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Dear Prof. Antoulas,

Attached is the paper "New relations between norms of system transfer functions" intended for publication in
the Elsevier journal - System and Control Letters.

Yours Sincerely,
Tzvetan Ivanov

*Abstract

It is well known that the H^2-norm and H^∞-norm
of a transfer function can differ arbitrarily since
both norms reflect fundamentally different properties.
However, if the pole structure of the transfer
function is known it is possible to bound the
H^∞-norm from above by a constant multiple
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we derive a tight bound for the H^∞-norm given
knowledge of the H^2-norm and the poles of a
transfer function. We compute the bound in closed
form for multiple input multiple output transfer
functions in continuous and discrete time. Futhermore
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given a weighted L^2-norm.

New relations between norms of system transfer functions[☆],[☆☆]

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Abstract

It is well known that the H^2-norm and H^∞-norm of a transfer function can differ arbitrarily since both norms reflect fundamentally different properties. However, if the pole structure of the transfer function is known it is possible to bound the H^∞-norm from above by a constant multiple of the H^2-norm. It is desirable to compute this constant as tightly as possible. In this article we derive a tight bound for the H^∞-norm given knowledge of the H^2-norm and the poles of a transfer function. We compute the bound in closed form for multiple input multiple output transfer functions in continuous and discrete time. Futhermore we derive a general procedure to compute the bound given a weighted L^2-norm.

Keywords: Robust Control; LQR; Confidence Region; Covariance ; Weighted H^2 norm; Supremum norm; H^∞ norm; Tight Bound; Reproducing Kernel

1. Introduction

Norms induced by inner products, such as the L^2-norm, are important because they lend themselves to computations and geometric interpretations. However in many applications the supremum norm is more meaningful. For example covariance matrices arising in statistical inference give rise to ellipsoidal

confidence regions, i.e., weighted 2-norm balls; however, from a robust control perspective, the confidence regions of interest are those given by supremum norm balls. Both norms also arise naturally in model order reduction: approximation criteria based on the 2-norm are appreciated for their rich structure since they lead to efficient model-reduction algorithms [8, 5, 3]. However, if one thinks of a transfer function as an operator acting on input signals, a supremum norm approximation criterion is more natural [4]. This is why it is of special interest to quantify the deviation of the optimal 2-norm approximation from an optimal supremum norm approximation.

The problem of bounding the H^∞ norm from above by a constant multiple of the H^2 norm has been first addressed in the engineering context in De Bruyne et al [2]. The main assumption there was that the function space consists of all strictly proper rational functions with poles ranging over a prescribed set of simple poles, where complex poles occur in complex conjugate pairs. Under the assumption that the coefficients of the numerator polynomial are real, as is obviously frequently the case, the bounds computed have been shown to be conservative, yet no indication was given on how to remove this conservatism. In [6] these results have been extended to vector-valued transfer functions which share the same fixed denominator. Both articles [2, 6] are constrained to the standard L^2-norm and cannot be applied in situations where a weighted L^2-norm is of interest.

In this article we provide a tight bound for the case of vector-valued transfer functions with fixed, possibly different, scalar denominator polynomials in each component. We distinguish between real-rational and complex-rational transfer functions and quantify the amount of conservatism when using the bound which applies to complex-rational transfer functions in the real-rational case. The bounds are given by closed form formulas for the H^2-norm case and by computational procedures in the weighted L^2-norm case. The methods presented in this article are actually general in the sense that they can be applied to any finite-dimensional Hilbert space consisting of vector-valued bounded functions.

The paper is structured as follows: In Section 2 we derive the bound for complex-rational transfer functions corresponding to continuous-time systems. In Section 3 we study the real-rational case. In Section 4 we comment on how to apply the results for the discrete-time case. In Section 5 we give a numerical example before we conclude in Section 6.

Notation: The letters $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively; $\jmath = \sqrt{-1}$ denotes the imaginary unit; $\bar{z}$ will be used to denote complex conjugate of $z \in \mathbb{C}$; M^{T} and M^{H} denote the transpose and conjugate transpose of a matrix $M \in \mathbb{C}^{m \times n}$ respectively; $\mathbb{H}_+ \subseteq \mathbb{C}$ and $\mathbb{D}^{\mathrm{c}} \subseteq \mathbb{C}$ denote the closed right half-plane and complement of the open unit disk respectively; $\mathbb{R}[s]$ and $\mathbb{C}[s]$ denote the polynomials in the indeterminant s with real and complex coefficients respectively; $\mathbb{R}(s)$ and $\mathbb{C}(s)$ denote real and complex rational functions respectively;

2. Bounds for complex-rational continuous-time transfer functions

Knowing the poles of a transfer function describing a stable continuous time system does not suffice to impose a constraint on the value of its frequency response at a given frequency. If, however, one also knows the "energy" of its impulse response or equivalently the 2-norm of its frequency response, it is possible to give a tight upper-bound on the absolute value of its frequency response for each given frequency. This problem appears to have been first considered in the engineering context by De Bruyne et al [2]. In Theorem 2 we generalize the existing results into several directions. We treat the case where the transfer functions is vector-valued. Moreover the 2-norm need not to be the standard H^2-norm but can be a weighted L^2-norm. Last but not least we remark that the bound we derive in (7) holds not only for $s \in \jmath\mathbb{R}$ but for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$, i.e., all s in the closed right half plane. This is particularly useful in applications like model reduction by interpolation where the interpolation points are typically not on the imaginary axis. In Theorem 8 we shall see that for systems with a real-valued impulse response it is possible to improve the the upper bound obtained in Theorem 2.

Setup: We consider transfer functions corresponding to asymptotically stable single input multiple output continuous-time systems. In other words we consider a Hurwitz-stable vector of transfer functions $M \in \mathbb{C}(s)^L$ of the form

$$M(s) = \begin{bmatrix} M_1(s) \\ \vdots \\ M_L(s) \end{bmatrix}, \quad M_l(s) = \frac{b_{l,0} + b_{l,1}s + \cdots + b_{l,n_l-1}s^{n_l-1}}{d_{l,0} + d_{l,1}s + \cdots + d_{l,n_l-1}s^{n_l-1} + s^{n_l}}. \quad (1)$$

Here $n_1, \ldots, n_L \geq 1$ are s.t. for all $l = 1, \ldots, L$ we have $M_l(s) = \frac{b_l(s)}{d_l(s)}$ with

$$b_l(s) = \sum_{k=0}^{n_l-1} b_{l,k}s^k \in \mathbb{C}[s], \quad d_l(s) = s^{n_l} + \sum_{k=0}^{n_l-1} d_{l,k}s^k \in \mathbb{C}[s], \quad (2)$$

and $d_l(s) \neq 0$ for all $s \in \mathbb{H}_+$ with $\mathbb{H}_+ := \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \geq 0\}$.

We consider a weighted L^2-norm $\|\cdot\|_W$ on $L^2(\jmath\mathbb{R})$ of the form

$$\|M\|_W^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{l=1}^{L} |M_l(\jmath\omega)|^2 \, W_l(\omega) \, \mathrm{d}\omega, \quad (3)$$

where $W_l : \mathbb{R} \to \mathbb{R}_{\geq 0}$ denotes a weight function for each $l = 1, \ldots, L$.

By assuming that $\|\cdot\|_W$ is a norm we implicitly impose constraints, like positivity and integrability, on the weights W_l.

Remark 1. The use of weighted L^2-norms is motivated by applications like system-identification. In the scalar case $M = \hat{P} - P$ corresponds to the error

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between the estimator $\hat{P}$ and the true plant P. The probability of the event E given by the ε-confidence ellipsoid

$$E := \{M \mid \|M\|_W^2 < \varepsilon\} \quad \text{with} \quad W = \frac{\Phi_v(\omega)}{\Phi_u(\omega)}, \tag{4}$$

where the weight W is given by the inverse of the signal-to-noise-ratio, can be explicitly calculated using standard results on asymptotic normality [7].

In the single-input multiple-output case we have $W_l = \Phi_{v_l}/\Phi_u$ where Φ_{v_l} is the spectrum of the noise-process v_l at the l-th output channel and Φ_u is the input spectrum.

Theorem 2. With respect to the setup described above define

$$B_l(s) := \frac{1}{d_l(s)}[s^0, \ldots, s^{n_l-1}]^{\mathrm{T}} \quad \forall l = 1, \ldots, L, \tag{5}$$

together with the functions $K_l : \mathbb{H}_+ \times \mathbb{H}_+ \to \mathbb{C}$ given by[1]

$$K_l(s, w) := B_l(w)^{\mathrm{H}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} B_l(\jmath\omega) B_l(\jmath\omega)^{\mathrm{H}} W_l(\omega) \mathrm{d}\omega \right)^{-1} B_l(s). \tag{6}$$

Then for all $l = 1, \ldots, L$ the expression $K_l(s, s)$ is a $\mathbb{R}$-valued function on $\mathbb{H}_+$ such that for all $M : \mathbb{H}_+ \to \mathbb{C}^L$ of the form (1) there holds

$$\forall s \in \mathbb{H}_+ : \quad \sum_{l=1}^{L} |M_l(s)|^2 \leq C(s) \cdot \|M\|_W^2, \quad C(s) := \max_{1 \leq l \leq L} K_l(s, s). \tag{7}$$

If $d_1, \ldots, d_L \in \mathbb{C}[s]$ are fixed, (7) yields a tight bound in the sense that there exists an $M \in \mathbb{C}(s)^L$ of the form (1) such that (7) holds with equality.

The proof of Theorem 2 will be given right after the proof of Lemma 7 stated below. Before we state Lemma 7, a few remarks are in order.

Corollary 3. For all $M \in \mathbb{C}(s)^L$ of the form (1) define

$$\|M\|_\infty^2 := \sup_{s \in \mathbb{H}_+} \sum_{l=1}^{L} |M_l(s)|^2. \tag{8}$$

Then there holds

$$\|M\|_\infty^2 \leq C \cdot \|M\|_W^2 \quad \text{with} \quad C := \sup_{s \in \mathbb{H}_+} C(s). \tag{9}$$

Moreover for $d_1, \ldots, d_L \in \mathbb{C}[s]$ fixed this bound cannot be improved, i.e., there exists an $M \in \mathbb{C}^L(s)$ of the form (1) for which (9) holds with equality.

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[1] The inverse in (6) exists because the components of B_l are linearly independent functions and $\|\cdot\|_W$ is a norm.

Remark 4. For the case where the 2-norm is given by the standard H^2-norm, i.e., $W \equiv 1$, the celebrated Christoffel-Darboux identity, see e.g. [1], states that

$$K_l(s, w) := \frac{1 - Q_l(s)\overline{Q_l(w)}}{s + \bar{w}} \quad \text{with} \quad Q_l(s) = \frac{\overline{d_l(-\bar{s})}}{d_l(s)}. \tag{10}$$

Remark 5. If $T_l \in \mathbb{C}^{n_l \times n_l}$ is invertible and $A_l = T_l B_l$ then it is easy to check that K_l given by (6) can equivalently be written as

$$K_l(s, w) = A_l(w)^{\mathrm{H}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} A_l(\jmath\omega) A_l(\jmath\omega)^{\mathrm{H}} W_l(\omega) \mathrm{d}\omega \right)^{-1} A_l(s). \tag{11}$$

In particular for $d_l(s) = \prod_{k=1}^{n_l}(s + a_k)$ with $a_k \neq a_{k'}$ for all $k \neq k'$ then

$$A_l = [1/(s + a_1), \ldots, 1/(s + a_{n_l})]^{\mathrm{T}} \in \mathbb{C}(s)^L, \tag{12}$$

is given by $A_l = T_l B_l$ where the non-singular matrix $T_l \in \mathbb{C}^{n_l \times n_l}$ is obtained via partial-fraction expansion. In [2] this special choice of basis, which is possible only if the pole locations are distinct, was used for the scalar unweighted case, i.e., for $L = 1$ and $W \equiv 1$. In this case the diagonal elements of (11) are given by

$$K_l(s, s) = \frac{2 \operatorname{Re}(a_1)}{|s - a_1|^2} + \cdots + \frac{2 \operatorname{Re}(a_{n_l})}{|s - a_{n_l}|^2}. \tag{13}$$

We remark that (13) is valid also in the case of multiple poles, a fact which can be verified directly from the Christoffel-Darboux formula (10).

Remark 6. The key ideas which will allow us to prove Theorem 2 are summarized in Lemma 7. It is possible to state the problem in the language of functional analysis. Using this language the statement of Lemma 7 is that, for $d_1, \ldots, d_L \in \mathbb{C}[s]$ fixed, the functions K_l, given by (6), form the reproducing kernel of the linear space consisting of all transfer functions of the form (1) with respect to the inner-product (14) induced by the weighted L^2-norm.

Lemma 7. Let $d_1, \ldots, d_L \in \mathbb{C}[s]$ denote fixed Hurwitz-stable polynomials and define

$$\langle M, N \rangle_W := \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{l=1}^{L} \overline{N_l(\jmath\omega)} M(\jmath\omega) \, W_l(\omega) \, \mathrm{d}\omega, \tag{14}$$

for all M, N of the form (1). Moreover for $s, w \in \mathbb{H}_+$ define

$$K(s, w) := \begin{bmatrix} K_1(s, w) & & \\ & \ddots & \\ & & K_L(s, w) \end{bmatrix} \in \mathbb{C}^{L \times L}, \tag{15}$$

where K_l is given by (6). For $w \in \mathbb{H}_+$ and $c \in \mathbb{C}^L$ define

$$K_{w,c}(s) := K(s, w)c. \tag{16}$$

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Then there exists $N \in \mathbb{C}(s)^L$ of the form (1), with the same fixed denominator polynomials $d_1, \ldots, d_L$, such that $K_{w,c} = N$. Moreover

$$\langle M, K_{w,c} \rangle_W = c^{\mathrm{H}} M(w) \quad \text{and} \quad \langle K_{w,c}, K_{w,c} \rangle_W = c^{\mathrm{H}} K(w,w) c, \qquad (17)$$

holds for all $M \in \mathbb{C}(s)^L$ of the form (1).

Proof. For $M = [M_1, \ldots, M_L]^{\mathrm{T}} \in \mathbb{C}(s)^L$ of the form (1), let the n_l-vector $m_l = [b_{l,1}, \ldots, b_{l,n_l}]^{\mathrm{T}} \in \mathbb{C}^{n_l}$ denote the coordinate-vector of M_l. In other words we have $M_l(s) = B_l(s)^{\mathrm{T}} m_l$. Let

$$G_l = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} B_l(\jmath\omega) B_l(\jmath\omega)^{\mathrm{H}} W_l(\omega) \mathrm{d}\omega \right)^{\mathrm{T}} \quad \text{for} \quad l = 1, \ldots, L.$$

It follows from (3) and (6) that

$$K_l(s,w) = B_l(s)^{\mathrm{T}} G_l^{-1} B_l(w)^{\mathrm{H,T}} \quad \text{and} \quad \|M\|_W^2 = \sum_{l=1}^{L} m_l^{\mathrm{H}} G_l m_l.$$

For $c \in \mathbb{C}^L$ there holds

$$K_{w,c}(s) = [B_1(s)^{\mathrm{T}} \cdot c_1 G_1^{-1} B_1(w)^{\mathrm{T,H}}, \ldots, B_L(s)^{\mathrm{T}} \cdot c_L G_L^{-1} B_L(w)^{\mathrm{T,H}}]^{\mathrm{T}},$$

which proves that $K_{w,c}$ is of the form (1) for any fixed $w \in \mathbb{H}_+$. In particular there holds that

$$\langle M, K_{w,c} \rangle_W = \sum_{l=1}^{L} (c_l \cdot G_l^{-1} B_l(w)^{\mathrm{T,H}})^{\mathrm{H}} G_l m_l$$

$$= \sum_{l=1}^{L} \bar{c}_l B_l(w)^{\mathrm{T}} G_l^{\mathrm{H},-1} G_l m_l = c^{\mathrm{H}} M(w),$$

since $G_l = G_l^{\mathrm{H}}$ and, by construction of m_l, $B_l(w)^{\mathrm{T}} m_l = M_l(w)$. This proves the first part of (17), and also the second part of (17), by using $M = K_{w,c}$. $\qquad \square$

Proof *of Theorem 2.* First we note a preliminary result which states that

$$\sum_{l=1}^{L} |M_l(w)|^2 = \sup_{c^{\mathrm{H}} c \leq 1} |c^{\mathrm{H}} M(w)|^2. \qquad (18)$$

This is easy to show if one uses the Cauchy-Bunyakovsky-Schwarz (CBS) inequality. Next, because as established in Lemma 7, there holds

$$c^{\mathrm{H}} M(w) = \langle M, K(\cdot, w) c \rangle_W,$$

the CBS inequality shows that, for fixed vector $c \in \mathbb{C}^L$, and for all functions M of the form (1) there holds

$$|c^{\mathrm{H}} M(w)|^2 \leq \|M\|_W^2 \cdot \|K(\cdot, w) c\|_W^2. \qquad (19)$$

Moreover this bound is tight over $M \in \mathbb{C}(s)^L$ of the form (1) since it becomes an equality for $M \in \mathbb{C}(s)^L$ given by $M(s) = K(s,w) c$. Now supremizing over all possible $c \in \mathbb{C}^L$ with norm 1 yields

$$\sup\{c^{\mathrm{H}} K(w,w) c \,|\, c \in \mathbb{C}^L, c^{\mathrm{H}} c \leq 1\} = \lambda_{\max}[K(w,w)]. \qquad (20)$$

Then, by using (19), equation (7) follows since $K(w,w)$ is a diagonal matrix. $\quad \square$

In Section 3 we address the question of how to improve the bound given by (7) for the case where the transfer functions are known to be real-rational.

3. Bounds for real-rational continuous-time transfer functions

The main result of this section is Theorem 8 which states that the bound in the real-rational case is up to two times smaller than in the unconstrained case we already discussed. Moreover Theorem 8 shows that the bound can be computed in a similar fashion to the bound in the complex case.

Theorem 8. Let $\tilde{M} \in \mathbb{R}(s)^L$ denote a real-rational Hurwitz-stable transfer function of the form (1) and assume that the matrix inverted in (6) is real-valued, i.e.,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} B_l(\jmath\omega) B_l(\jmath\omega)^{\mathrm{H}} W_l(\omega) \mathrm{d}\omega \quad \text{is a } n_l \times n_l \text{ real-valued matrix,} \qquad (21)$$

for all $l = 1, \ldots, L$. We note that this is the case for $W_l \equiv 1$ and weights given by $W_l = |R_l|^2$ where $R_l \in \mathbb{R}(s)$ is real-rational. Then for all $s \in \mathbb{H}_+$ there holds

$$\sum\nolimits_{l=1}^{L} |\tilde{M}_l(s)|^2 \leq \tilde{C}(s) \cdot \|\tilde{M}\|_W^2, \quad \tilde{C}(s) := \max_{1 \leq l \leq L} \frac{K_l(s,s) + |K_l(s,\bar{s})|}{2}, \qquad (22)$$

with $\|\cdot\|_W$ and K_l given by (3) and (6), respectively. For fixed $d_1, \ldots, d_L \in \mathbb{R}[s]$ this bound is tight in the sense that there exists a $\tilde{M} \in \mathbb{R}(s)^L$ of the form (1) such that (22) holds with equality. Moreover for all $s \in \mathbb{H}_+$ there holds that

$$C(s)/2 \leq \tilde{C}(s) \leq C(s), \qquad (23)$$

with $C(s)$ given by (7). In particular for $\|\cdot\|_\infty$ defined by (8) there holds

$$\|\tilde{M}\|_\infty^2 \leq \tilde{C} \|\tilde{M}\|_W^2 \quad \text{with} \quad \tilde{C} := \sup_{s \in \mathbb{H}_+} \tilde{C}(s). \qquad (24)$$

Moreover, for $d_1, \ldots, d_L \in \mathbb{R}[s]$ fixed, this bound cannot be improved, i.e., there exists an $\tilde{M} \in \mathbb{R}(s)^L$ of the form (1) for which (24) holds with equality.

Proof. The proof of Theorem 8 is based on Lemma 9 and is located right after the proof of this Lemma. $\qquad \square$

Lemma 9. Let $d_1, \ldots, d_L \in \mathbb{R}[s]$ denote Hurwitz-stable polynomials with real coefficients, and let $K_l(s,w)$ be defined as in the statement of Theorem 2. For $l = 1, \ldots, L$ we define real-valued 2×2 matrices via

$$K_l^{(2)}(s,w) = \begin{bmatrix} \mathrm{Re}\left(\frac{K_l(s,w) + \overline{K_l(\bar{s},w)}}{2}\right) & -\mathrm{Re}\left(\frac{K_l(s,w) - \overline{K_l(\bar{s},w)}}{2\jmath}\right) \\ \mathrm{Im}\left(\frac{K_l(s,w) + \overline{K_l(\bar{s},w)}}{2}\right) & -\mathrm{Im}\left(\frac{K_l(s,w) - \overline{K_l(\bar{s},w)}}{2\jmath}\right) \end{bmatrix}. \qquad (25)$$

And futher define the block-diagonal matrix $K^{(2)}(s,w) \in \mathbb{R}^{2L \times 2L}$ via

$$K^{(2)}(s,w) := \begin{bmatrix} K_1^{(2)}(s,w) & & \\ & \ddots & \\ & & K_L^{(2)}(s,w) \end{bmatrix}, \tag{26a}$$

$$K_{w,r}^{(2)}(s) := \begin{bmatrix} 1 & \jmath & \\ & \ddots & \\ & & 1 & \jmath \end{bmatrix} K^{(2)}(s,w)r \quad \forall r \in \mathbb{R}^{2L}, s, w \in \mathbb{H}_+. \tag{26b}$$

Then $K_{w,r}^{(2)} \in \mathbb{R}(s)^L$ is of the form (1), and for all $\tilde{M} \in \mathbb{R}(s)$ of the form (1) there holds that

$$\left\langle \tilde{M}, K_{w,r}^{(2)} \right\rangle_W = \sum_{l=1}^L [r_{2l-1}, r_{2l}] \begin{bmatrix} \operatorname{Re} \tilde{M}_l(w) \\ \operatorname{Im} \tilde{M}_l(w) \end{bmatrix} \quad \forall \, r \in \mathbb{R}^{2L}, w \in \mathbb{H}_+. \tag{27}$$

The eigenvalues λ_+ and λ_- of $K_l^{(2)}(w,w)$ are real, non-negative and given by

$$\lambda_\pm \left[K_l^{(2)}(w,w) \right] = \frac{K_l(w,w) \pm |K_l(w,\bar{w})|}{2} \quad \text{for all} \quad w \in \mathbb{H}_+. \tag{28}$$

In particular the maximum eigenvalue $\lambda_{\max}$ of $K^{(2)}(w,w)$ is given by

$$\lambda_{\max} \left[K^{(2)}(w,w) \right] = \max_{1 \le l \le L} \frac{K_l(w,w) + |K_l(w,\bar{w})|}{2} \quad \text{for all} \quad w \in \mathbb{H}_+. \tag{29}$$

Proof. For all real rational functions $\tilde{M}, \tilde{U}, \tilde{V} \in \mathbb{R}(s)^L$ of the form (1) there holds

$$\left\langle \tilde{M}, \tilde{U} + \jmath \tilde{V} \right\rangle_W = \underbrace{\left\langle \tilde{M}, \tilde{U} \right\rangle_W}_{\in \mathbb{R}} - \jmath \underbrace{\left\langle \tilde{M}, \tilde{V} \right\rangle_W}_{\in \mathbb{R}}, \tag{30}$$

by assumption (21). Let $e_1, \dots, e_L$ denote the standard basis of $\mathbb{R}^L$ and, for fixed $w \in \mathbb{H}_+$, define

$$\tilde{U}_l(s) = \frac{K_l(s,w) + \overline{K_l(\bar{s},w)}}{2} e_l \quad \text{and} \quad \tilde{V}_l(s) = \frac{K_l(s,w) - \overline{K_l(\bar{s},w)}}{2\jmath} e_l.$$

From this it follows that $\tilde{U}_l, \tilde{V}_l \in \mathbb{R}(s)^L$ are of the form (1) and $K_{w,e_l} = \tilde{U}_l + \jmath \tilde{V}_l$. By property (17) of K_{w,e_l} and the decomposition (30) it follows that

$$\left\langle \tilde{M}, \tilde{U}_l \right\rangle_W = \operatorname{Re} M_l(w) \quad \text{and} \quad \left\langle \tilde{M}, -\tilde{V}_l \right\rangle_W = \operatorname{Im} M_l(w).$$

Moreover

$$\tilde{U}_l(s) = K_{w,e_{2l-1}}^{(2)}(s) \quad \text{and} \quad \tilde{V}_l(s) = K_{w,e_{2l}}^{(2)}(s),$$

which follows from the defining equations (25) and (26). We may thus conclude that (27) indeed holds. We now show that (28) holds. From (6) it follows that

for all $s, w \in \mathbb{H}_+$ there holds $\overline{K_l(s, w)} = K_l(w, s)$. Writing $\mathrm{Re}(a)$ as $(a + \bar{a})/2$, $\mathrm{Im}(a)$ as $(a - \bar{a})/2\jmath$ and utilizing the fact that $\overline{K(s, w)} = K(w, s)$ we can rewrite (25) as

$$[K_l^{(2)}(s, w)]_{1,1} = \frac{K(s, w) + K(w, s) + K(w, \bar{s}) + K(\bar{s}, w)}{4}$$

$$[K_l^{(2)}(s, w)]_{2,2} = \frac{K(s, w) + K(w, s) - K(w, \bar{s}) - K(\bar{s}, w)}{4}$$

$$[K_l^{(2)}(s, w)]_{2,1} = \frac{K(s, w) - K(w, s) + K(w, \bar{s}) - K(\bar{s}, w)}{4\jmath}$$

and $[K_l^{(2)}(s, w)]_{1,2} = [K_l^{(2)}(w, s)]_{2,1}$ for all $s, w \in \mathbb{H}_+$. In particular for $w = s$ the expression for (25) simplifies and one obtains

$$K_l^{(2)}(w, w) = \frac{1}{2} \begin{bmatrix} K_l(w, w) & \\ & K_l(w, w) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathrm{Re}\, K_l(w, \bar{w}) & \mathrm{Im}\, K_l(w, \bar{w}) \\ \mathrm{Im}\, K_l(w, \bar{w}) & -\mathrm{Re}\, K_l(w, \bar{w}) \end{bmatrix},$$

for which an easy calculation shows that (28) is satisfied. The result of (29) follows from the fact that $K^{(2)}(w, w)$ is block-diagonal. $\qquad\square$

Proof *of Theorem 8.* We adapt the proof of Theorem 2. First we note that

$$\sum\nolimits_{l=1}^{L} |\tilde{M}_l(w)|^2 = \sup_{r^\mathrm{T} r \leq 1} \left| \sum\nolimits_{l=1}^{L} r_{2l-1} \cdot \mathrm{Re}\, \tilde{M}_l(w) + \sum\nolimits_{l=1}^{L} r_{2l} \cdot \mathrm{Im}\, \tilde{M}_l(w) \right|^2, \quad (31)$$

which follows, just like (18), from the CBS inequality. By the CBS inequality one also verifies that, for fixed $r \in \mathbb{R}^{2L}$, we have

$$\left| \sum_{l=1}^{L} r_{2l-1} \cdot \mathrm{Re}\, \tilde{M}_l(w) + \sum_{l=1}^{L} r_{2l} \cdot \mathrm{Im}\, \tilde{M}_l(w) \right|^2 \leq \|\tilde{M}\|_W^2 \cdot \|K_{w,r}^{(2)}(s)\|_W^2, \quad (32)$$

for all $\tilde{M} \in \mathbb{R}(s)$ of the form (1). Here we used that $K_{w,r}^{(2)}$ given by (26) satisfies (27), which we proved in Lemma 9.

Moreover this bound is tight over $\tilde{M} \in \mathbb{R}(s)^L$ of the form (1) since it becomes an equality for $\tilde{M} \in \mathbb{R}(s)^L$ given by $\tilde{M} = K_{w,r}^{(2)}$. By utilizing property (27) we see that

$$\left\langle K_{w,r}^{(2)}, K_{w,r}^{(2)} \right\rangle_W = r^\mathrm{T} K^{(2)}(w, w) r \qquad (33)$$

which, when supremized over all possible $r \in \mathbb{R}^{2L}$ with norm 1, yields

$$\sup\{r^\mathrm{T} K^{(2)}(w, w) r \mid r \in \mathbb{R}^{2L}, r^\mathrm{T} r \leq 1\} = \lambda_{\max}[K^{(2)}(w, w)]. \qquad (34)$$

From this and (29), which we proved in Lemma 9, it follows that (22) holds. It is clear, e.g., from (33), that for all $w \in \mathbb{H}_+$ the matrix $K^{(2)}(w, w)$ in $\mathbb{R}^{2L \times 2L}$ is positive-semidefinite. This, together with (28), implies that

$$|K_l(w, \bar{w})| \leq K_l(w, w) \quad \text{for all} \quad l = 1, \ldots, L. \qquad (35)$$

In particular we verified that (23) indeed holds. $\qquad\square$

	continuous-time	discrete-time
1)	$M(s) \in \mathbb{C}^L$ for $s \in \mathbb{H}_+$ $d_1(s), \ldots, d_L(s)$ Hurwitz-stable	$M(z) \in \mathbb{C}^L$ for $z \in \mathbb{D}^{\mathrm{c}}$ $d_1(z), \ldots, d_L(z)$ Schur-stable
2)	$\int_{-\infty}^{\infty} \sum_{l=1}^{L} |M_l(\jmath\omega)|^2 W_l(\omega)\,\mathrm{d}\omega$	$\int_{-\pi}^{\pi} \sum_{l=1}^{L} |M_l(\mathrm{e}^{\jmath\omega})|^2 W_l(\omega)\,\mathrm{d}\omega$
3.1)	$\dfrac{1 - Q_l(s)\overline{Q_l(w)}}{s + \bar{w}}$	$-\dfrac{1 - Q_l(z)\overline{Q_l(w)}}{1 - \bar{w}z}$
3.2)	$Q_l(s) = \overline{d_l(-\bar{s})}/d_l(s)$	$Q_l(z) = z^{n_l}\overline{d_l(\bar{z}^{-1})}/d_l(z)$

Table 1: Correspondence between continuous-time and discrete-time constructions. 1) Defines a stable transfer function, 2) the weighted L^2-norm $\|\cdot\|_W$, 3.1) the Christophel-Darboux kernel K_l, with symbol Q_l given by 3.2).

4. Bounds for discrete-time transfer functions

In Section 2 and Section 3 the bounds for Hurwitz-stable continuous-time systems are given by Theorem 2 in the complex-rational case, and by Theorem 8 in the real-rational case.

These bounds apply mutatis mutandis in the discrete time case. The right half-plane $\mathbb{H}_+$ has to be replaced by $\mathbb{D}^{\mathrm{c}} = \{z \in \mathbb{C} \,|\, |z| \geq 1\}$ which denotes the complement of the open unit disc. In other words Hurwitz-stability gets replaced by Schur-stability. All other modifications are listed in Table 1.

5. Numerical Example

We consider the class of Hurwitz-stable transfer functions of the form

$$M(s) = \left[\frac{b_{1,0}}{s+1}, \frac{b_{2,0} + b_{2,1}s}{25/4 + 3s + s^2}\right]^{\mathrm{T}}. \tag{36}$$

In Fig. 1 we illustrate the behavior of the bound in four cases which correspond to real or complex-valued numerator polynomials and H^2-norm ($W_{a,l} := W_a \equiv 1$ for $l = 1, 2$) or weighted L^2-norm with weight $W_{b,l}(\omega) = W_b(\omega) = 4 \cdot (1 + 4\omega^2)^{-1}$ for $l = 1, 2$, i.e., we assign the same weight to both output channels.

Compared to the uniform weight W_a, the weight W_b emphasizes the behavior of the transfer function for low frequencies. This is why, in the low frequency region, the bound C_b corresponding to W_b is smaller than the bound C_a corresponding to W_a. Similarly, compared to W_b, the weight W_a emphasizes high frequencies. Therefore, in the high frequency region the bound C_a corresponding to W_a is smaller than the bound C_b corresponding to W_b. Finally we remark that inequality (23) is indeed satisfied for $C = C_a$ and $C = C_b$.
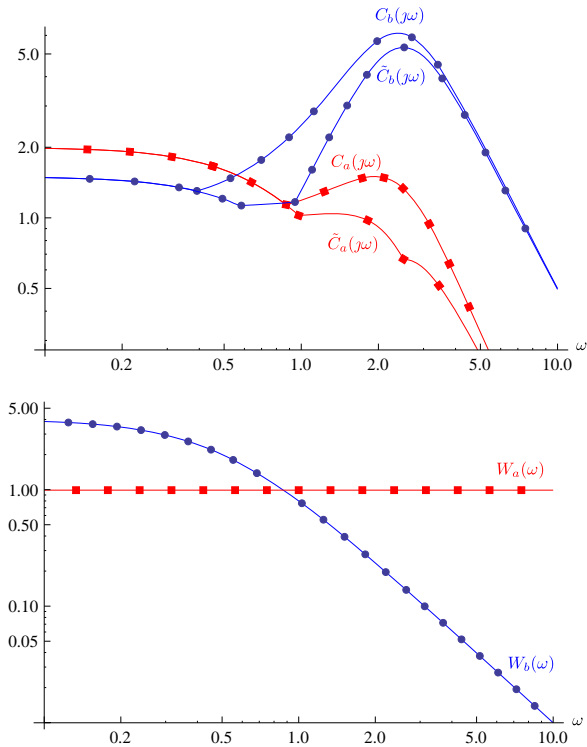
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Figure 1: *Bottom:* weight-functions $W_a \equiv 1$ and $W_b(\omega) = 4 \cdot (1 + 4\omega^2)^{-1}$. *Top:* C_a and C_b denote the bounds for complex-rational case, given in (7), with weight W_a and W_b respectively. Similarly $\tilde{C}_a$ and $\tilde{C}_b$ denote bounds for the real-rational case, given in (22), with weight W_a and W_b respectively. The fact that C_a and $\tilde{C}_a$ coincide for $\omega < 0.9$ and similarly that C_b and $\tilde{C}_b$ coincide for $\omega < 0.3$ is a particular feature of this example and has no general implications.

6. Conclusions

We have addressed the problem of bounding the value of a vector-valued transfer function at a given interpolation point, e.g., a frequency point, via its weighted L^2-norm under the assumption that the poles of the transfer function are known. We have done this for continuous-time and discrete-time single input multiple output systems in the complex-rational and real-rational case. We have derived a constructive method to compute the bound for the real-rational case using the result of the complex-rational case. Using this we have been able to show that the bound for the complex-rational case can be up to twice as conservative as the bound in the real-rational case. We have established the connection to reproducing kernel Hilbert space theory which has proven to be of great interest due to the availability of Christoffel-Darboux type formulas which clarify the dependence of the bound on the pole structure.

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