning stability and realization [1], [2].

The modern or time-domain approach turns around the axiomatic concept of state. The method is exact in defining the notion of dynamical systems and also describes all internal couplings among the system variables [3], [4]. Nevertheless, the procedure became somewhat disappointing due to the necessity of finding state-variable models and to the implicit assumption that all state-variables are accessible for direct measurement. This assumption is justified in mechanical or electrical systems but it is not generally satisfied for plants in chemical, gas, paper, and other industries.

These considerations were responsible for the comeback of transfer function methods [5]–[7].

On the other hand, the enormous increase in the use of digital computers in process control has stimulated studies in the field of discrete systems for both types of representation. See [8]–[10] and also the above-mentioned references. All of them are concerned with constant sampling period, which is convenient for the simplicity of implementation and mathematical treatment. However, the general case of aperiodic sampling is *a priori* capable of more favorable solutions to the problem of control and/or identification of dynamical systems, and it is also feasible with modern time-sharing equipment.

In this work, an input/output modeling technique for aperiodic sampling linear systems has been developed. The external description includes the sampling sequence among the variables to be handled. The system is described by input/output data according to the actual experimentation conditions. Although the multivariable case is covered, the complexity of the polynomial matrix theory is avoided.

The procedure is believed to be a basic formulation for its later application to the synthesis of linear control systems sampled in an aperiodic way, since most of these techniques for nonperiodically sampled systems rely exclusively on the state-space equations [11]–[13].

I. Basic Assumptions

Our discussion is restricted to the following:

1) linear time-invariant multivariable dynamical systems of finite order;

2) systems whose transfer function is a $p \times m$ matrix (m-inputs, p-outputs), where the different entries are strictly proper rational functions.

We end this preliminary section with the following statement.

Statement: Let (G_l) be a family of vector functions

$$G_l;\ |R^n \to |R^n\ G_l \in C^\infty(|R^n, |R^n) \qquad (l=0, 1, \cdots, n)$$

$C^\infty(|R^n, |R^n)$ being the set of infinitely differentiable functions on $|R^n$.

If the following conditions are verified:

a) there exists an integer $r \leqslant n$ such that the elements $(G_r(z))$ are linearly independent for all $z \in |R^n$.

b) there exists an integer $k > r$ such that $G_k(z)$ depends linearly on $(G_0(z), \cdots, G_r(z))$.

Then, there are functions $f_0, f_1, \cdots, f_n \in C^\infty(|R^n, |R)$ such that the following expression holds:

$$\sum_{l=0}^{n} f_{n-l}(z) G_l(z) = O \qquad \forall z \in |R^n. \tag{1.1}$$

The previous result is a direct consequence of the Cramer Rule; for more details see [15].

II. External Description for Nonperiodically Sampled Linear Systems

A. Input/Output Modeling Technique

Let $H(s)$ be the matrix transfer function of a linear time-invariant multivariable system.

$$H(s) = (H_{rq}(s)) \qquad (r=1, \cdots, p), (q=1, \cdots, m) \tag{2.1}$$

let us rewrite $H(s)$ as

$$H(s) = \frac{N(s)}{d(s)} \tag{2.2}$$

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The author is with the Instituto de Electrónica de Comunicaciones (C.S.I.C.), Madrid, Spain.
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