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Optimal Experiment Designs with Respect to the Intended Model Application*

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By optimally designing the inputs during the identification of a transfer function model, the performance degradation due to the error in the transfer function estimates can be minimized.

Key Words—Identification; optimization; frequency domain.

Abstract—The purpose of the design of identification experiments is to make the collected data maximally informative with respect to the intended use of the model, subject to constraints that might be at hand. When the true system is replaced by an estimated model, there results a performance degradation that is due to the error in the transfer function estimates. Using some recent asymptotic expressions for the bias and the variance of the estimated transfer function, it is shown how this performance degradation can be minimized by a proper experiment design. Several applications, where it is beneficial to let the experiment be carried out in closed loop, are highlighted.

1. INTRODUCTION

THE IDENTIFICATION of transfer functions from experimental data is a fundamental problem in applications of control theory. A typical approach to such identification can be outlined as follows.

Suppose that the true system can be described as

$$\begin{aligned} y(t) &= G_0(q)u(t) + v_0(t) \\ &= \left[\sum_{k=1}^{\infty} g_0(k)q^{-k}\right]u(t) + v_0(t) \\ &= \sum_{k=1}^{\infty} g_0(k)u(t-k) + v_0(t) \qquad (1.1) \end{aligned}$$

Here, $y(t)$ and $u(t)$ are the scalar output and input, respectively, q^{-1} is the backward shift operator, $G_0(q)$ is the transfer function operator, and $g_0(k)$ are the impulse response coefficients. Moreover, $\{v_0(t)\}$ is assumed to be a zero-mean stationary stochastic process with spectral density $\phi_v(\omega)$. The following representation of $v_0(t)$ shall frequently be used:

$$v_0(t) = H_0(q)e(t) = e(t) + \sum_{k=1}^{\infty} h_0(k)e(t-k), \qquad (1.2)$$

where $\{e(t)\}$ is a sequence of independent random variables with zero mean values and variances σ^2. Then

$$\phi_v(\omega) = \sigma^2|H_0(e^{i\omega})|^2. \qquad (1.3)$$

The function $G_0(e^{i\omega})$ is the transfer function of the system (1.1). In order to estimate G_0 and H_0 from observed data $Z^N = (u(1), y(1), \ldots, u(N), y(N))$, one often proceeds as follows:

Postulate a set of candidate models

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t); \theta \in D_{\mathcal{M}}. \qquad (1.4)$$

For a fixed value of θ, $G(q,\theta)$ and $H(q,\theta)$ denote a particular model.

Let $\hat{y}(t\,|\,\theta)$ denote the one-step ahead prediction according to the model corresponding to the value θ:

$$\hat{y}(t\,|\,\theta) = H^{-1}(q,\theta)[G(q,\theta)]u(t) + [1 - H^{-1}(q,\theta)]y(t), \qquad (1.5)$$

and let $\varepsilon(t,\theta)$ be the prediction error $\varepsilon(t,\theta) = y(t) - \hat{y}(t\,|\,\theta)$. Then let the estimate be

$$\hat{\theta}_N = \arg\min_{\theta \in D_{\mathcal{M}}} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta). \qquad (1.6)$$

In this way the estimates

$$\hat{G}_N(e^{i\omega}) = G(e^{i\omega},\hat{\theta}_N),\ \hat{H}_N(e^{i\omega}) = H(e^{i\omega},\hat{\theta}_N) \qquad (1.7)$$

are obtained. Variants of the (prediction error) identification method (1.6) that employ prefilter

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and/or k-step ahead predictions can also be formulated as (1.6) but with the noise model set $H(e^{i\omega}, \theta)$ replaced by a modified noise model set $\bar{H}(e^{i\omega}, \theta)$, as shown in Wahlberg and Ljung (1986); such variants are consequently included in this treatment.

The quality of the estimates $\hat{G}_N$ and $\hat{H}_N$ can be evaluated in terms of the differences

$$\Delta T_N(e^{i\omega}) \triangleq \hat{T}_N(e^{i\omega}) - T_0(e^{i\omega}), \tag{1.8}$$

where

$$\hat{T}_N(e^{i\omega}) \triangleq \begin{pmatrix} \hat{G}_N(e^{i\omega}) \\ \hat{H}_N(e^{i\omega}) \end{pmatrix}; \; T_0(e^{i\omega}) \triangleq \begin{pmatrix} G_0(e^{i\omega}) \\ H_0(e^{i\omega}) \end{pmatrix}. \tag{1.9}$$

It is natural to split up the error $\Delta T_N(e^{i\omega})$ into a random part and a bias part:

$$\Delta T_N(e^{i\omega}) = \hat{T}_N(e^{i\omega}) - E\hat{T}_N(e^{i\omega}) + E\hat{T}_N(e^{i\omega}) - T_0(e^{i\omega}). \tag{1.10}$$

A scalar criterion can be formed as

$$J_N = \int_{-\pi}^{\pi} \mathrm{tr}[Q(\omega)\Pi_N(\omega)]\mathrm{d}\omega, \tag{1.11}$$

where

$$\Pi_N(\omega) = E\Delta T_N(e^{-i\omega})\Delta T_N^T(e^{i\omega}) \tag{1.12}$$

is the mean square error of the stimates (a 2×2 matrix function) formed by expectation w.r.t. the random vector $\hat{\theta}_N$. Moreover

$$Q(\omega) = \begin{pmatrix} Q_{11}(\omega) & Q_{12}(\omega) \\ Q_{21}(\omega) & Q_{22}(\omega) \end{pmatrix} \tag{1.13}$$

is a weighting matrix that reflects the intended use of the model $\hat{T}_N$. In Ljung (1985) several examples of Q corresponding to typical model applications such as simulation, prediction and control are given. Specific examples will be given in Sections 6–8.

The value of the criterion (1.11) will depend on a number of design variables that are at the user's disposal. His objective is thus to choose these so as to minimize (1.11) subject to whatever constraints are imposed. A comprehensive treatment of this problem is given in Ljung (to appear). Here, the authors concentrate on input design issues. Suppose that the user can choose the input spectrum and possibly also some feedback mechanism. The input may thus be generated as

$$u(t) = -F(q)y(t) + w(t), \tag{1.14}$$

where F is the feedback regulator and $w(t)$ is some extra input signal. Suppose that F and the spectrum of w, $\Phi_w(\omega)$, is at the designer's disposal, subject to constraints to be specified later. Minimization of the criterion (1.11) with respect to these design variables is thus desired. This is the problem discussed in the present paper.

This problem relates closely to questions discussed in some other papers. Asymptotic variance expressions for the random term in (1.10) were derived in Ljung (1985), where some open loop experiment design questions are also treated. Expressions for the bias term are discussed in Wahlberg and Ljung (1986). Open loop design is studied in Yuan and Ljung (1985) and some further results are given in Ljung (1986). Note also that the area of off-line experiment design has been somewhat neglected during the last decade. An excellent survey of off-line results can be found in Mehra (1974). Some connections with earlier frequency domain results will be pointed out in this paper.

The paper is organized as follows. Section 2 gives a summary of the asymptotic expressions for the bias and the variance of the transfer function estimates. In Section 3 the connections between the mean square error of the transfer function estimates and the resulting performance degradation of the identified model when it is used in a particular application is established. The formulas show how the performance degradation depends on the input design. The performance degradation can be split up into a term due to the bias of the transfer function estimates and one due to the variance of these estimates. The minimization of the bias contribution is examined in Section 4, while Section 5 studies the minimization of the variance contribution. In the last three sections optimal experiment designs are obtained that minimize the variance contribution of the performance for a number of different applications. Section 6 examines the case where only the I/O transfer function $G(e^{i\omega})$ is of interest. Section 7 briefly states some optimal input design results for the case where the model is to be used either for simulation purposes or prediction. Finally, some interesting new results and insights are established in Section 8: it deals with the case where the intended use of the model is the design of a minimum variance regulator.

2. ASYMPTOTIC RESULTS FOR TRANSFER FUNCTION ESTIMATES

Basic convergence results for the estimate $\hat{\theta}_N$, given by (1.6) can be quoted from Ljung (1978) and Ljung and Caines (1979) as follows

$$\hat{\theta}_N \to \theta^* \text{w.p.1} \qquad \text{as } N \to \infty \tag{2.1}$$

$$\theta^* = \arg\min_{\theta \in D} E\varepsilon^2(t, \theta) \tag{2.2}$$

$$\sqrt{N}(\hat{\theta}_N - \theta^*) \underset{N\to\infty}{\overset{d}{\to}} N(0, P)$$

(convergence in distribution) (2.3)

An expression for P can be given, but this is not detailed here.

If one is mainly interested in the transfer function estimate $\hat{T}_N(e^{i\omega})$, then it is natural to translate (2.1)–(2.3) into frequency domain expressions for the bias and the variance of these estimates. By applying Parseval's relationship to (2.2) one can readily show (cf. Ljung, to appear) that

$$\theta^* = \arg\min_{\theta\in D} \int_{-\pi}^{\pi} B^{T}(e^{i\omega}, \theta)\phi(\omega)B(e^{-i\omega}, \theta) \frac{1}{|H(e^{i\omega}, \theta)|^2} d\omega, \quad (2.4)$$

where

$$B(e^{i\omega}, \theta) = \begin{pmatrix} G(e^{i\omega}, \theta) \\ H(e^{i\omega}, \theta) \end{pmatrix} - \begin{pmatrix} G_0(e^{i\omega}) \\ H_0(e^{i\omega}) \end{pmatrix} \quad (2.5)$$

and

$$\phi(\omega) = \begin{pmatrix} \phi_u(\omega) & \phi_{ue}(\omega) \\ \phi_{eu}(\omega) & \sigma^2 \end{pmatrix}. \quad (2.6)$$

Here $\phi_u(\omega)$ is the input spectrum and $\phi_{ue}(\omega)$ is the cross-spectrum between the input u and the white noise sequence e applied during identification to the system (1.1)–(1.2):

$$\phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau)e^{-i\tau\omega},$$

$$\phi_{ue}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{ue}(\tau)e^{-i\tau\omega}, \quad (2.7)$$

where the following limits are assumed to exist:

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{t=1}^{N} u(t)u(t-\tau) = R_u(\tau) \quad (2.8)$$

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{t=1}^{N} u(t)e(t-\tau) = R_{ue}(\tau);$$

$$R_{ue}(\tau) = 0 \qquad \text{for } \tau < 0. \quad (2.9)$$

Clearly, for an experiment performed in open loop, $\phi_{ue}(\omega) \equiv 0$. Note that expression (2.4) determines the asymptotic bias of the transfer function estimates.

The asymptotic normality result (2.3) can be translated to a result on the asymptotic distribution of $\tilde{T}_N(e^{i\omega}) \triangleq \hat{T}_N(e^{i\omega}) - T^*(e^{i\omega})$ $[T^*(e^{i\omega}) \triangleq T(e^{i\omega}, \theta^*)]$, but this gives, in general, quite a complex expression for the corresponding covariance matrix. For a wide class of model sets, however, a simple asymptotic expression can be derived. Suppose that the model set is subject to the following shift property

$$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \dim\theta_k = s \quad (2.10a)$$

$$\frac{\partial}{\partial\theta_k} T(q, \theta) = q^{-k+1} \frac{\partial}{\partial\theta_1} T(q, \theta). \quad (2.10b)$$

In Ljung (1985) it is shown that (2.10) is the typical structure for an nth order linear, black box model, involving s different polynomials in the delay operator. Then the following result holds:

$$\sqrt{N}(\hat{T}_N(e^{i\omega}) - T^*(e^{i\omega})) \underset{N\to\infty}{\overset{d}{\to}} N(0, P_n(\omega)) \quad (2.11a)$$

$$\lim_{n\to\infty} \frac{1}{n} P_n(\omega) = \phi_v(\omega)\cdot[\phi(\omega)]^{-1} \triangleq \bar{P}(\omega), \quad (2.11b)$$

where $\phi(\omega)$ is given by (2.6), and $\phi_v(\omega)$ by (1.3).

Heuristically one could rewrite (2.11) as

$$\operatorname{cov}(\hat{T}_N(e^{i\omega})) \simeq \frac{n}{N}\bar{P}(\omega) = \frac{n}{N}\phi_v(\omega)[\phi(\omega)]^{-1}. \quad (2.12)$$

For a further discussion on the validity and accuracy of (2.12), see Ljung (1985).

3. PERFORMANCE DEGRADATION AND THE INPUT DESIGN PROBLEM

Errors in the transfer function estimates will of course degrade the performance whenever the model is used. In this section some basic formulas for performance degradation due to the error $\hat{T}_N(e^{i\omega}) - T_0(e^{i\omega})$ are derived and it will be shown how this degradation can be kept small by proper input design.

Let $s(t)$ be a signal derived from a model application. It could be, e.g. the output of the system when a minimum variance regulator, computed using the model, is applied. Specific examples will be given in Sections 6–8. Conceptually one could write

$$s(t) = f(T(q))w(t) \quad (3.1)$$

to denote that the transfer functions T, as well as some additional signal (reference signals and/or noise) used to determine $s(t)$. If the true transfer function $T_0(q)$ is used in (3.1), the "true" or "best" result

$$s_0(t) = f(T_0(q))w(t) \quad (3.2)$$

is obtained. When, instead, $\hat{T}_N$ is used the result is

$$\hat{s}_N(t) = f(\hat{T}_N(q))w(t). \quad (3.3)$$

Similarly the expected transfer function estimate T^* gives

$$s^*(t) = f(T^*(q))w(t). \tag{3.4}$$

It is now of interest to evaluate the *performance degradation* due to the mean square error of the estimates. Let

$$\Delta s_N(t) \triangleq \hat{s}_N(t) - s_0(t). \tag{3.5}$$

When the error $\Delta T_N(q) = \hat{T}_N(q) - T_0(q)$ is small, Taylor's expansion can be used to derive

$$\Delta s_N(t) = \Delta T_N^{\mathrm{T}}(q)F(q)w(t), \tag{3.6}$$

where

$$F(q) = \frac{\partial}{\partial T} f(T)|_{T=T_0(q)} (2 \times r\,\text{matrix}; r = \dim w). \tag{3.7}$$

The expected spectrum of $\Delta s_N(t)$ is,

$$\phi_{\Delta s_N}(\omega) = E\{\Delta T_N^{\mathrm{T}}(e^{i\omega})F(e^{i\omega})\phi_w(\omega)F^T(e^{-i\omega})\Delta T_N(e^{-i\omega})\} = \mathrm{tr}[\Pi_N(\omega)Q(\omega)], \tag{3.8}$$

where, as before

$$\Pi_N(\omega) = E\Delta T_N(e^{-i\omega})\Delta T_N^T(e^{i\omega})(2 \times 2\,\text{matrix}) \tag{3.9}$$

and

$$Q(\omega) = F(e^{i\omega})\phi_w(\omega)F^{\mathrm{T}}(e^{-i\omega})(2 \times 2\,\text{matrix}). \tag{3.10}$$

The mean square performance degradation is

$$E\Delta s_N^2(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{\Delta s_N}(\omega)\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}[\Pi_N(\omega)Q(\omega)]\mathrm{d}\omega. \tag{3.11}$$

The expression (3.11) justifies the use of (1.11) as a design criterion. The use of the trace criterion (1.11) in connection with the minimization of a performance degradation was apparently first introduced by Payne (1974).

Notice that $\Pi_N(\omega)$ is the mean square error of $\hat{T}_N$, while $Q(\omega)$ reflects the intended use of the model. This is illustrated with the following example. *Example* 3.1. *Prediction.* Suppose the model is to be used as a predictor on a set of data $\{u^*(t)\}$ and denote by $\phi_u^*(\omega)$ and $\phi_{ue}^*(\omega)$ the input spectrum and the input-to-noise cross-spectrum to which the predictor will be applied. (Here a * superscript is used to distinguish these spectra from those used during identification; there is no connection, of course, with the minimizing value θ^*.) The true one-step ahead predictor is then given by

$$s_0(t) = \hat{y}(t\,|\,t-1) = H_0^{-1}(q)G_0(q)u^*(t) + [1 - H_0^{-1}(q)]y^*(t). \tag{3.12}$$

This corresponds to (3.2) with

$$w(t) = \begin{pmatrix} u^*(t) \\ y^*(t) \end{pmatrix} \text{ and } f(T) = [H^{-1}G \quad 1 - H^{-1}]. \tag{3.13}$$

This gives, after some calculations,

$$Q(\omega) = \frac{1}{|H_0(e^{i\omega})|^2}\begin{pmatrix} \phi_u^*(\omega) & \phi_{ue}^*(\omega) \\ \phi_{ue}^*(-\omega) & \sigma^2 \end{pmatrix}. \tag{3.14}$$

It has been assumed that the data (e^*, u^*, y^*) are also subject to the true model (1.1)–(1.2).

Returning to (3.8)–(3.11), it makes sense to select the input spectra $\phi_u(\omega)$ and $\phi_{ue}(\omega)$ so as to minimize the mean square performance degradation or, more generally, a weighted frequency norm of the error spectrum:

$$J(\phi) = \int_{-\pi}^{\pi} \phi_{\Delta s_N}(\omega)\alpha(\omega)\mathrm{d}\omega, \tag{3.15}$$

where $\alpha(\omega)$ is a weighting function that reflects the importance of having a small error spectrum at certain frequencies. For simplicity $\alpha(\omega) \equiv 1$ will be taken in the sequel. The argument ϕ (defined by (2.6)) has been added to stress that the interest in this paper will be to minimize (3.15) with respect to ϕ, i.e. w.r.t. to the design variables $\phi_u(\omega)$ and $\phi_{ue}(\omega)$:

$$\min_{\phi} J(\phi) \tag{3.16}$$

The approximation

$$E\hat{T}_N(e^{i\omega}) \approx T^*(e^{i\omega}) = T(e^{i\omega}, \theta^*) \tag{3.17}$$

is now introduced, which is reasonable, since typically (see Ljung, to appear, chapter 9).

$$E\hat{T}_N(e^{i\omega}) = T^*(e^{i\omega}) + o\left(\frac{1}{\sqrt{N}}\right). \tag{3.18}$$

The mean square error $\Pi_N(\omega)$ can then be split into a bias contribution and a variance contribution and, using the approximation (2.12),

$$\Pi_N(\omega) \approx B(e^{-i\omega}, \theta^*)B^{\mathrm{T}}(e^{i\omega}, \theta^*) + \frac{n}{N}\bar{P}(\omega). \tag{3.19}$$

Now (3.16) can be replaced by

$$\min_{\phi} [J_B(\phi) + J_V(\phi)] \tag{3.20}$$

with

$$J_B(\phi) = \int_{-\pi}^{\pi} B^T(e^{i\omega}, \theta^*(\phi))Q(\omega)B(e^{-i\omega}, \theta^*(\phi))d\omega \tag{3.21a}$$

$$J_V(\phi) = \frac{n}{N}\int_{-\pi}^{\pi} \mathrm{tr}[Q(\omega)\phi^{-1}(\omega)]\phi_v(\omega)d\omega. \tag{3.21b}$$

The argument ϕ has been appended to θ^* to stress that the parameter vector to which $\hat{\theta}_N$ converges (and hence the bias) is a function of the design variables.

In the general case it will not be possible to give a solution to problem (3.20). Therefore first, in Section 4, the problem

$$\min_{\phi} J_B(\phi),$$

is considered, then in Section 5, the problem

$$\min_{\phi} J_V(\phi)$$

is studied.

4. MINIMIZING THE BIAS CONTRIBUTION

The expression (2.4) is not all that easy to study, due to the complicated dependence on θ. Assuming that (2.4) has a unique minimum, (2.4) can be written as

$$\theta^* = \underset{\theta \in D_m}{\mathrm{sol}} \left\{\frac{d}{d\theta}[V_1\theta) + V_2(\theta)] = 0\right\} \tag{4.1a}$$

$$V_1(\theta) = \int_{-\pi}^{\pi} B^T(e^{i\omega}, \theta)\phi(\omega)B(e^{-i\omega}, \theta) \frac{1}{|H(e^{i\omega}, \theta^*)|^2} d\omega \tag{4.1b}$$

$$V_2(\theta) = \int_{-\pi}^{\pi} B^T(e^{i\omega}, \theta^*)\phi(\omega)B(e^{-i\omega}, \theta^*) \frac{1}{|H(e^{i\omega}, \theta)|^2}. \tag{4.1c}$$

(Here "sol$[f(x) = 0]$" means the solution to the equation $f(x) = 0$.)

To see that (4.1) is equivalent to (2.4), differentiate both expressions w.r.t. θ and compare. Of course, (4.1) cannot be used for the actual computation of θ^* since V_1 and V_2 depend on θ^*. However suppose, as a simplifying assumption, that

$$\theta^* \simeq \arg\min_{\theta \in D_m} V_1(\theta). \tag{4.2}$$

(This is certainly true when a fixed noise model is applied.) Under this approximation it is possible to solve the bias minimization problem: $\min_{\phi} J_B(\phi)$.

Result 4.1. Assume that (4.2) holds with equality. Then the solution to $\min_{\phi} J_B(\phi)$, with $J_B(\phi)$ given by (3.21a), is

$$\phi^{\mathrm{opt}}(\omega) = \alpha Q(\omega)|H(e^{i\omega}, \theta^*)|^2, \tag{4.3a}$$

where α is a positive scalar, provided (4.3a) belongs to the set of admissible designs for some α.

Proof. This result follows directly from lemma 6.1 of Yuan and Ljung (1985).

Notice that the estimate θ^* is a function of the design variables ϕ: $\theta^*(\phi)$. Therefore the relationship (4.3a), that defines the optimal choices of ϕ_u, ϕ_{ue} and the noise model H, is an implicit relationship:

$$\phi^{\mathrm{opt}}(\omega) = \alpha Q(\omega)|H(e^{i\omega}, \theta^*(\phi^{\mathrm{opt}}))|^2. \tag{4.3b}$$

The solution of (4.3b) may therefore not be immediate.

There are consequently several reservations associated with Result 4.1: it is based on the *ad hoc* assumption (4.2), and it may be difficult to realize. Nevertheless, the result is instructive and suggestive. It may be used with success in a pragmatic fashion, just as in Wahlberg and Ljung (1986).

Comparing $\phi^{\mathrm{opt}}(\omega)/|H(e^{i\omega}, \theta^*)|^2$ with the expressions for $Q(\omega)$ describing the intended model application (see e.g. (3.14) for the application of Example 3.1), Result 4.1 shows that "the design variables should be chosen so that the identification experiment as much as possible resembles the conditions under which the model is going to be used". In particular, if there is a cost $Q_{12}(\omega)$ associated with the cross-product term between the bias of G^* and the bias of H^*, then one should have $\phi_{ue}(\omega) \neq 0$, i.e. the experiment should be performed under output feedback.

5. MINIMIZING THE VARIANCE CONTRIBUTION

Now consider the problem

$$\min_{\phi} J_V(\phi)$$

with $J_V(\phi)$ given by (3.21b). With $Q(\omega)$ as in (1.13)

$$X(\omega) \triangleq \mathrm{tr}[\bar{P}(\omega)Q(\omega)] \tag{5.1a}$$

$$= \frac{\{\sigma^2 Q_{11}(\omega) - 2\mathrm{Re}[Q_{12}(\omega)\phi_{ue}(-\omega)] + Q_{22}(\omega)\phi_u(\omega)\}}{\sigma^2\phi_u(\omega) - |\phi_{ue}(\omega)|^2}\phi_v(\omega) \tag{5.1b}$$

Recall that

$$\frac{n}{N}\bar{P}(\omega) \simeq \operatorname{cov}(\hat{T}_N(e^{i\omega})) = \operatorname{cov}(\tilde{T}_N(e^{i\omega})), \quad (5.2)$$

where $\tilde{T}_N(e^{i\omega}) \triangleq \hat{T}_N(e^{i\omega}) - T^*(e^{i\omega})$. Then

$$J_V(\phi) = \frac{n}{N}\int_{-\pi}^{\pi} X(\omega)d\omega$$

$$\simeq \int_{-\pi}^{\pi} \phi_{\tilde{s}_N}(\omega)d\omega = 2\pi \operatorname{Var}(\tilde{s}_N(t)), \quad (5.3)$$

where $\tilde{s}_N(t) \triangleq \hat{s}_N(t) - s^*(t)$ and $\phi_{\tilde{s}_N}(\omega)$ is the spectrum of $\tilde{s}_N(t)$:

$$\phi_{\tilde{s}_N}(\omega) = \operatorname{tr}\{E[\tilde{T}_N(e^{i\omega})\tilde{T}_N^T(e^{-i\omega})]Q(\omega)\} \simeq \frac{n}{N}X(\omega) \quad (5.4)$$

(see (3.10)–(3.11)). If the identification has been performed in open loop, then $\phi_{ue} \equiv 0$. If it has been performed under the feedback law (1.14), where $\{w(t)\}$ is an additive stationary stochastic process with spectrum $\phi_w(\omega)$, independent of $\{e(t)\}$, then (deleting arguments):

$$\phi_{ue} = -\frac{FH_0}{1 + FG_0}\sigma^2 \quad (5.5)$$

$$\phi_u = \left|\frac{FH_0}{1 + FG_0}\right|^2 \sigma^2 + \left|\frac{1}{1 + FG_0}\right|^2 \phi_w. \quad (5.6)$$

The experiment design task is therefore to minimize $J_v(\phi)$ with respect to ϕ_u and ϕ_{ue} (or equivalently F and ϕ_w) for different applications under either input variance constraint or output variance constraint

$$\int_{-\pi}^{\pi} \phi_u(\omega)d\omega \leqslant C \quad (5.7a)$$

$$\int_{-\pi}^{\pi} \phi_y(\omega)d\omega \leqslant C. \quad (5.8a)$$

More generally, one may want to constrain the input or output power in some specific frequency ranges. In that case (5.7a) and (5.8a) can be replaced by

$$\phi_u(\omega) \leqslant D(\omega) \text{ with } \int_{-\pi}^{\pi} D(\omega)d\omega = C \quad (5.7b)$$

$$\phi_y(\omega) \leqslant D(\omega) \text{ with } \int_{-\pi}^{\pi} D(\omega)d\omega = C. \quad (5.8b)$$

Notice that

$$\phi_y = \left|\frac{H_0}{1 + FG_0}\right|^2 \sigma^2 + \left|\frac{G_0}{1 + FG_0}\right|^2 \phi_w. \quad (5.9)$$

In the next three sections, this minimization problem will be solved for a variety of applications (simulation, prediction, minimum variance control,...), each leading to a particular set of weights $Q_{ij}(\omega)$, $i, j = 1, 2$ in (5.1). In particular some applications in which the use of feedback during identification has a beneficial effect will be highlighted. Before going into specific applications, in this section two general preliminary results will be stated. For simplicity, in the following, assume that $\sigma^2 = 1$.

Result 5.1. If $Q_{12} = 0$ and *the input variance is constrained*, then open loop operation is optimal, i.e. $\phi_{ue}(\omega) = 0$. This follows directly from (5.1). It also follows that the input spectrum is

$$\phi_u^{\text{opt}}(\omega) = \mu\sqrt{Q_{11}(\omega)\phi_v(\omega)}, \quad (5.10)$$

where μ is adjusted so that equality holds in (5.7) (see Yuan and Ljung, 1985).

Result 5.2. If the matrix Q is singular and the input variance is constrained, then feedback operation is optimal during identification, i.e. $\phi_{ue}^{\text{opt}}(\omega) \neq 0$.

Proof. If Q has rank 1, then necessarily (see (3.10))

$$Q = \alpha\begin{bmatrix} |Z|^2 & Z \\ \bar{Z} & 1 \end{bmatrix} \quad \text{or} \quad Q = \alpha\begin{bmatrix} 1 & Z \\ \bar{Z} & |Z|^2 \end{bmatrix}$$

for some $Z = Z(\omega)$. Therefore

$$X(\omega) = \alpha\frac{|Z(\omega)|^2 - 2\operatorname{Re}[Z(\omega)\phi_{ue}(-\omega)] + \phi_u(\omega)}{\phi_u(\omega) - |\phi_{ue}(\omega)|^2}\phi_v(\omega). \quad (5.11)$$

First assume that no feedback is applied, i.e. $\phi_{ue}(\omega) = 0$ (see (5.5)). Then

$$X(\omega) = \alpha\left(1 + \frac{|Z|^2}{\phi_u}\right)\phi_v. \quad (5.12)$$

Assume now that $\phi_{ue}(\omega) = Z(\omega)$ is taken. Substituting in (5.11) yields $X(\omega) = \alpha\phi_v(\omega)$. Hence feed-back reduces $X(\omega)$ and therefore $\operatorname{var}(\tilde{s}_N(t))$, unless $Z = 0$.

6. ESTIMATION OF THE I/O DYNAMICS $G(e^{i\omega})$ ONLY, WITH POINTWISE POWER CONSTRAINTS

6.1. *The optimal input design*

Suppose the objective is to minimize $\operatorname{Var}(\hat{G}_N)$ only: this is sometimes referred to as G-optimality (Mehra, 1974). It is easy to show that *if the input* *power is constrained* as in (5.7b), the optimal strategy is to maximize the input power in open loop operation. Consider now the case where *the output power is constrained* as in (5.8b).

Suppose there is a delay d in the plant, i.e.

$$G_0(q) = \sum_{k=d}^{\infty} g_0(k)q^{-k}. \tag{6.1}$$

Then $H_0(q)$ is decomposed as

$$H_0(q) = H_0^{(1)}(q) + H_0^{(2)}(q), \tag{6.2a}$$

where

$$H_0^{(1)}(q) = 1 + h_1 q^{-1} + \cdots + h_{d-1} q^{-d+1};$$
$$H_0^{(2)}(q) = \sum_{k=d}^{\infty} h_k q^{-k}. \tag{6.2b}$$

Then (5.8b) can be replaced by

$$\phi_y(\omega) = |G_0|^2\phi_u + |H_0^{(1)}|^2 + |H_0^{(2)}|^2 + 2\mathrm{Re}(G_0 \bar{H}_0^{(2)}\phi_{ue}) \leqslant D(\omega), \tag{6.3}$$

where $\bar{H}_0^{(2)}$ is the complex conjugate of $H_0^{(2)}$. (The arguments ω have been dropped for convenience.) Therefore $\mathrm{Var}(\hat{G}_N(e^{i\omega}))$ must be minimized subject to (6.3). By (2.12) and (5.1), this is equivalent to minimizing

$$X(\omega) = \frac{\phi_v}{\phi_u - |\phi_{ue}|^2} \tag{6.4}$$

subject to (6.3). The optimal solution is

$$\phi_{ue}^{\mathrm{opt}}(\omega) = -\frac{H_0^{(2)}(e^{i\omega})}{G_0(e^{i\omega})};$$
$$\phi_u^{\mathrm{opt}}(\omega) = |\phi_{ue}^{\mathrm{opt}}(\omega)|^2 + \sqrt{\frac{\phi_v(\omega)}{\mu |G_0(e^{i\omega})|^2}}, \tag{6.5}$$

where μ is a Lagrange multiplier. This corresponds to the feedback law

$$u(t) = -\frac{H_0^{(2)}(q)}{G_0(q)} y(t) + w(t), \tag{6.6}$$

with an arbitrary signal $\{w(t)\}$ independent of $\{e(t)\}$. The regulator (6.6) is the minimum variance control law, which is of course applicable only if $G_0(q)$ has minimum phase. If the power $\phi_w(\omega)$ of the external signal $w(t)$ is adjusted so as to satisfy the constraint (6.3) with equality, then the corresponding optimal variance of $\hat{G}_N(e^{i\omega})$ can be approximated by (see (2.12)):

$$\mathrm{Var}(\hat{G}_N(e^{i\omega})) \simeq \frac{n}{N}\frac{\phi_v}{\phi_u - |\phi_{ue}|^2} = \frac{n}{N}\phi_v \frac{|G_0|^2}{D - |H_0^{(1)}|^2}. \tag{6.7}$$

Comment 6.1. It is interesting to compare (6.7) with the best achievable error variance using open loop identification. With $\phi_{ue} = 0$ and ϕ_u chosen such that $\phi_y = D$, the error variance becomes

$$\mathrm{Var}(\hat{G}_N(e^{i\omega})) \simeq \frac{n}{N}\frac{\phi_v}{\phi_u} = \frac{n}{N}\phi_v \frac{|G_0|^2}{D - |H_0|^2}, \tag{6.8}$$

where $|H_0| \geqslant |H_0^{(1)}|$ (see (6.2)). The conclusion is that, if the system has minimum phase and if the output power is constrained, one should use the minimum variance control law (6.6) to identify $G(e^{i\omega})$. The benefit over open loop identification is indicated by a comparison between (6.8) and (6.7).

Comment 6.2. The result above is close to earlier results on optimal input design obtained using different techniques and a different optimality criterion. It follows from Theorem 4.2 of Ng *et al.* (1977) that if $G(q)$ is parametrized with a vector θ, $G(q, \theta)$, and if the determinant of the Information Matrix is maximized (this is often referred to as D-optimality), then the optimal input design under output variance constraint is to use a minimum variance regulator, provided again $G_0(q)$ is minimum phase. It is interesting to see that one arrives at the same criterion using two very different input design criteria: D-optimality and G-optimality. This was also noted in Mehra (1974).

6.2. *Comparison between switching regulators and external input*

It is well known that, in order to achieve identifiability under feedback conditions, an alternative to using an external signal $w(t)$ as in (6.6) is to switch between different regulators: see, e.g. Söderström *et al.* (1976). Here we want to compare these two strategies in terms of their effect on $\mathrm{Var}(\hat{G}_N(e^{i\omega}))$, using the asymptotic expression (2.12).

Case 1. Consider first the case where a constant feedback law (1.14) is used for the identification of $G(q)$. Then

$$y(t) = \frac{G_0}{1 + G_0 F} w(t) + \frac{H_0}{1 + G_0 F} e(t) \tag{6.9a}$$

$$u(t) = \frac{1}{1 + G_0 F} w(t) - \frac{F H_0}{1 + F G_0} e(t)$$
$$= \tilde{w}(t) - \frac{F H_0}{1 + F G_0} e(t). \tag{6.9b}$$

Therefore, by (2.12)

$$\mathrm{Var}(\hat{G}_N(e^{i\omega})) \simeq \frac{n}{N}|1 + G_0 F|^2 \frac{\phi_v}{\phi_w} = \frac{n}{N}\frac{\phi_v}{\phi_{\tilde{w}}}. \tag{6.10}$$

The output spectrum is given by (5.9) with $\sigma^2 = 1$.

Suppose now that the minimum variance regulator is used, denoted by $F^*(q)$, and assume for simplicity that the delay d in (6.1) is $d = 1$. Then F^* is determined by

$$1 + F^*G_0 = H_0 \tag{6.11}$$

and (6.10) becomes

$$\text{Var}\,\hat{G}_N \simeq \frac{n}{N}\frac{\phi_v^2}{\phi_w}. \tag{6.12}$$

(Recall that $\phi_v = |H|^2$, since σ^2 is assumed to be 1.) The first term in (5.9) becomes 1: it is the output spectrum obtained with a minimum variance regulator with no external input. The second term is denoted by $\tilde{\phi}_y$: it is the increase in the output spectrum due to the external input $w(t)$. It follows from (6.10) and (6.11) that $\phi_y = 1 + \tilde{\phi}_y$ where

$$\tilde{\phi}_y = \frac{|G_0|^2}{|1 + G_0F^*|^2}\phi_w \simeq \frac{n}{N}|G_0|^2\frac{\phi_v}{\text{Var}\,\hat{G}_N}. \tag{6.13}$$

Equivalently

$$\tilde{\phi}_y(\omega)\text{Var}\,\hat{G}_N(e^{i\omega}) \simeq \frac{n}{N}|G_0(e^{i\omega})|^2\phi_v(\omega). \tag{6.14}$$

Comment 6.3. Expression (6.10) (or (6.12)) shows that it is the external input that accounts for all the accuracy of the transfer function, while (6.14) shows that any decrease in Var $\hat{G}_N$ (achieved by a larger ϕ_w) results in an inversely proportional increase in $\tilde{\phi}_y(\omega)$. The relationship holds frequency by frequency, but for any given frequency the quantity on the right hand side of (6.14) is fixed: it depends only on the true system.

Case 2. Consider now the case where no external input is used in (1.14), but where the regulator is switched between two control laws $F_1(q)$ and $F_2(q)$, each used 50% of the time. Denote

$$L = \frac{|1 + F_1G_0|^2|1 + F_2G_0|^2}{|H_0|^2} \tag{6.15}$$

and

$$S_i = -\frac{F_iH_0}{1 + F_iG_0} \qquad i = 1, 2 \qquad (\text{see}\,(6.9\text{b})). \tag{6.16}$$

Assuming that the effects on ϕ_u and ϕ_{ue} of the transients at the switchings are negligible, it then follows from (2.12) that

$$\text{Var}(\hat{G}_N(e^{i\omega})) \simeq \frac{n}{N}\frac{4\phi_v}{|S_2 - S_1|^2} = \frac{n}{N}\frac{4\phi_v L}{|F_2 - F_1|^2}. \tag{6.17}$$

The output spectrum is

$$\phi_y = \frac{1}{2}\frac{|H_0|^2}{|1 + F_1G_0|^2} + \frac{1}{2}\frac{|H_0|^2}{|1 + F_2G_0|^2}. \tag{6.18}$$

Comment 6.4. Expression (6.17) shows that it is the "distance" $\Delta F \triangleq |F_2 - F_1|$ between the two regulators that gives the accuracy of the transfer function estimate. This distance therefore plays a role similar to the external noise power ϕ_w in the case of switching regulators. However, a comparison of the respective effects of ϕ_w and ΔF on ϕ_y does not appear possible, and an expression similar to (6.14) is not available in this case.

7. SIMULATION AND PREDICTION

7.1. *Simulation*

Suppose the objective is to use the model $\hat{G}_N(q)$ to simulate the output with an input signal with spectrum $\phi_u^*(\omega)$. The question is: what is the optimal input design during identification? The simulated output will be

$$\hat{y}_N(t) = \hat{G}_N(q)u^*(t). \tag{7.1}$$

It differs from the output simulated with the "expected model" $G^*(q)$. The error is

$$\tilde{y}_N(t) = \tilde{G}_N(q)u^*(t). \tag{7.2}$$

The corresponding spectrum is

$$\phi_{\tilde{y}}(\omega) = E|\tilde{G}_N(e^{i\omega})|^2\phi_u^*(\omega). \tag{7.3}$$

This corresponds to (3.10)–(3.11) with $Q_{11} = \phi_u^*$, $Q_{12} = Q_{21} = Q_{22} = 0$.

With *constrained input variance*, the optimal experiment design is to identify in open loop. This follows from Result 5.1. With *output power constraint*, one can show that the solution for $\phi_{ue}(\omega)$ is identical to that obtained in Section 6.1, and the same conclusions apply: applying a minimum variance controller during identification is optimal if the system has minimum phase.

7.2. *Prediction*

Suppose the objective is to use the model as a predictor on a new set of input data with possibly different spectra. The deviation between the prediction error $\hat{\varepsilon}_N(t)$ obtained from the model and its expected value $\varepsilon^*(t)$ can be approximated as

$$\tilde{\varepsilon}_N(t) \triangleq \hat{\varepsilon}_N(t) - \varepsilon^*(t) = \frac{1}{H_0}(\tilde{G}_N(q)u(t) + \tilde{H}_N(q)e(t)). \tag{7.4}$$

This means that $Q(\omega)$ in (3.21b) takes the form

$$Q(\omega) = \frac{1}{|H_0|^2}\begin{bmatrix} \phi_u^*(\omega) & \phi_{ue}^*(\omega) \\ \phi_{ue}^*(-\omega) & 1 \end{bmatrix}, \quad (7.5)$$

where the stars indicate that these are the spectra of the data to which the predictor will be applied. Therefore

$$X(\omega) = \frac{\phi_u(\omega) + \phi_u^*(\omega) - 2\mathrm{Re}(\phi_{ue}^*(\omega)\phi_{ue}(-\omega))}{\phi_u(\omega) - |\phi_{ue}(\omega)|^2}. \quad (7.6)$$

Suppose the predictor is to be applied to an open loop situation, i.e. $\phi_{ue}^* = 0$. Then, under constrained input variance, it is optimal to identify the system in open loop (Ljung, 1985). However, under constrained output variance, one can show that the optimal strategy is to use feedback, i.e. $\phi_{ue}^{\mathrm{opt}} \neq 0$ even though $\phi_{ue}^* = 0$.

8. THE MINIMUM VARIANCE REGULATOR APPLICATION

An interesting case is where the identified model is to be used for the design of a minimum variance regulator. It will be shown that, in the case of a constraint on either the input or the output variance, the optimal strategy is to identify the model under minimum variance control feedback, (which of course will require knowledge of the true system, so that it can only be implemented approximately).

8.1. *Minimum variance control with input power constraint*

Suppose the system has minimum phase and the delay in $G_0(q)$ is $d = 1$. Then the true minimum variance regulator is

$$u(t) = \frac{1 - H_0(q)}{G_0(q)} y(t). \quad (8.1)$$

In practice, this would be replaced by

$$u(t) = \frac{1 - \hat{H}_N(q)}{\hat{G}_N(q)} y(t). \quad (8.2)$$

Instead of $y(t) = e(t)$, this would produce an output

$$y_N(t) \simeq e(t) + \frac{1}{H_0(q)}\left\{\frac{H_0(q) - 1}{G_0(q)}[\tilde{G}_N(q) + G^*(q) - G_0(q)] - [\tilde{H}_N(q) + H^*(q) - H_0(q)]\right\}e(t), \quad (8.3)$$

where $\tilde{G}_N = \hat{G}_N - G^*, \tilde{H}_N = \hat{H}_N - H^*$ and we have assumed $\tilde{G}_N$, $\tilde{H}_N$, $G^* - G_0$ and $H^* - H_0$ to be small. Considering again only the variance contribution to the performance degradation and denoting

$$Z_0 \triangleq \frac{1 - H_0}{G_0}$$

yields the following expression for $Q(\omega)$:

$$Q(\omega) = \frac{1}{|H_0|^2}\begin{bmatrix} |Z_0|^2 & Z_0 \\ \bar{Z}_0 & 1 \end{bmatrix}, \quad (8.4)$$

where $\bar{Z}_0 \triangleq Z_0(e^{-i\omega})$. Inserting (8.4) into (5.1b), gives, after some calculations (compare (5.11)–(5.12))

$$X(\omega) = 1 + \frac{|Z_0(e^{i\omega}) - \phi_{ue}(\omega)|^2}{\phi_u(\omega) - |\phi_{ue}(\omega)|^2}. \quad (8.5a)$$

This is minimized for

$$\phi_{ue}(\omega) = Z_0(e^{i\omega}) \quad (8.5b)$$

regardless of $\phi_u(\omega)$. Therefore, provided the choice $\phi_{ue}(\omega) = Z_0(e^{i\omega})$ does not violate the given constraint, this is the optimal solution. It corresponds to

$$u(t) = Z_0(q)e(t) + w(t) = Z_0(q)y(t) + w(t) \quad (8.6)$$

regardless of the extra input $w(t)$. Therefore a necessary condition for optimality of $\phi_{ue} = Z_0$ is that

$$\int_{-\pi}^{\pi} |Z_0(e^{i\omega})|^2 \mathrm{d}\omega \leqslant C. \quad (8.7)$$

8.2. *Discussion and further results*

(a) The exact minimum variance control law is $u(t) = Z_0(q)y(t)$. Therefore, if the intent is to apply this controller to the system, condition (8.7) will always be satisfied. Hence, (8.6) is the optimal design strategy during identification.

(b) One might think that the optimal strategy is to adjust the spectrum of $w(t)$ in (8.6) so that the constraint (5.7) is satisfied with equality. As a matter of fact, it follows from (8.5a) that if $\phi_{ue} = Z_0$, then $X(\omega) = 1$ regardless of ϕ_u (i.e. of ϕ_w). The conclusion is: if one wants to identify a system in view of applying a minimum variance controller to it, the optimal strategy is to identify it under minimum variance control $u(t) = Z_0(q)y(t)$; surprisingly, addition of an external input $w(t)$ does not contribute to a decrease of the error variance.

(c) With the optimal feedback design $u(t) = Z_0(q)y(t)$, the spectrum of the error $y_N(t) - y^*(t)$ is given by

$$\phi_{\tilde{y}}(\omega) \sim \frac{n}{N}. \quad (8.8)$$

This is the minimum of $\phi_{\tilde{y}}$ for all ϕ_u and ϕ_{ue}. By comparison with open loop identification, using (5.4) and (8.4):

$$\phi_{\tilde{y}}(\omega) \simeq \frac{n}{N}\left(1 + \frac{|Z_0|^2}{\phi_u(\omega)}\right). \tag{8.9}$$

Hence, in open loop, the same minimum can only be achieved with infinite input energy. In the general case where $E\{e^2(t)\} = \sigma^2$, the expressions (8.8) and (8.9) must be multiplied by σ^2, the minimum value of the error spectrum becomes $\phi_{\tilde{y}}(\omega) \simeq \frac{n}{N}\sigma^2$. Recall that this is not the spectrum of the output of the minimum variance controlled system, but the spectrum of the additional error on that output signal that is due to the variance of the transfer function estimates.

The reason why an external input $w(t)$ does not decrease the error variance is as follows. Suppose the system is described by an ARMAX model

$$A_0(q)y(t) = B_0(q)u(t) + C_0(q)e(t). \tag{8.10}$$

Then the true minimum variance regulator is

$$u(t) = \frac{A_0(q) - C_0(q)}{B_0(q)}y(t). \tag{8.11}$$

(d) By applying more input power, the estimates $\hat{A}_N(q)$, $\hat{B}_N(q)$ and $\hat{C}_N(q)$ are improved, but not those of $(\hat{A}_N - \hat{C}_N)/\hat{B}_N$, which is all that is needed to compute the minimum variance regulator (MVR). The simulations will illustrate this point.

8.3. *Numerical illustration*

The theoretical results have been tested by a series of simulations on the following system.

$$\mathscr{S}: A_0(q)y(t) = B_0(q)u(t) + C_0(q)e(t) \tag{8.12}$$

with

$$A_0(q) = 1 - 1.5q^{-1} + 0.7q^{-2}$$

$$B_0(q) = q^{-1}(1 + 0.5q^{-1})$$

$$C_0(q) = 1 - q^{-1} + 0.2q^{-2}.$$

The true MVR is

$$\text{MVR}: (1 + 0.5q^{-1})u(t)$$
$$= (-0.5 + 0.5q^{-1})y(t) + w(t) \tag{8.13}$$

with $w(t) \equiv 0$. One aim of the simulations was to check whether the asymptotic results apply to low order models (n small) and for reasonably short data lengths. Five simulations have been performed, numbered 1–5 in the sequel. In the first three, the system is identified from I/O data obtained while the MVR (8.13) acts on the system: $\{u(t)\}$ and $\{y(t)\}$ are generated through (8.12)–(8.13), where $\{e(t)\}$ is white Gaussian noise of zero mean and unit variance, WGN(0, 1), while $\{w(t)\}$ is WGN(0, σ_w), where σ_w takes on three different values. In the last two simulations, the system is identified in open loop: $\{e(t)\}$ is WGN(0, 1), $\{u(t)\}$ is WGN(0, σ_u) with two different values for σ_u and $\{y(t)\}$ is generated through (8.12). For each simulation, 10 independent runs have been performed using 10 independent sequences $\{e(t)\}$ of 500 points each.

For each of the 10 runs in simulations 1–5, the polynomials $\hat{A}_N(q)$, $\hat{B}_N(q)$, $\hat{C}_N(q)$ have been estimated by maximum likelihood estimation using IDPAC, assuming a second order ARMAX model. From these, the estimated MVR parameters have been calculated

$$u(t) = \frac{\hat{A}_N(q) - \hat{C}_N(q)}{\hat{B}_N(q)}y(t)$$
$$= \frac{\hat{\alpha}_1 + \hat{\alpha}_2 q^{-1}}{1 + \hat{\beta}q^{-1}}y(t) \tag{8.14}$$

with

$$\alpha_1 = \frac{a_1 - c_1}{b_1}, \alpha_2 = \frac{a_2 - c_2}{b_1}, \beta = \frac{b_2}{b_1}. \tag{8.15}$$

Table 1 gives the true values of three of the system parameters and of the three regulator parameters, as well as the average (over 10 runs) of their estimates and their standard deviations. The other system parameters are excluded for lack of space, but their behaviour is identical.

TABLE 1. TRUE AND ESTIMATED VALUES (WITH STANDARD DEVIATIONS) OF a_1, b_1, c_1 AND α_1, α_2, β FOR 5 DIFFERENT EXPERIMENTAL CONDITIONS, TOGETHER WITH OUTPUT VARIANCE AND INPUT VARIANCE. THE ESTIMATES ARE AVERAGES OVER 10 MONTE CARLO RUNS OF 500 DATA EACH. (NOTE, C.L. = CLOSED LOOP EXPERIMENT; O.L. = OPEN LOOP EXPERIMENT)

		Simulation number and experimental conditions				
		1	2	3	4	5
	True value	C.L. $\sigma_w = 0.01$	C.L. $\sigma_w = 0.1$	C.L. $\sigma_w = 1$	O.L. $\sigma_u = 10$	O.L. $\sigma_u = 1$
a_1	−1.5	0.546	−1.459	−1.509	−1.499	−1.506
		±2.443	±0.565	±0.009	±0.001	±0.009
b_1	1	−0.987	1.118	1.002	1.000	0.997
		±3.840	±0.565	±0.056	±0.005	±0.058
c_1	−1	0.052	−0.895	−1.014	−1.007	−1.012
		±0.666	±0.479	±0.049	+0.035	±0.031
α_1	−0.5	−0.505	−0.508	−0.494	−0.493	−0.436
		±0.026	±0.037	±0.035	±0.036	±0.041
α_2	0.5	0.509	0.517	0.500	0.499	0.501
		±0.085	±0.045	±0.048	±0.051	±0.052
β	0.5	0.484	0.482	0.470	0.502	0.488
		±0.051	±0.099	±0.049	±0.007	±0.088
σ_y^2		1.000	1.034	4.409	1889	20.401
σ_u^2		1.000	1.013	2.333	100	1

Table 1 confirms several points made in Section 8.2. Simulations 1–3 show that an increase in the external signal power increases the accuracy of the a_i, b_i and c_i, but does not affect the accuracy of the regulator parameters. To obtain the same accuracy with open loop identification, one needs to allow for much more output variance. In fact, output variances such as those obtained in simulations 1 and 2 cannot be achieved in open loop, since the noise $\{e(t)\}$ by itself contributes an output variance of 1.521.

Assuming now that the regulator (8.14), corresponding to a set of estimated values $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\beta}$, is applied to the system, one can then compute the output variance V_y of the closed loop system

$$y(t) = \frac{\hat{B}_N(q)C_0(q)}{A_0(q)\hat{B}_N(q) - B_0(q)\hat{A}_N(q) + B_0(q)\hat{C}(q)} e(t) \tag{8.16}$$

and compare this with the variance of the output that would have been obtained with the exact minimum variance regulator, namely $\sigma^2 = 1$. The difference $V_{\tilde{y}} = (V_y - \sigma^2)$ then measures the performance degradation due to the variance in the parameter estimates. This difference has been computed for each of the 10 runs in each simulation, in order to compare it with the theoretical value given by (8.8) and (8.9); the averages $\hat{V}_{\tilde{y}} = \frac{1}{10}\sum_{1}^{10} \hat{V}_{\tilde{y}}^{(i)}$ over 10 runs are presented in Table 2. In these 5 simulations $n/N = 2/500 = 0.004$. Note the larger standard deviations of $\hat{V}_{\tilde{y}}$ in simulations 1 and 2. This is due to the fact that, in two or three of the runs in Simulations 1 and 2, the parameter estimation algorithm runs into numerical problems due to insufficient excitation. The corresponding values of $\hat{V}_y$ in these runs are way out of the range of the $\hat{V}_y$ values for the other runs. These few "outliers" have a dramatic impact on the average value of the performance degradation and its standard deviation. These simulations show that there is a practical (i.e. numerical) reason for maintaining a certain level of external input signal; this does not alter the theoretical conclusions of Section 8.2.

TABLE 2. COMPARISON OF THE MEAN PERFORMANCE DEGRADATION OVER 10 RUNS WITH THE THEORETICAL PERFORMANCE DEGRADATION. EXPERIMENTAL PERFORMANCE DEGRADATION = AVERAGE OVER 10 RUNS OF $(\hat{V}_y^{(i)} - 1)$, TOGETHER WITH THE STANDARD DEVIATION, WHERE $\hat{V}_y^{(i)}$ IS THE SAMPLE OUTPUT VARIANCE OF (8.16) OVER 500 DATA FOR THE ith RUN. THEORETICAL PERFORMANCE DEGRADATION: GIVEN BY (8.8) AND (8.9).

	Simulation number				
	1	2	3	4	5
Average experimental performance degradation $\frac{1}{10}\sum_{1}^{10} \hat{V}_{\tilde{y}}^{(i)}$	0.030 ±0.018	0.011 ±0.007	0.005 ±0.002	0.004 ±0.002	0.006 ±0.002
Theoretical performance degradation $\phi_{\tilde{y}}$	0.004	0.004	0.004	0.004	0.008

8.4. *Minimum variance regulator with output power constraint*

The optimal experiment design for this case is also (8.6), because it yields the smallest possible value for $\phi_{\tilde{y}}$, and because the output spectrum is $\phi_y(\omega) = \sigma^2 + \phi_{\tilde{y}}(\omega)$.

9. CONCLUSIONS

Identification experiments should be designed as a function of the intended use of the identified model. When the unknown "true" model is replaced in a specific application by an estimated model, there results a performance degradation due to the error in the I/O transfer function and noise model estimates. Using recent asymptotic expressions for the bias and the variance of transfer function estimates, it has been shown how this performance degradation can be minimized by a proper experiment design. It is beneficial to let the experiment be carried out in closed loop not only in all cases where the output variance is constrained, but in several input variance constraint cases as well. Several of the results obtained here are consistent with earlier experiment design results obtained under different sets of assumptions (see e.g. Gustavsson *et al.*, 1981; Ng *et al.*, 1977).

A special case of particular interest is when the intended use of the model is to design a minimum variance regulator. The performance degradation due to the variance of the model parameters is always larger than or equal to the model order divided by the number of data collected during the experiment, regardless of the design. The optimal design is to perform the identification under minimum variance control and, somewhat surprisingly, the addition of an external input signal does not improve the performance. Because the optimal design strategies always depend on the unknown system, one might argue that these results are not very practical. However, they are qualitatively important and provide further justification for the use of self-tuning regulators.

REFERENCES

Gustavsson, I., L. Ljung and T. Söderström (1981). Choice and effect of different feedback configurations. In Eykhoff, P. (Ed.) *Trends and Progress in System Identification*. Pergamon Press, Oxford.

Ljung, L. (1978). Convergence analysis of parametric identification methods. *IEEE Trans. Aut. Control*, AC-23, 770–783.

Ljung, L. (1985). Asymptotic variance expressions for identified black-box transfer functions models. *IEEE Trans. Aut. Control*, AC-30, 834–844.

Ljung, L. (1986). Parametric methods for identification of transfer functions of linear systems. In Leondes, C. T. (Ed.) *Advances in Control Vol. XXIV*. Academic Press, New York.

Ljung, L. (to appear). *System Identification — Theory For the User.* Prentice-Hall, Englewood Cliffs, New Jersey.

Ljung, L. and P. E. Caines (1979). Asymptotic normality of prediction error estimation for approximate system models. *Stochastics*, **3**, 29–46.

Mehra, R. K. (1974). Optimal input signals for parameter estimation in dynamic systems — survey and new results. *IEEE Trans. Aut. Control*, **AC-19**, 753–768.

Ng, T. S., G. C. Goodwin and T. Söderström (1977). Optimal experiment design for linear systems with input–output constraints. *Automatica*, **13**, 571–577.

Payne, R. L. (1974). Optimal experiment design for dynamic system identification. Ph.D. Thesis, Imperial College, London.

Söderström, T., L. Ljung and I. Gustavsson (1976). Identifiability conditions for linear multivariable systems operating under feedback. *IEEE Trans. Aut. Control*, **AC-21**, 837–840.

Wahlberg, B. and L. Ljung (1986). Design variables for bias distribution in transfer function estimation. *IEEE Trans. Aut. Control*, **AC-31**.

Yuan, Z. D. and L. Ljung (1985). Unprejediced optimal open loop input design for identification of transfer functions, *Automatica*, **21**, 697–708.

Yuan, Z. D. and Ljung (1984). Black-box identification of multivariable transfer functions — asymptotic properties and optimal input design. *Int. J. Control*, **40**, 233–256.