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Unstable Ones in Understood Algebraic Questions of Modelling for Control Design

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ABSTRACT

The use of a necessarily approximate system model for the design of a feedback controller for an actual plant introduces strictures on the quality of the model which are different from those pertaining in open loop. This has been the focus of a raft of recent papers on closed loop modelling, model validation, and iterative control design where appropriate measures of approximation quality have been proposed, analysed and implemented. Our attention here will concentrate on underlying implicit or understood questions of an algebraic nature which need to be resolved prior to any quantitative analysis of approximation. For example, it should be evident that a model $\hat{P}$ is useful as a design input for a controller for the actual plant P only if the pair $(\hat{P}, P)$ is simultaneously stabilizable. Yet this question has not really been asked. For given P and T we characterize the set of admissible models $\{\hat{P}\}$, where admissible means that a controller designed from $\hat{P}$ and T yields a stable closed loop. Necessary conditions on (P, T) are derived for this set to be nonempty; they connect the unstable poles and zeros of P on the positive real axis with the unstable zeros and ones of T on that axis.

Keywords: modelling for control, model validation, simultaneous stabilization.

1 INTRODUCTION

It has been the conventional wisdom, in the small but growing circle of platonic friends who spend their nights dreaming about modelling for control, that a model need not necessarily be a very accurate description of the true system for it to deliver a high performance controller. The important feature is that the model should describe with high precision the dynamical characteristics that are essential for control design. Examples have flourished where excellent control performance was achieved on the

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actual plant even though the controller was based on a model that had large open-loop errors in some frequency bands.

This observation naturally leads one to ask the question *What is a 'good' model for control design?* It is commonly acknowledged that the model should be accurate around the cross-over frequency of the closed loop system to be designed. To go beyond such common sense rules and to obtain a precise characterization of all models that are 'good for control design' turns out to be a difficult and deep question.

One reasonable definition of a 'good' model is if:

(i) the controller derived from that model stabilizes the actual plant;

(ii) this controller achieves a performance on the actual plant that is close to the performance achieved on the model, i.e. the predicted or nominal performances are close.

The qualification of a 'good' model depends on two ingredients: the unknown plant and the control design criterion. Thus, the question of qualifying 'good' models for control design is a question that involves three players:

1. the unknown plant;
2. the control design criterion;
3. the set of models available for consideration.

Understanding the connections between these three players is, we believe, fundamental in addressing the question "When is a model good for control design?" In addition, the solution to the question "What are the good models $\{\hat{P}\}$ for a given P and a given criterion?" forces one to examine the compatibility of the design criterion with the known features of the plant P. As we shall reveal, there are combinations of plants and control criteria for which the set of models that satisfy the 'goodness' criterion (i) above is empty.

To make the problem of assessing goodness of models revealing and tractable we will adopt the following framework.

- The system is noise-free;
- A model reference control design is used;
- The robust stability problem is considered only, i.e. performance is not treated.

Thus, we consider the situation where there is an unknown 'true' system with transfer function $P(s)$, a stable reference model $T(s)$ and a model reference control design criterion which, for the model $\hat{P}(s)$, computes a corresponding controller $C(\hat{P}, T)(s)$ from

$$\frac{\hat{P}(s)C(s)}{1+\hat{P}(s)C(s)} = T(s). \tag{1}$$

Note that the relationship (1) uniquely defines a controller for any couple $(\hat{P}, T)$, since $C = \frac{T}{(1-T)\hat{P}}$. However, for such a controller to be proper, the relative degree of T must be larger than or equal to that of $\hat{P}$. We shall come back to these relative degree constraints later in the paper.

We then ask the question: "What is the set of models $\mathcal{P} \triangleq \{\hat{P}(s)\}$ for which the corresponding controllers $C(\hat{P}, T)(s)$ stabilizes the true system P ?" We shall call such models 'stabilizing models'.

The answer to the above question is our central contribution. It is a first - and admittedly modest - contribution to the question posed earlier of characterizing all models that are 'good' for control design. However, our analysis will reveal much more. We will show that the existence of 'stabilizing' models imposes necessary conditions on the connection between the pole-zero pattern of P and the zero-one pattern of T on the extended positive real axis $R_{+\infty}$. Before we proceed further, we introduce some notations.

Notations

$T(s)$ is the designed complementary sensitivity function, or reference model, defined by $T = \frac{\hat{P}C}{1+\hat{P}C}$.[1]

$S(s)$ is the associated designed sensitivity function $S = 1 - T = \frac{1}{1+\hat{P}C}$.

$\mathcal{RH}_\infty$ is the set of stable, proper transfer functions.

$R(s)$ is the set of all real transfer functions in s.

$R_{+\infty}$ is the extended positive real axis in the s-plane.

Stable, stabilizing loops or controllers or models refer to the internal stability of the loop, i.e. the stability of all four transfer functions in the map

$$H(P, C) = \frac{1}{1+PC}\begin{pmatrix} PC & P \\ C & 1 \end{pmatrix}.$$

Unstable zero of a transfer function $G(s)$ is a value α with $Re(\alpha) \geq 0$ where $G(\alpha) = 0$.

Unstable one of transfer function $G(s)$ is a value α with $Re(\alpha) \geq 0$ where $G(\alpha) = 1$.

$\delta(G)$ represents the relative degree of transfer function $G(s)$, i.e. the denominator degree minus the numerator degree.

[1]For ease of notation we will from now on drop the argument s from all transfer functions except when we won't.

Proper transfer function $G(s)$ is one where $\delta(G) \geq 0$.

Biproper transfer function $G(s)$ is one where both G and G^{-1} are proper, i.e. $\delta(G) = 0$.

Bistable transfer function $G(s)$ where both G and G^{-1} are stable.

With some simplification, our results can be summarized as follows.

- We shall show that the problem is equivalent to the problem of stabilizing the plant $\frac{1}{PT}$ by a controller that has no unstable poles and zeros except at finitely many possible specific right half plane locations. The difficulty of this problem is essentially equivalent to that of stabilizing a plant by a *bistable controller*, for which no tractable necessary and sufficient conditions are known: see [2], [3]. However, useful necessary conditions are known under which a plant is stabilizable by a bistable controller. These will lead us to show the following.
- The solution set $\{\hat{P}\}$ of stabilizing models is non empty only if:
  - * T has an unstable $R_{+\infty}$ -zero between any pair of $R_{+\infty}$ -poles of P between which P has an uneven number of $R_{+\infty}$ -zeros, and
  - * T has an unstable $R_{+\infty}$ -one between any pair of $R_{+\infty}$ -zeros of P between which P has an uneven number of $R_{+\infty}$ -poles.
- These necessary conditions for the existence of stabilizing models are automatically satisfied, and are also sufficient, in the following practically relevant cases:
  - * P has no unstable poles (i.e. P is stable);
  - * P has no unstable zeros;
  - * P has at most one unstable zero and one unstable pole.

Thus, our results show that 'difficult plants' P put constraints on the set of admissible nominal closed loop systems T for there to exist a nonempty set of 'good', i.e. stabilizing, models $\hat{P}$. This is not entirely surprising, of course. It has been well known for some time now that plants with right half plane poles and zeros pose specific constraints on the *achievable* closed loop performance; the understanding of these constraints has been the object of much of the research of the last 15 years [5], [6], [1]. However, we do not know of any results connecting unstable poles and zeros of P with the *designed* closed loop performance in the context of modelling for control.

The question addressed in this paper is that of characterizing the plant models $\{\hat{P}\}$ that are stabilizing for a given plant P, in that the controllers designed on the basis of a $\hat{P}$ stabilize the true P. It is related to, but significantly harder than, the question of characterizing the set of plants $\{P\}$ that are stabilized by a controller $C(\hat{P}, T)$ designed on the basis of a model $\hat{P}$ and a reference model T. This last question is easily solved using the dual Youla parametrization [10], [9], [7], [8].

Another closely related research area is that of simultaneous stabilization of several plants by a single controller, and indeed we have already foreshadowed that some techniques and results of simultaneous stabilization emerge as a result of our analysis. In hindsight, this is not surprising since the first quality of a 'good' model $\hat{P}$ is that the controller resulting from that model (which of course must stabilize $\hat{P}$) should also stabilize P. However, the appearance of simultaneous stabilization conditions in the context of modelling for control is certainly new — at least it was new to the authors.

Our scenario will unfold as follows. In Section 2 we shall state the problem of characterizing all stabilizing plant models for a given plant and a given model reference control design. In Section 3 we use the Youla parametrization of all controllers stabilizing the plant P and the dual Youla parametrization of all models $\hat{P}$ stabilized by all such controllers to reformulate our problem as a simultaneous stabilization problem for P and $\hat{P}$. In Section 4 we give a parametrization of the plant model set and the corresponding controller set as a function of a Youla parameter that must obey an algebraic relation, which is essentially equivalent to a bistable stabilization problem. This leads us, in Section 5, to produce necessary conditions on the chosen reference model T for the solution set of stabilizing models to be non-empty. These conditions are in terms of the pole-zero pattern of the plant P on the positive real axis. We then illustrate in Section 6 how our conditions can be used to design an adequate reference model T and to construct stabilizing models.

2 STATEMENT OF THE PROBLEM

For simplicity we shall consider the system to be scalar, i.e. it is a single-input/single-output system. In addition, we present our results in the continuous time domain, but they can be transposed without any added difficulty to discrete-time systems.

There are of course compatibility constraints between $\hat{P}$ and the admissible T, because the nominal closed loop system $H(\hat{P}, C)$ must be stable. This means that the product $\hat{P}C$ cannot contain any unstable pole-zero cancellations. It requires that T must be zero at the unstable zeros of $\hat{P}$, and that T must be one at the unstable poles of $\hat{P}$. If these two interpolation constraints are satisfied, then the controller $C(\hat{P}, T)$ defined by (1) is stabilizing.

The problem addressed in this paper can then be formulated very simply as follows. We shall call it 'boxed problem #1'[2], and give equivalent, more technical formulations, later.

[2] One denotes the problem as '#1' rather than as '1' because one is careful not to confuse it with a 'one' as in 'unstable one,' which one might equally write 'one.'

Boxed problem #1
Given a proper plant P and a stable, proper reference model T, characterize the set $\mathcal{P} = \{\hat{P}\}$ of all plant models for which there exists a controller $C(\hat{P}, T)$ such that the following three conditions hold:

(A) $C(\hat{P}, T)$ satisfies $\frac{\hat{P}C}{1+\hat{P}C} = T$;

(B) $C(\hat{P}, T)$ stabilizes $\hat{P}$;

(C) $C(\hat{P}, T)$ stabilizes P.

We shall call $\mathcal{P}$ the set of all *stabilizing models*. Note that we do not, at this stage, impose that the plant models $\hat{P}$ or the controllers C are proper. This allows us at first to use the unconstrained formalism of the Youla parametrization. At the final stage, we shall constrain our parametrization to produce proper models and controllers, as should be the case in any realistic situation.

Comments

1. The boxed problem statement above is probably the simplest possible formulation of the problem of characterizing all 'good' models for control design described in the introduction, where the control design criterion has been chosen as simply as possible (model reference control) and where the qualification 'good' has been restricted here to the robust stabilization property. The obvious extension, which is the object of present research, is to also include some notion of distance between the achieved and designed closed loop systems in the problem statement.

2. Condition (A) above determines the diagonal elements of $H(\hat{P}, C)$, while (B) implies that the off-diagonal elements must also be stable.

3. The problem statement involves three players, P, $\{\hat{P}\}$ and T, with $\{C(\hat{P}, T)\}$ being just a function of the latter two. Alternative and closely related problem formulations could have been enunciated involving combinations of P, $\hat{P}$, C and the control criterion T. Our formulation is closest to that of model validation for control, which is the central theme of this special issue.

4. Our boxed problem will reveal the constraints that P imposes on the choice of reference model T for the set $\mathcal{P}$ to be non empty.

5. Conditions (B) and (C) above clearly indicate that our problem can be viewed in the framework of simultaneous stabilization of two plants, a well-known problem for which tractable necessary and sufficient conditions exist [10], [4]. However, condition (A) complicates things considerably, as we shall discover.

6. The question of the simultaneous stabilizability of P and $\hat{P}$ is the first question that should be asked in evaluating the quality of a model $\hat{P}$ that is used for the design of a controller for P. Strangely, however, the assumption of simultaneous stabilizability seems to be understood; at least, we are unaware of it being raised in the model validation for control context.

3 THE PATH TO A SOLUTION

Since $\hat{P}$ and P must be stabilized by the same controller, the last observation suggests to take as a starting point the Youla-Kucera parametrization of the set $\mathcal{C}$ of all controllers that stabilize P, and then to consider all models $\hat{P}$ that are stabilized by any member of this set $\mathcal{C}$.

Let $N, D \in \mathcal{R}H_\infty$ be a coprime factorisation of P in $\mathcal{R}H_\infty$, i.e. $P = ND^{-1}$ with N and D proper, stable and having no common unstable zeros in the extended closed right half plane. Let C be any stabilizing controller of P; then C admits a factorization $C = XY^{-1}$ with $X, Y \in \mathcal{R}H_\infty$, coprime, such that [9]

$$NX + DY = 1. \tag{2}$$

The set of all controllers C that stabilize P is given by the Youla-Kucera parametrization (see e.g. [10], [9])

$$\mathcal{C} = \{C(V) \triangleq \frac{X - DV}{Y + NV}, \qquad \forall V \in \mathcal{R}H_\infty\}. \tag{3}$$

Now consider a particular controller $C(V)$ in the set $\mathcal{C}$. The set of all plant models $\hat{P}$ that are stabilized by this controller is parametrized by the dual Youla-Kucera parametrization

$$\mathcal{P}_1(V) = \{\hat{P}(Q, V) \triangleq \frac{N - Q(Y + NV)}{D + Q(X - DV)}, \qquad \forall Q \in \mathcal{R}H_\infty\}. \tag{4}$$

Fixing V fixes a particular controller; then fixing Q fixes a particular model $\hat{P}$. Therefore the set of all plants $\hat{P}$ that are stabilized by some stabilizing controller of P is the union of all $\mathcal{P}_1(V)$ for all $V \in \mathcal{R}H_\infty$:

$$\hat{\mathcal{P}} = \{\hat{P}(Q, V) \triangleq \frac{N - Q(Y + NV)}{D + Q(X - DV)} = \frac{\hat{N}(Q, V)}{\hat{D}(Q, V)}, \qquad \forall Q, V \in \mathcal{R}H_\infty\}. \tag{5}$$

Observe that (5) is a parametrization of the set $\hat{\mathcal{P}}$ of all models that are simultaneously stabilizable with P.

Since any member in $\hat{\mathcal{P}}$ satisfies conditions (B) and (C), we now examine condition (A). First notice that

$$\frac{\hat{P}(Q,V)C(V)}{1+\hat{P}(Q,V)C(V)} = [N - Q(Y+NV)](X-DV). \tag{6}$$

The boxed problem #1 can thus be reformulated as an equivalent boxed problem.

Boxed problem #2

Given $N, D \in \mathcal{R}H_\infty$, coprime, with $P \triangleq ND^{-1}$ proper, given $X, Y \in \mathcal{R}H_\infty$, coprime, satisfying $NX + DY = 1$, and given $T \in \mathcal{R}H_\infty$, does there exist $(Q, V) \in \mathcal{R}H_\infty \times \mathcal{R}H_\infty$ such that

$$[N - Q(Y+NV)](X-DV) = T. \tag{7}$$

If so, can we parametrize all solutions ?

The following lemma gives a first constrained parametrization of the solution set of boxed problem # 2.

Lemma 1 *The solution set of boxed problem # 2 is described by*

$$V = \frac{XK - YT}{NT + DK} \tag{8}$$

$$Q = \frac{NS - DK}{Y + NV}, \tag{9}$$

where $S \triangleq 1 - T$ *and* K *is any rational transfer function such that*
(a) $NT + DK \in \mathcal{R}H_\infty$
(b) the unstable zeros of $NT + DK$ *are a subset of the unstable zeros of* T
(c) $V \in \mathcal{R}H_\infty$ *and* $Q \in \mathcal{R}H_\infty$.

Proof:
The boxed problem above has a solution if and only if the following two conditions hold:
(i) there exist $\hat{N}, V \in \mathcal{R}H_\infty \times \mathcal{R}H_\infty$ such that

$$\hat{N}(X - DV) = T \tag{10}$$

(ii) the solution

$$Q \triangleq \frac{N - \hat{N}}{Y + NV} \in \mathcal{R}H_\infty. \tag{11}$$

Rewrite (10) as

$$X\hat{N} - DV\hat{N} = T. \tag{12}$$

From (2) we have the identity $X(NT) + D(YT) = T$. In addition, since $X(DK) + D(-XK) = 0$ for all $K \in R(s)$, the set of real rational transfer functions, it follows that the solution set of (12) over $R(s)$ (i.e. without the stability constraint on $\hat{N}$ and V) is described by

$$X(NT + DK) + D(YT - XK) = T \qquad \forall K \in R(s). \tag{13}$$

Comparing (12) and (13), any solution $(\hat{N}, V)$ of (10) must take the form

$$\hat{N} = NT + DK \tag{14}$$

$$V = \frac{XK - YT}{NT + DK} \tag{15}$$

for some $K \in R(s)$. Substituting (14) in (11) yields the expression (9) for Q. From $\hat{N} = N - Q(Y + NV)$ it follows that $\hat{N}$ must be stable. From (10) and the stability of $X - DV$, $\hat{N}$ cannot have any unstable zeros that are not zeros of T. ■

The constraints on K in Lemma 1 are very indirect. In our main result of this section, we give a new parametrization of the solution set of Boxed Problem #2. To simplify things somewhat, we shall from now on make a standing assumption which is generically satisfied.

Genericity Assumption: (D, T) and (N, S) are coprime in $\mathcal{R}H_\infty$, i.e. the unstable poles of P are not zeros of the designed T, and the unstable zeros of P are not zeros of the designed $S = 1 - T$.

Theorem 1 *The solution set $\{Q, V\} \in \mathcal{R}H_\infty \times \mathcal{R}H_\infty$ of Boxed Problem #2 is described by (8)-(9), where K is any stable real rational transfer function ($K \in \mathcal{R}H_\infty$) such that*

- *the unstable zeros of $NT + DK$ are a subset of the unstable zeros of T;* (16)
- *the unstable zeros of K are a subset of the unstable zeros and the unstable ones of T.* (17)

Any solution takes the form $K = K_1K_2$ such that
(I) $K_1 \in \mathcal{R}H_\infty$ is biproper and the zeros of K_1 are a subset of the unstable zeros of T;
(II) $K_2 \in \mathcal{R}H_\infty$, and the unstable zeros of K_2 are a subset of the unstable zeros of S;
(III) $NT_2 + DK_2$ has no unstable zeros, where $T_2 \triangleq \frac{T}{K_1}$.

Proof :

1. First we show that K must be stable using the conditions of Lemma 1. Suppose K had an unstable pole at $\alpha \in Re(s) \geq 0$. Then it follows from condition (a) that α must be an unstable zero of D. It also follows from (8) and (c) that α must either be a zero of X or a zero of $NT + DK$. It cannot be a zero of X because D and X are coprime by (2). If it were a zero of $NT + DK$, it would be a zero of T by (b). But this is again impossible because D and T would both have a zero at α, which is ruled out by our Genericity Assumption. Conversely, if K is stable, then (a) is satisfied.

2. Next we show, using (c), that any unstable zero of K must be either a zero of T or a zero of $S \triangleq 1 - T$. Suppose now that $K(\alpha) = 0$ for some $\alpha \in Re(s) \geq 0$. Then by (b) either $T(\alpha) = 0$ (in which case the result is proved) or $N(\alpha)T(\alpha) \neq 0$. Substituting (8) in (9) yields

$$Q = \frac{(NS - DK)(NT + DK)}{K}. \tag{18}$$

Since $N(\alpha)T(\alpha) \neq 0$, Q would be unstable unless $N(\alpha)S(\alpha) = 0$, which implies $S(\alpha) = 0$. Conversely, if any unstable zero of K is a zero of either T or S, then (18) shows that Q is stable, and (8) shows that V is stable because, by (16), any unstable zero of $NT + DK$ must be a zero of T and hence of K. This proves the first part of the result.

3. It then follows from steps 1 and 2 above that the set of all admissible solutions K are generated by all families $K = K_1K_2$, where K_1 contains any subset of the unstable zeros of T (and can be taken biproper without loss of generality), and where K_2 is any $K_2 \in \mathcal{R}H_\infty$ that satisfies conditions (II) and (III). ∎

Comment

The set of solutions $\{K\}$ of our boxed problem #2 is obtained as a finite family of sets, each one determined by the particular subset of unstable zeros of the designed transfer function T (if there are any) that is included in the 'free parameter' K. These finite number of possible subsets determine a finite number of possible factors K_1, whose denominators are immaterial, except that they must be stable. For each selection of K_1, the solution is then obtained as the whole family of solutions $K_2 \in \mathcal{R}H_\infty$ of the following equations:

$$(i) K_2 \in \mathcal{R}H_\infty, \text{ and the unstable zeros of } K_2 \text{ are a subset of the unstable zeros of } S; \tag{19}$$

$$(ii) N(\frac{T}{K_1}) + DK_2 \text{ has no unstable zeros.} \tag{20}$$

The solution set may be empty, and we shall give an example in Section 6. We shall return in Section 5 to a discussion of the solutions of this problem, but before we do this we present the parametrization of all solutions of our initial boxed problem, because this sheds interesting light on the solution structure and on the reasons for the constraints imposed on the parametrization by K in our previous result.

4 A PARAMETRIZATION OF ALL STABILIZING MODELS

We now return to the initial problem formulation, and present a parametrization of all models $\hat{P}$ that solve the initial boxed problem.

Theorem 2 *Let P be any proper real rational transfer function, and let T be any proper stable real rational reference model, such that the Genericity Assumption holds. Then the set of models $\mathcal{P} = \{\hat{P}\}$ and corresponding controllers $\{C(\hat{P}, T) : \hat{P} \in \mathcal{P}\}$ that solves Boxed Problem #1 is described by*

$$\hat{P} = \frac{K}{1-T} = \frac{K}{S} \tag{21}$$

$$C(\hat{P}, T) = \frac{T}{K} \tag{22}$$

where K is any stable proper real rational transfer function of the form $K = K_1 K_2$, with K_1 and K_2 satisfying conditions (I) to (III) of Theorem 1.

Proof:

Let K be any solution $K = K_1 K_2$ satisfying conditions (I) to (III) of Theorem 1. Then it follows from (5) that $\hat{P} = \frac{\hat{N}}{\hat{D}}$. Using (8), (9) and (18), we have

$$\hat{N} = NT + DK \tag{23}$$

$$\begin{aligned} \hat{D} &= D + Q\left(X - \frac{D(XK - YT)}{NT + DK}\right) \\ &= D + \frac{QT}{NT + DK} = D + \frac{(NS - DK)T}{K} \\ &= \frac{(DK + NT)S}{K}. \end{aligned} \tag{24}$$

The result (21) follows immediately. The corresponding controller set is obtained similarly from (3). $C(\hat{P}, T) = \frac{\hat{X}}{\hat{Y}}$ where

$$\begin{aligned} \hat{X} &= X - \frac{D(XK - YT)}{NT + DK} \\ &= \frac{T}{\hat{N}} \end{aligned} \tag{25}$$

$$\begin{aligned}\hat{Y} &= Y + \frac{N(XK - YT)}{NT + DK} \\ &= \frac{K}{\hat{N}},\end{aligned} \tag{26}$$

from which (22) follows immediately. ■

We make a few trivial observations, which show that our results are not easily invalidated by 'idiot tests'.

Comments

1. First observe that
$$\hat{P}C = \frac{T}{S} = \frac{T}{1-T}, \tag{27}$$
as it should be, and that no unstable pole-zero cancellations can occur in forming the product $\hat{P}C$. Indeed, if $K(\alpha) = 0$ at some $\alpha \in \mathcal{R}e(s) \geq 0$, then by Theorem 1 either $T(\alpha) = 0$ (and hence $S(\alpha) = 1$), implying that the plant model has an unstable zero at α that is not cancelled by a pole of the controller, or $S(\alpha) = 0$ (and hence $T(\alpha) = 1$), implying that the controller has an unstable pole at α that is not cancelled by a zero of the plant model. Conversely, any unstable zero of T that is not a zero of K yields an unstable zero of the controller that is not cancelled by a pole of the plant model, since S and T cannot be zero at the same point.

2. Notice that the solution set $\mathcal{P}$ (if it is not empty) depends only, and in a very simple way, on T and K, and hence on T and P, as it should.

Properness constraints

We now examine the constraints on the solution imposed by the properness requirements on the plant models $\hat{P}$ and the controllers C.

It follows from (21) and (22) that the properness of $\hat{P}$ and C impose the following relative degree conditions on K:

$$\delta(1-T) \leq \delta(K) \leq \delta(T). \tag{28}$$

A necessary condition for this is that $\delta(1-T) = 0$, i.e. the reference model T cannot have a 'one' at infinity. Assuming that this constraint on T is satisfied, then the solution set $\{K\}$ of (16)-(17) must be restricted to those solutions for which

$$\delta(K) \leq \delta(T). \tag{29}$$

We thus have the following result.

Theorem 3 *Let P be any proper real rational transfer function, and let T be any proper stable real rational reference model, such that the Genericity Assumption holds. Then Boxed Problem #1 has a solution only if $T(\infty) \neq 1$. The set of proper models $\mathcal{P} = \{\hat{P}\}$ and corresponding proper controllers $\{C(\hat{P}, T) : \hat{P} \in \mathcal{P}\}$ that solves Boxed Problem #1 is described by (21) and (22) where K is any stable proper real rational transfer function of the form $K = K_1K_2$, with $\delta(K) \leq \delta(T)$, where K_1 and K_2 satisfy conditions (I) to (III) of Theorem 1.*

5 NECESSARY CONDITIONS ON THE REFERENCE MODEL

The conditions (I)-(III) of Theorem 1 are necessary and sufficient. However, condition (III) remains opaque, since it requires the search for solutions $K_2 \in \mathcal{R}H_\infty$ for which $NT_2 + DK_2$ has no unstable zeros with N, D and T_2 given. Regarding (D, NT_2) as a coprime factorization, we see that this translates into a stabilization problem of $\frac{1}{PT_2}$ by a stable controller, K_2, whose unstable zeros (if any) may only be unstable ones of T. This is now essentially a 'bistable stabilization problem' for $\frac{1}{PT_2}$ for which there exist necessary conditions for the existence of a solution K_2, which we now develop in this context for some special cases.

The reference model T has no unstable zeros

In this case T_2 is T. The conditions of Theorem 1 become: find all $K \in \mathcal{R}H_\infty \cap$ {unstable zeros of K are unstable ones of T} such that $NT + DK$ has no unstable zeros.

This is equivalent to saying that the system $\frac{D}{NT} = \frac{1}{PT}$ must be stabilizable by a *stable controller* K whose zeros can only be at specific and isolated locations (i.e. the unstable ones of T). The problem of stabilizing a plant by a stable controller without any restriction on its zeros is known as the *strong stabilization* problem, and a necessary and sufficient condition for strong stabilization of the system $\frac{1}{PT}$ is that it obeys the *parity interlacing property*.[3] Since T is assumed stable and inverse stable this is equivalent to P^{-1} satisfying the PIP. This means that P must have an even number of zeros on $R_{+\infty}$ between any two poles on $R_{+\infty}$. Since there are constraints on the unstable zeros of K, the parity interlacing property of $\frac{1}{P}$ is necessary but not sufficient for the existence of a solution K.

The reference model T has no unstable zeros and no unstable ones

In this case we must find all stable and inverse stable K such that $NT + DK$ has no unstable zeros. This means that $\frac{1}{PT}$ must be stabilizable by a stable and inverse stable controller K. With the added constraint of biproperness of K, this is known on the shopfloor as the *bistable stabilization* problem, which can be shown (see e.g.

[3] Well known in control engineering circles as the PIP.

[3]) to be reducible to the simultaneous stabilization of three different plants by the same controller. It has been shown in [2] that necessary and sufficient conditions for the simultaneous stabilization of three plants cannot be obtained by performing rational operations, sign tests and logical combinations of these on the parameters of the plants. However, a necessary condition for a plant P to be stabilized by a bistable controller is that P has the *even interlacing property*, i.e. P must have an even number of $R_{+\infty}$ poles between any pair of $R_{+\infty}$ zeros of P, and an even number of $R_{+\infty}$ zeros between any pair of $R_{+\infty}$ poles. It is known that in general this condition is not sufficient. Again, the condition is only on the $R_{+\infty}$ pole-zero pattern of P because T is assumed to have only stable poles and zeros.

Analysis of the general case

We now examine the situation where the designed closed loop transfer function T can have both unstable zeros and unstable ones.

By Theorem 1, K must be such that the unstable zeros of $NT + DK$, if any, are a subset of the unstable zeros of T. Suppose a solution K exists and denote

$$\hat{N} \triangleq NT + DK \tag{30}$$

$$R \triangleq \frac{NT}{\hat{N}} = 1 - \frac{DK}{\hat{N}}. \tag{31}$$

Consider first a point $\alpha \in Re(s) \geq 0$ at which $\hat{N}(\alpha) = 0$. Then $T(\alpha) = 0$, and thus $K(\alpha) = 0$ by the Genericity Assumption. Therefore R is a stable rational function, whose unstable zeros are exactly the unstable zeros of N plus those unstable zeros of T that are not zeros of $\hat{N}$, and whose unstable ones are exactly the unstable zeros of D plus those unstable zeros of K that are not zeros of T. The latter can only be unstable ones of T, by (17).

Therefore, a solution K exists if we can construct a stable rational transfer function $R(s)$ such that

- $R(s) = 0$ at the unstable zeros of N and any subset of the unstable zeros of T, and nowhere else in the right half plane; (32)
- $R(s) = 1$ at the unstable zeros of D and any subset of the unstable ones of T, and nowhere else in the right half plane. (33)

We first give a necessary condition for the existence of such a $R(s)$, and we then present some cases of 'well-behaved' plants P for which a solution always exists.

Theorem 4 *The following are necessary conditions for Boxed Problem #1 to have a solution.*

1. *If P has 2 or more poles on $R_{+\infty}$, then T must have a $R_{+\infty}$-zero between any pair of $R_{+\infty}$-poles of P between which P has an odd number of zeros.*

2. *If P has 2 or more zeros on $R_{+\infty}$, then T must have a $R_{+\infty}$-one between any pair of $R_{+\infty}$-zeros of P between which P has an odd number of poles.*

Proof:
Let α_1 and α_2 be two successive zeros of D on $R_{+\infty}$; then $R(\alpha_1) = R(\alpha_2) = 1$. Suppose there is an odd number of zeros of N in the interval $[\alpha_1, \alpha_2]$. Then the rational function $R(s)$ cannot have the same sign at the two end points of that interval. Thus, for a solution R to exist, T must have a zero on that interval that is not incorporated as a zero of $\hat{N}$. This zero is then not taken as a zero of K (see Theorem 1).

Conversely, let β_1 and β_2 be two successive zeros of N on $R_{+\infty}$. Then $R(\beta_1) = R(\beta_2) = 0$. Then a rational function R can only pass an even number of times through the value 1 on the interval $[\beta_1, \beta_2]$. This means that, if D has an odd number of zeros on this interval, a solution R exists only if on that interval K has a zero that is not a zero of $\hat{N}$. By Theorem 1, this requires that T has an unstable 1 on that interval. ■

We observe that, if a stable rational function R can be constructed that interpolates the unstable zeros and ones as required by (32) and (33), then a solution K exists that obeys the conditions of Theorem 1. Hence, a necessary and sufficient condition for the existence of a solution to Boxed Problem #1 is the existence of a stable rational function $R(s)$ satisfying these interpolation conditions.

Special cases of interest

The following are special cases of practical interest for which the necessary conditions are automatically satisfied.

1. P has no unstable poles;
2. P has no unstable zeros;
3. P has at most one unstable zero and one unstable pole.

It is shown in [3] that these conditions are then also sufficient for the existence of a bistable stabilizing K_2 of $\frac{1}{PT_2}$ (see condition (III) of Theorem 1), and therefore the solution set $\mathcal{P}$ is then nonempty.

6 EXAMPLES

We present a series of examples that illustrate the main results of our paper. We start with a 'difficult' plant, namely one that has more than one unstable pole and more than one unstable zero.

Example 1
Consider the 'true plant'

$$P = \frac{s-1}{(s-2)(s+1)},$$

and a model reference control design based on a nominal model with reference model

$$T = \frac{s^2+6}{(s+2)(s+3)}.$$

Observe that the plant has two $R_{+\infty}$ zeros (at $s = 1$ and $s = +\infty$) with only one $R_{+\infty}$ pole in between them. Therefore we know by Theorem 4 that the set of stabilizing models $\mathcal{P}$ is empty because T does not have a one on the interval $(1, +\infty)$. Thus, for this choice of reference model, there does not exist any model $\hat{P}$ which will produce a stabilizing controller for the actual plant.

Example 2
Consider the 'true plant'

$$P = \frac{s}{(s-1)(s+1)},$$

which has two $R_{+\infty}$ zeros (at $s = 0$ and $s = \infty$) with one $R_{+\infty}$ pole in between them. In order to ensure a non-empty solution set $\mathcal{P}$, we must choose a T which contains a one somewhere on $(0, \infty)$. Thus take

$$T = \frac{3s+1}{(s+1)^2},$$

which is equal to one at $s = 1$. By Theorem 4 a solution for K might therefore exist. Let P be factored as $P = ND^{-1}$, with $N = \frac{s}{(s+1)^2}$ and $D = \frac{s-1}{s+1}$.

Following our 'analysis of the general case', we know that a solution K satisfying conditions (I) to (III) of Theorem 1 exists if we can construct a stable rational transfer function $R(s)$ such that

- $R = 0$ at the unstable zeros of N and any subset of the unstable zeros of T, and nowhere else in the right half plane. Thus, R must have a zero at $s = 0$ and at $s = +\infty$ (with possible multiplicity two) and nowhere else in the right half plane.
- $R = 1$ at the unstable zeros of D and any subset of the unstable ones of T, and nowhere else in the right half plane. Thus, R must be one at $s = 1$, possibly also at $s = 0$, and nowhere else in the right half plane.

This is an interpolation problem yielding a class of solutions. One solution is given by

$$R = \frac{4s}{(s+1)^2},$$

yielding

$$\begin{aligned}\hat{N} &= \tfrac{NT}{R} &= \tfrac{3s+1}{4(s+1)^2},\\ K &= \tfrac{\hat{N}-NT}{D} &= \tfrac{(3s+1)(s-1)}{4(s+1)^3},\\ \hat{P} &= \tfrac{K}{1-T} &= \tfrac{3s+1}{4s(s+1)},\\ C &= \tfrac{T}{K} &= \tfrac{4(s+1)}{s-1}.\end{aligned}$$

Example 3

Finally, we present an example of a 'well-behaved plant' that does not run foul of the interlacing constraints. Take

$$P = \frac{1}{s-2}, \quad T = \frac{1}{s+1}.$$

Here P has an unstable zero at $+\infty$ and an unstable pole at $s = 2$, while T has an unstable zero at $+\infty$ and an unstable one at $s = 0$.

The R construction above yields conditions that R must be a stable transfer function having a zero at $+\infty$ (with possible multiplicity two), and a one at $s = 2$ and possibly at $s = 0$, but nowhere else in the right half plane. The rational function $R = \frac{3}{s+1}$ satisfies these requirements and leads to a solution $\hat{P} = \frac{1}{3s}$ and $C = 3$.

For $\alpha > 2$ the rational function $R = \frac{\alpha}{s+(\alpha-2)}$ also satisfies the requirements. This leads to a set $\{\frac{1}{\alpha s} : \alpha > 2\}$ of stabilizing models. Note that the controller $C(\hat{P}, T)$ associated to $\hat{P} = \frac{1}{\alpha s}$ is simply given by $C = \alpha$. The set $\mathcal{P} = \{\frac{1}{\alpha s} : \alpha > 2\}$ is the set of stabilizing models leading to a proportional controller.

7 CONCLUSION

We have posed a suite of seemingly obvious (but understood) questions of model validity for control design from the stance of simultaneous stabilization. The outcome has been a sequence of algebraic constraints on the reference model without which no model, however cleverly identified, could yield a stabilizing controller for the real plant. The technicalities are more complex than, but an outgrowth of, the questions of simultaneous, strong and bistable stabilization. These are unfamiliar but fundamental requirements in the analysis of the validity of a model set for control design.

The necessary conditions relating the unstable pole-zero pattern of the plant to demands on the reference model are clear and simple. They focus attention on the interplay between the plant to be controlled and the control objective in analysing the validity of plant models.

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