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Relating H_2 and H_∞ bounds for finite-dimensional systems†

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Abstract

For a linear time invariant system, the infinity-norm of the transfer function can be used as a measure of the gain of the system. This notion of system gain is ideally suited to the frequency domain design techniques such as H_∞ optimal control. Another measure of the gain of a system is the H_2 norm, which is often associated with the LQG optimal control problem. The only known connection between these two norms is that, for discrete time transfer functions, the H_2 norm is bounded by the H_∞ norm. It is shown in this paper that, given precise or certain partial knowledge of the poles of the transfer function, it is possible to obtain an upper bound of the H_∞ norm as a function of the H_2 norm, both in the continuous and discrete time cases. It is also shown that, in continuous time, the H_2 norm can be bounded by a function of the H_∞ norm and the bandwidth of the system.

Keywords: H_2 norm; H_∞ norm; Norm bounds

1. Introduction

In order to derive certain relations between the H_2 and H_∞ norms of stable transfer functions, we make the assumption that we know the poles of the transfer functions. (The assumption will later be relaxed somewhat.) We consider single input single output (SISO) systems with rational transfer functions (without much loss of generality) and we distinguish between the continuous and discrete time case. Bilinear transformations *cannot* be used to relate H_2 norms of continuous and discrete transfer functions, and the results are not completely parallel.

In the *continuous time case*, we make the assumption that the considered transfer functions are strictly proper, stable, and of the following form:

$$M(s) = \sum_{i=1}^{N} \frac{b_i}{s + a_i}, \tag{1.1}$$

where $\operatorname{Re} a_i > 0$. In addition, we assume (for convenience) that these poles are all distinct.

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A first measure of the "gain" of such a system is given by the *infinity-norm*

$$\|M\|_\infty = \sup_\omega |M(\mathrm{j}\omega)|.$$

It is finite since $M(s)$ is proper and has no poles on the imaginary axis.

Another measure of the gain of a system is given by the *2-norm* of its transfer function, which is the expected root-mean square (RMS) value of the output when the input is a realization of unit variance white noise. We can define the 2-norm of a system as

$$\|M\|_2^2 = \int_0^\infty |M(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(\mathrm{j}\omega)|^2\,\mathrm{d}\omega,$$

where the last equality holds by Parseval's theorem. The first equality shows that the 2-norm of a system can also be interpreted as the $\mathscr{L}_2$ norm of its response to a unit impulse. [2]

The 2-norm of $M(s)$ is also finite since $M(s)$ is strictly proper and has no poles on the imaginary axis. It can be calculated easily when taking into account the following relations

$$\|M\|_2^2 = \frac{1}{2\pi\mathrm{j}}\int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} M^\star(s)M(s)\,\mathrm{d}s = \frac{1}{2\pi\mathrm{j}}\oint M^\star(s)M(s)\,\mathrm{d}s,$$

where $M^\star(s) = \bar{M}(-s)$. Here $\bar{M}(s)$ denotes the complex conjugate of $M(s)$ and is obtained from $M(s)$ by replacing each coefficient by its complex conjugate. By the residue theorem, and with $M(s)$ as defined in (1.1), $\|M\|_2^2$ equals the sum of the residues of $M^\star(s)M(s)$ at its poles in the left half plane [4]:

$$\|M\|_2^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \lim_{s\to -a_i} (s+a_i)\frac{b_i}{s+a_i}\frac{b_j^\star}{-s+a_j^\star} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{b_i b_j^\star}{a_i+a_j^\star}.$$

Remark. The following example shows that in continuous time, the 2-norm of a transfer function can be much larger than its infinity-norm or vice versa. Take $M(s) = \varepsilon/(s+\varepsilon)$, $\forall\varepsilon > 0$. Then $\|M\|_\infty = 1$, while $\|M\|_2 = \sqrt{\varepsilon/2}$. Alternatively, with $M(s) = \sqrt{\varepsilon}/(s+\varepsilon)$ we have $\|M\|_\infty = 1/\sqrt{\varepsilon}$, while $\|M\|_2 = 1/\sqrt{2}$. This shows that each of these two norms can be unbounded while the other is not.

We will show, however, that complete or partial knowledge of the poles a_i and the degree N allows us to bound the infinity-norm of $M(s)$ by a function of the 2-norm and of the poles a_i.

In the *discrete time case*, we assume that the transfer functions are of the following form

$$M(z) = b_0 + \sum_{i=1}^{N}\frac{b_i}{1-a_i z}, \quad |a_i| > 1. \tag{1.2}$$

It is assumed again that the a_i are distinct. The *infinity-norm* of $M(z)$ is given by

$$\|M\|_\infty = \sup_\omega |M(\mathrm{e}^{\mathrm{j}\omega})|.$$

The remarks that were made about the infinity-norm in the continuous time remain valid. The definition of the *2-norm* of $M(z)$ is very similar to the definition in the continuous time case:

$$\|M\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} M^\star(\mathrm{e}^{\mathrm{j}\omega})M(\mathrm{e}^{\mathrm{j}\omega})\,\mathrm{d}\omega = \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1} M^\star(z)M(z)z^{-1}\,\mathrm{d}z.$$

Using the residue theorem again, and with $M(z)$ defined as in (1.2), we obtain

$$\|M\|_2^2 = |b_0|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\lim_{z\to 1/a_i}\left(z-\frac{1}{a_i}\right)\frac{-b_i/a_i}{z-1/a_i}\frac{b_j^\star z}{z-a_j^\star}z^{-1}$$

$$= |b_0|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{b_i b_j^\star}{a_i a_j^\star - 1}.$$

The infinity-norm and the 2-norm of $M(z)$ are finite, $M(z)$ being proper and having no poles on the unit circle. Notice that in the discrete time case, the 2-norm remains finite even with a direct feedthrough term in the transfer function, in contrast to the continuous time case.

Remark. In the discrete time case, one can easily show that the 2-norm of a transfer function is bounded from above by its infinity-norm:

$$\|M\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |M(e^{j\omega})|^2 \, d\omega \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|M\|_\infty^2 \, d\omega = \|M\|_\infty^2 . \tag{1.3}$$

On the other hand, the choice $M(z) = \sqrt{\varepsilon}/(1-(1+\varepsilon)z)$ yields $\|M\|_\infty = 1/\sqrt{\varepsilon}$ and $\|M\|_2 = 1/\sqrt{\varepsilon+2}$. Evidently, with $\|M\|_2$ bounded, a bound on $\|M\|_\infty$ is not guaranteed.

Results connecting different norms of finite-dimensional systems are rather scarce. Some inequalities connecting the l_1-norm and the H_∞ norm can be found in [3], while [3] and [5] establish inequalities between the H_∞-norm and the Hankel singular values of the system. Except for the obvious inequality (1.3) noted in [3], no results seem to be available connecting $\|M\|_2$ and $\|M\|_\infty$.

2. Relating H_2 and H_∞ norms for continuous time systems

With $M(s)$ defined as in (1.1), we establish the following theorem that is valid for all b_i.

Theorem 2.1. *Consider $M(s)$ as in (1.1). Define*

$$\beta_N = \begin{bmatrix} \frac{1}{2\,\mathrm{Re}\,a_1} & \frac{1}{a_1^\star + a_2} & \cdots & \frac{1}{a_1^\star + a_N} \\ \frac{1}{a_1 + a_2^\star} & \frac{1}{2\,\mathrm{Re}\,a_2} & \cdots & \frac{1}{a_2^\star + a_N} \\ \vdots & & \ddots & \vdots \\ \frac{1}{a_1 + a_N^\star} & \frac{1}{a_2 + a_N^\star} & \cdots & \frac{1}{2\,\mathrm{Re}\,a_N} \end{bmatrix}, \tag{2.1}$$

$$C_N(j\omega) = \begin{bmatrix} \frac{1}{-j\omega + a_1^\star} & \frac{1}{-j\omega + a_2^\star} & \cdots & \frac{1}{-j\omega + a_N^\star} \end{bmatrix}^{\mathrm{T}}. \tag{2.2}$$

Then $\forall b_i$ and $\forall \omega$,

$$|M(j\omega)|^2 \leqslant \|M\|_2^2\, C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega), \tag{2.3}$$

and also

$$\|M\|_\infty^2 \leqslant \|M\|_2^2 \|C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)\|_\infty . \tag{2.4}$$

Equality is achieved in (2.4) if the b_i are given by

$$[b_1\ b_2\ \cdots\ b_N]^{\mathrm{T}} = \frac{\beta_N^{-1} C_N(j\bar{\omega}_0)}{\sqrt{C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0)}}\|M\|_2 = \frac{\beta_N^{-1} C_N(j\bar{\omega}_0)}{C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0)}\|M\|_\infty , \tag{2.5}$$

where $\bar{\omega}_0$ maximizes $C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)$ along the $j\omega$-axis.

Comments.

- We note that the quantities β_N and $C_N(j\omega)$ depend only on the poles of the system. Therefore, given a set of transfer functions with a fixed set of poles, equality is achieved in (2.4) only for one particular transfer function in that set, i.e. for a specific set of zeros.
- Even if the a_i occur in complex conjugate pairs, the b_i defined by (2.5) do not necessarily lead to a real rational transfer function, as we shall illustrate in Section 5.

Before proving the theorem itself, we need a preliminary result.

Lemma 2.2 (Anderson [1]). *Let x, y be vectors in $\mathbb{C}^n$ and let A be a positive definite Hermitian matrix. Let k be positive real. Then*

(1) $|x^\star y|^2 \leqslant x^\star A x\, y^\star A^{-1} y$;

(2) $\max_x |x^\star y|^2$ *subject to* $x^\star A x \leqslant k$ *is solved by* $x = [A^{-1}y/\sqrt{y^\star A^{-1} y}]\sqrt{k}$ *and is* $k y^\star A^{-1} y$.

Proof of Theorem 2.1. Denote $b^\star = [b_1^\star\, b_2^\star \cdots b_N^\star]$ and recognize that $\|M\|_2^2 = b^\star \beta_N b$. It follows that β_N is positive definite Hermitian. It now follows from Lemma 2.2 that

$$|M(j\omega)|^2 = |b^\star C_N(j\omega)|^2 \leqslant b^\star \beta_N b\, C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega) = \|M\|_2^2\, C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega).$$

In particular, if $\tilde{\omega}_0$ denotes the frequency for which $|M(j\omega)|$ is maximum over ω, there holds $\|M\|_\infty^2 \leqslant \|M\|_2^2 C_N^\star(j\tilde{\omega}_0)\beta_N^{-1} C_N(j\tilde{\omega}_0) \leqslant \|M\|_2^2 \|C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)\|_\infty$. Note that the suprema of $|M(j\omega)|$ and $|C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)|$ are not necessarily achieved for the same ω. It follows from the second part of Lemma 2.2 that, for any fixed ω_0, and subject to $\|M\|_2^2 \leqslant k$, we have $\max_{b_i} |M(j\omega_0)|^2 = k C_N^\star(j\omega_0)\beta_N^{-1} C_N(j\omega_0)$ with, at the maximum, $\|M\|_2^2 = k$ and

$$b \triangleq \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} = \frac{\beta_N^{-1} C_N(j\omega_0)}{\sqrt{C_N^\star(j\omega_0)\beta_N^{-1} C_N(j\omega_0)}} \|M\|_2 = b(\omega_0).$$

Now let $\bar{\omega}_0$ be the frequency that maximizes $|C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)|$ over ω. Define

$$\bar{b} \triangleq \frac{\beta_N^{-1} C_N(j\bar{\omega}_0)}{\sqrt{C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0)}} \|M\|_2,$$

and define a new transfer function $M^+(j\omega) \triangleq \bar{b}^\star C_N(j\omega)$. Notice that $\|M^+\|_2^2 = \bar{b}^\star \beta_N \bar{b} = \|M\|_2^2$, i.e. the 2-norms of $M(s)$ and $M^+(s)$ are identical. Observe also that

$$|M^+(j\bar{\omega}_0)|^2 = |\bar{b}^\star C_N(j\bar{\omega}_0)|^2 = \|M^+\|_2^2 C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0).$$

To establish that $M^+(s)$ satisfies (2.4) with equality, it remains to be proved that $|M^+(j\bar{\omega}_0)| = \|M^+\|_\infty = \sup_\omega |M^+(j\omega)|$. Now

$$|M^+(j\omega)| = |C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\omega)| \frac{\|M^+\|_2}{\sqrt{C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0)}}.$$

By the Cauchy–Schwartz inequality,

$$\begin{aligned} |C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\omega)| &\leqslant (C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0))^{1/2}\, (C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega))^{1/2} \\ &\leqslant C_N^\star(j\bar{\omega}_0)\beta_N^{-1} C_N(j\bar{\omega}_0), \end{aligned}$$

where the last inequality follows from the fact that $\bar{\omega}_0$ maximizes $C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)$ over ω. This concludes the proof of Theorem 2.1. □

Comment. It follows from the proof of the theorem that $|M^+(j\omega)|$ and $C_N^\star(j\omega)\beta_N^{-1} C_N(j\omega)$ reach their maximum at the same ω, but this is not true for general $M(j\omega)$ as our example in Section 5 will demonstrate.

The next theorem constitutes the main result connecting H_2 and H_∞ norms for continuous time systems. It gives a simplified expression of the quantity $C_N^\star(j\omega)\beta_N^{-1}C_N(j\omega)$ in terms of the real and imaginary parts of the poles of the system.

Theorem 2.3. *Let* $a_i = \sigma_i + j\omega_i$, $i = 1, \dots, N$ *with* $\sigma_i > 0$. *Then*

$$C_N^\star(j\omega)\beta_N^{-1}C_N(j\omega) = \sum_{i=1}^{N} \frac{2\sigma_i}{(\omega+\omega_i)^2+\sigma_i^2}, \tag{2.6}$$

where β_N *and* $C_N(j\omega)$ *are defined as in* (2.1) *and* (2.2) *respectively. Moreover, with* $M(s)$ *defined as in* (1.1), *there holds, no matter what the values of* b_i *are,*

$$\|M\|_\infty^2 \leqslant \|M\|_2^2 \left\| \sum_{i=1}^{N} \frac{2\sigma_i}{(\omega+\omega_i)^2+\sigma_i^2} \right\|_\infty. \tag{2.7}$$

Equality is achieved in (2.7) *if the* b_i *are given by* (2.5).

The proof of this theorem, which can be obtained from the authors, is based on the following induction step:

$$C_n^\star(j\omega)\beta_n^{-1}C_n(j\omega) = C_{n-1}^\star(j\omega)\beta_{n-1}^{-1}C_{n-1}(j\omega) + \frac{2\sigma_n}{(\omega+\omega_n)^2+\sigma_n^2}, \tag{2.8}$$

where β_n and $C_n(j\omega)$ are defined as follows:

$$\beta_n = \begin{bmatrix} \beta_{n-1} & \mathscr{B}_{n-1} \\ \mathscr{B}_{n-1}^\star & \dfrac{1}{2\sigma_n} \end{bmatrix}, \quad \text{where } \mathscr{B}_{n-1} = \left[\frac{1}{a_1^\star + a_n} \cdots \frac{1}{a_{n-1}^\star + a_n} \right]^{\mathrm{T}},$$

$$C_n(j\omega) = \left[\frac{1}{-j\omega + a_1^\star} \ \frac{1}{-j\omega + a_2^\star} \cdots \frac{1}{-j\omega + a_n^\star} \right]^{\mathrm{T}}.$$

The following corollary follows immediately from expression (2.7).

Corollary 2.4. *Let* $a_i = \sigma_i + j\omega_i$, $i = 1, \dots, N$ *with* $\sigma_i > 0$. *Then with* $M(s)$ *defined as in* (1.1)

$$\|M\|_\infty^2 \leqslant \|M\|_2^2 \sum_{i=1}^{N} \frac{2}{\sigma_i}.$$

There exists a choice of b_i *for which equality is achieved, i.e.*

$$\|M\|_\infty^2 = \|M\|_2^2 \sum_{i=1}^{N} \frac{2}{\sigma_i}$$

if and only if $\omega_i = 0$ $\forall i$. *In addition, if* $\sigma_i > \sigma$ $\forall i$, *then*

$$\|M\|_\infty^2 \leqslant \|M\|_2^2 \frac{2N}{\sigma}.$$

Finally, if $|\omega_i| \leqslant \sigma_i$ $\forall i$, *and if* $M(s)$ *is real rational (i.e. the* a_i *occur in complex conjugate pairs), then*

$$\sup_\omega \sum_{i=1}^{N} \frac{2\sigma_i}{(\omega+\omega_i)^2+\sigma_i^2} = \sum_{i=1}^{N} \frac{2\sigma_i}{\omega_i^2+\sigma_i^2}.$$

Therefore,

$$\|M\|_\infty^2 \leqslant \|M\|_2^2 \sum_{i=1}^{N} \frac{2\sigma_i}{\omega_i^2 + \sigma_i^2}.$$

Proof. The first part is obvious from (2.7). The second part follows because $\omega_i = 0\ \forall i$ is necessary and sufficient for

$$\sup_\omega \left[\sum_{i=1}^{N} \frac{2\sigma_i}{(\omega + \omega_i)^2 + \sigma_i^2} \right] = \sum_{i=1}^{N} \frac{2}{\sigma_i}.$$

The third part is obvious. To prove the last part, consider the function

$$f(\omega) = \frac{2\sigma_i}{(\omega + \omega_i)^2 + \sigma_i^2} + \frac{2\sigma_i}{(\omega - \omega_i)^2 + \sigma_i^2},$$

where σ_i and ω_i are real and positive. It is easy to show that this function assumes a *global* maximum at $\omega = 0$ if $|\omega_i| \leqslant \sigma_i$. □

3. Relating H_2 and H_∞ norms for discrete time systems

We now present the corresponding relations between $\|M\|_2^2$ and $\|M\|_\infty^2$ in the discrete time case. $M(z)$ is defined as in (1.2). The proofs are similar to those for the continuous time problem and are omitted.

Theorem 3.1. *Consider* $M(z)$ *as in* (1.2). *Define*

$$\beta_N = \begin{bmatrix} \frac{1}{a_1^\star a_1 - 1} & \frac{1}{a_1^\star a_2 - 1} & \cdots & \frac{1}{a_1^\star a_N - 1} \\ \frac{1}{a_2^\star a_1 - 1} & \frac{1}{a_2^\star a_2 - 1} & \cdots & \frac{1}{a_2^\star a_N - 1} \\ \vdots & & \ddots & \vdots \\ \frac{1}{a_N^\star a_1 - 1} & \frac{1}{a_N^\star a_2 - 1} & \cdots & \frac{1}{a_N^\star a_N - 1} \end{bmatrix}, \tag{3.1}$$

$$C_N(\mathrm{e}^{\mathrm{j}\omega}) = \left[\frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega} a_1^\star}, \ldots, \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega} a_N^\star} \right]^\mathrm{T}. \tag{3.2}$$

Then for all b_i,

$$\|M\|_2^2 \leqslant \|M\|_\infty^2 \leqslant \|M\|_2^2 \|1 + C_N^\star(\mathrm{e}^{\mathrm{j}\omega})\beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\omega})\|_\infty. \tag{3.3}$$

Equality is achieved in the right-hand side of the previous relation for coefficients b_i *given by*

$$\begin{bmatrix} b_0 \\ \vdots \\ b_N \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ \beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\bar\omega_0}) \end{bmatrix}}{\sqrt{1 + C_N^\star(\mathrm{e}^{\mathrm{j}\bar\omega_0})\beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\bar\omega_0})}} \|M\|_2 = \frac{\begin{bmatrix} 1 \\ \beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\bar\omega_0}) \end{bmatrix}}{1 + C_N^\star(\mathrm{e}^{\mathrm{j}\bar\omega_0})\beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\bar\omega_0})} \|M\|_\infty, \tag{3.4}$$

where $\bar\omega_0$ *maximizes* $C_N^\star(\mathrm{e}^{\mathrm{j}\omega})\beta_N^{-1} C_N(\mathrm{e}^{\mathrm{j}\omega})$ *over* ω.

Theorem 3.2. *Let $a_i = \sigma_i + j\omega_i$, $i = 1, \dots, N$ with $|a_i| > 1$. Then with C_N and β_N as defined in Theorem 3.1,*

$$C_N^*(e^{j\omega})\beta_N^{-1}C_N(e^{j\omega}) = \sum_{i=1}^{N} \frac{|a_i|^2 - 1}{1 + |a_i|^2 + 2\omega_i \sin\omega - 2\sigma_i \cos\omega}. \tag{3.5}$$

In addition with $M(z)$ defined as in (1.2),

$$\| M \|_2^2 \leqslant \| M \|_\infty^2 \leqslant \| M \|_2^2 \left\| 1 + \sum_{i=1}^{N} \frac{|a_i|^2 - 1}{1 + |a_i|^2 + 2\omega_i \sin\omega - 2\sigma_i \cos\omega} \right\|_\infty. \tag{3.6}$$

The following special cases are interesting. They follow from (3.6).

Corollary 3.3. *If all a_i are either purely real ($\omega_i = 0$, $\forall i$), or purely imaginary ($\sigma_i = 0$, $\forall i$), then we have the following result:*

$$\sup_\omega \sum_{i=1}^{N} \frac{|a_i|^2 - 1}{1 + |a_i|^2 + 2\omega_i \sin\omega - 2\sigma_i \cos\omega} = \sum_{i=1}^{N} \frac{|a_i| + 1}{|a_i| - 1}.$$

Therefore, with $M(z)$ defined as in (1.2),

$$\| M \|_2^2 \leqslant \| M \|_\infty^2 \leqslant \| M \|_2^2 \left[1 + \sum_{i=1}^{N} \frac{|a_i| + 1}{|a_i| - 1} \right].$$

If, in addition, $|a_i| \geqslant c$, $\forall i$, then

$$\| M \|_2^2 \leqslant \| M \|_\infty^2 \leqslant \| M \|_2^2 \left[1 + N\frac{c + 1}{c - 1} \right].$$

4. Bounding the H_2 norm by the H_∞ norm and the bandwidth

Theorem 4.1. *Let $M(s)$ be a continuous time stable strictly proper transfer function. Define α by*

$$\alpha = \inf\left\{ \alpha_1 \,\middle|\, |M(j\omega)| \leqslant \left| \frac{\alpha_1}{j\omega + \alpha_1} \right| \| M \|_\infty \ \forall \omega > \alpha_1 \right\}.$$

Then

$$\| M \|_2 < \sqrt{\frac{\alpha(\pi + 4)}{4\pi}}\, \| M \|_\infty. \tag{4.1}$$

We observe that α is like the bandwidth of the system. For $\omega > \alpha$, $|M|$ falls off, relative to its maximum value, at 20 dB/decade (or faster).

Proof. Observe that

$$\| M \|_2^2 \leqslant \| M \|_\infty^2 \frac{1}{\pi}\left[\int_0^\alpha d\omega + \int_\alpha^\infty \frac{\alpha^2}{\omega^2 + \alpha^2} d\omega \right] = \frac{\alpha(\pi + 4)}{4\pi} \| M \|_\infty^2,$$

which concludes the proof of Theorem 4.1. □

The previous theorem shows that continuous time wide bandwidth systems can have large 2-norms for their transfer function, given unity H_∞ norm. An obvious case is the first example given in the introduction.

5. Numerical example

To illustrate the previously established inequalities, let us take the s-domain transfer function.

$$M(s) = \frac{1}{s+1} + \frac{1}{s+0.5-2\mathrm{j}} + \frac{1}{s+0.5+2\mathrm{j}} = \frac{3s^2+4s+5.25}{s^3+2s^2+5.25s+4.25}.$$

It can easily be calculated that $\|M\|_2 = 1.8915$ and that $|M(\mathrm{j}\omega)|$ achieves its maximum for $\omega = 2.0964$. The corresponding maximum value is $\|M\|_\infty = 2.3653$. Fig. 1 shows a plot of $|M(\mathrm{j}\omega)|$ and its upper bound $\|M\|_2(C_N^\star(\mathrm{j}\omega)\beta_N^{-1}C_N(\mathrm{j}\omega))^{1/2}$ as a function of ω, see (2.4).

The b_i for which equality is achieved in (2.4) can be calculated using (2.5), where $\bar{\omega}_0 = 2$ is the frequency at which $C_N^\star(\mathrm{j}\omega)\beta_N^{-1}C_N(\mathrm{j}\omega)$ achieves its maximum value 4.4615 over ω, see Fig. 2. With these values of b_i, we obtain a transfer function

$$M^+(s) = \frac{-0.77-0.89\mathrm{j}}{s+1} + \frac{2.24-0.09\mathrm{j}}{s+0.5-2\mathrm{j}} + \frac{-0.23+0.16\mathrm{j}}{s+0.5+2\mathrm{j}}$$

$$= \frac{(1.24-0.83\mathrm{j})s^2+(2.76+4.13\mathrm{j})s-1.76+1.17\mathrm{j}}{s^3+2s^2+5.25s+4.25},$$

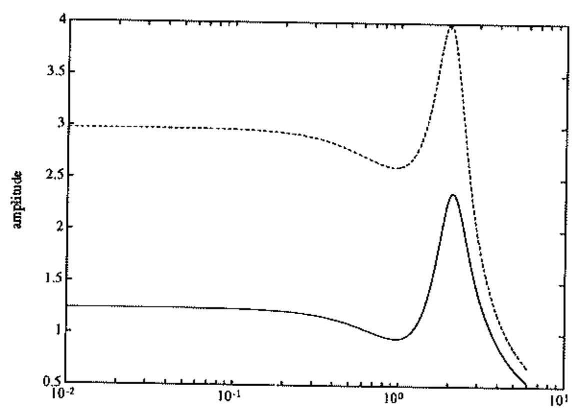

Fig. 1. Plot of $|M(\mathrm{j}\omega)|$ (——) and $\|M\|_2\ (C_N^\star(\mathrm{j}\omega)\beta_N^{-1}C_N(\mathrm{j}\omega))^{1/2}$ (⋯).

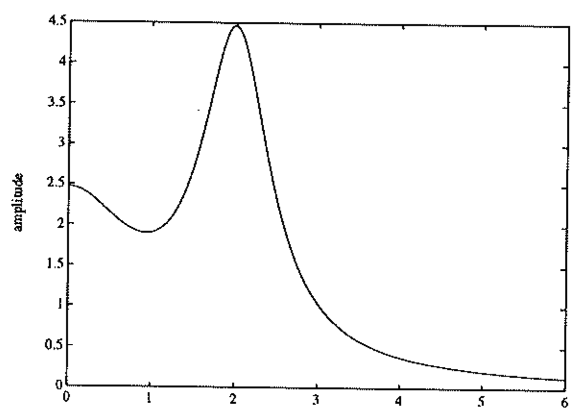

Fig. 2. Determination of $\bar{\omega}_0$: plot of $C_N^\star(\mathrm{j}\omega)\beta_N^{-1}C_N(\mathrm{j}\omega)$.

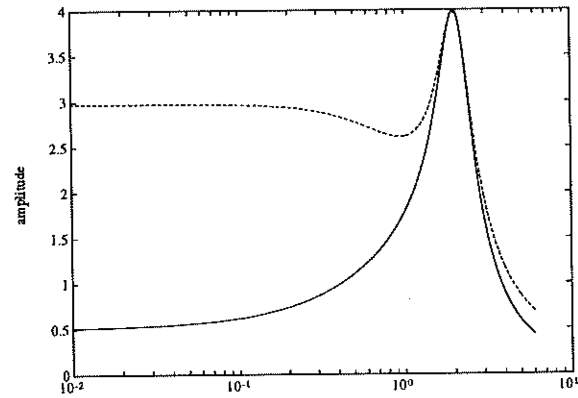

Fig. 3. Plot of $|M^+(j\omega)|$ (——) and $\|M^+\|_2\,(C_N^\star(j\omega)\beta_N^{-1}C_N(j\omega))^{1/2}$ (···).

that achieves equality in relation (2.4); note that it is not real. $\|M^+\|_\infty = 3.996$ while $\|M^+\|_2$ remains unchanged by construction: see the proof of Theorem 2.1. $|M^+(j\omega)|$ and $\|M^+\|_2(C_N^\star(j\omega)\beta_N^{-1}C_N(j\omega))^{1/2}$ are both plotted in Fig. 3. They achieve their maximum for $\bar{\omega}_0 = 2$, as should be.

6. Conclusion

In this work, we have presented bounds on the H_∞ norm of a transfer function in terms of the H_2 norm and a quantity computed using exact or partial knowledge of the poles. A result (without knowledge of the poles apart form stability) bounding the H_2 norm in terms of the H_∞ norm is available in discrete time, essentially because the integration interval used in computing the H_2 norm is finite. In continuous time, such a result is not available; it is possible, however, to bound the H_2 norm by a function of the H_∞ norm and the bandwidth of the system.

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