

Uniquely Identifiable State-space and ARMA Parametrizations for Multivariable Linear Systems*

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Uniquely identifiable 'overlapping' parametrizations for state-space and ARMA models are derived from a set of invariants defined on the Hankel matrix of Markov parameters.

Key Words—Multivariable systems; invariance; identifiability; identification; parametrizations; linear systems.

Abstract—Multivariable systems can be represented, in a uniquely identifiable way, either by canonical forms or by so-called overlapping forms. The advantage of the latter is that they do not require the *a priori* estimation of a set of structural invariants (e.g. Kronecker invariants). We show here how to define uniquely identifiable overlapping parametrizations for state-space and ARMA models. We show that these parametrizations are all related to a set of intrinsic invariants, which are obtained from the Markov parameters of the system. Different forms of overlapping ARMA parametrizations are derived and their properties discussed.

1. INTRODUCTION

AN IMPORTANT and widely studied problem in the theory of identification of multivariate stationary finite-dimensional stochastic processes is that of determining the structure of the state-space or ARMA model for that process such that the model parameters become uniquely identifiable. Two different lines of thought have been followed for this problem. The first idea is to use canonical (state-space or ARMA) forms (Luenberger, 1967; Denham, 1974; Rissanen, 1974; Guidorzi, 1981). To any finite dimensional process one can associate a canonical form in a unique way by specifying a selection procedure. Different selection procedures will lead to different canonical forms, but the parameters in any two canonical representations of

a given process are related by a bijective relationship. The structure of a canonical representation of a process is determined by a set of 'structural invariants' (e.g. the Kronecker invariants) which are again uniquely defined by the process and the selection procedure. The disadvantage when using canonical forms is that the estimation of those structural invariants is very critical: the parameter estimates are not consistent if the structural invariants have been wrongly estimated (Caines and Rissanen, 1974).

In recent years an alternative approach has been proposed, namely that of using 'overlapping parametrizations'. This concept was first suggested by Glover and Willems (1974), and further studied by Ljung and Rissanen (1976); Van Overbeek and Ljung (1982); Picci (1982); Rissanen (1981); Deistler and Hannan (1982); Wertz, Gevers and Hannan (1982); Gevers and Wertz (1982); Guidorzi and Beghelli (1982); Corrêa and Glover (1982). It has been recognized that the set of all finite dimensional systems can be represented through a finite number of uniquely identifiable parametrizations. Loosely speaking, a parametrization describing a given system is called 'uniquely identifiable' if the parametrized model is only able to describe the input-output behaviour of this system for a unique value of the parameter vector. To each of these uniquely identifiable parametrizations there corresponds a set of integers called 'structure indices'. Each of these parametrizations defines a representation, which is able to represent almost all finite dimensional systems of a specified order, each system can normally be represented by more than one such representation, and any two parametrizations for a given process are related by a linear transformation which corresponds to a coordinate transformation in Euclidean space; hence the use of the word 'overlapping' parametrizations. So far, most of the

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work on overlapping parametrizations has been on state-space representations. These were obtained as the result of a specific but rather *ad hoc* choice of a state-vector, which is taken as a basis of the space spanned by the predicted values of the observation process.

The objective of this paper is to give a tutorial and unifying presentation of overlapping parametrizations for both state-space and ARMA models. The unifying viewpoint is achieved by formulating the parametrization problem as one of defining equivalence classes on the set of all parametrized probability maps defined on the observation process. For a p -dimensional observation process of order n , this defines a quotient space parametrized by a set of $2np$ invariants, called 'intrinsic invariants', because they are independent of the choice of a particular finite-dimensional representation. The main contribution of this paper is to derive uniquely identifiable state-space and ARMA parametrizations from these $2np$ intrinsic invariants. While the overlapping state-space representations contain exactly the $2np$ intrinsic invariants, the overlapping ARMA forms contain in general more than $2np$ parameters. The parameters of the moving average part are in fact related by certain constraints. This may cause some input-output equations taken independently, to become noncausal if these constraints are not taken into account. Therefore, we also introduce two other overlapping ARMA parametrizations that guarantee causality of each input-output equation.

The outline of the paper is as follows. Section 2 sets the problem of canonical and overlapping parametrizations in the framework of invariants for the equivalence relation defined on the set of linear systems of minimal order n . In section 3, we show how to define the $2np$ intrinsic invariants. We also recall a result, proved in Wertz, Gevers and Hannan (1982), on the invariance of the determinant of the Fisher information matrix. The state-space and ARMA representations derived from the set of $2np$ intrinsic invariants are introduced in Section 4, while Section 5 studies the two other ARMA representations.

Finally we want to comment on the differences between using canonical forms and overlapping forms in identification applications. If canonical forms are used to model linear systems, then any given system can be represented by only one canonical form. The canonical parametrization is determined by the set of 'structural invariants', which must be estimated first from the data. This is a very critical procedure, as stated above. On the other hand, a canonical form often has slightly less than $2np$ parameters. If the same system is to be represented by an overlapping form, then any one of the finite number of overlapping parametrizations is

usually able to represent this system. Therefore, in an identification context, one can choose an arbitrary set of 'structure indices', estimate the parameters of the corresponding overlapping form in the coordinate space defined by these indices and, if numerical problems arise, change to another overlapping form using a coordinate transformation [see Van Overbeek and Ljung (1982) for an application of this idea]. The fact that no preliminary estimation of structural invariants is required with overlapping forms is a major advantage, even if this is at the expense of a few additional parameters to be estimated.

2. PARAMETRIZATION OF MULTIVARIATE PROCESSES

Representations

Assumption 2.1. From now on, we consider a real p -dimensional wide-sense stationary purely non deterministic full rank zero-mean Gaussian* stochastic process $\{y_t\}$ with rational spectrum $\Phi(z)$. We also assume that $\Phi(z)$ is positive definite on $|z| = 1$ and that the linear least-squares predictor of $\{y_t\}$ given the past history of the process is of full rank [see (3)].

Impulse response representation. It is well known that $\{y_t\}$ can be uniquely described by the so-called Wold decomposition (Hannan, 1970; Gevers and Anderson, 1981)

$$y_t = \sum_{i=0}^{\infty} H_i e_{t-i} = \mathbf{H}E^t \quad (1)$$

where $\{e_t\}$ is a p -dimensional white Gaussian sequence with positive definite covariance matrix Q , $H_0 = I_p$, H_i are $p \times p$ matrices called Markov parameters or impulse response matrices, and

$$\sum_0^{\infty} |H_i Q H_i^T| < \infty.$$

If $H(z)$ is defined as

$$H(z) \triangleq \sum_0^{\infty} H_i z^{-i},$$

then $H(z)$ is analytic in $|z| \geq 1$, the inverse $G(z) \triangleq H^{-1}(z)$ exists and is analytic in $|z| \geq 1$. Finally, \mathbf{H} and E^t denote

$$\mathbf{H} = [H_0 \quad H_1 \quad H_2 \dots], \quad E^t = \begin{bmatrix} e_t \\ e_{t-1} \\ e_{t-2} \\ \vdots \end{bmatrix}.$$

* The Gaussian assumption is made for simplicity; we shall see later that all results hold, up to second order, for non-Gaussian processes.

The representation (1) is often called the *innovations representation*. It is causally invertible [since the inverse filter $G(z)$ is stable]: the process $\{e_t\}$, called innovation process, can be obtained from $\{y_t\}$ as follows:

$$e_t = \sum_{i=0}^{\infty} G_i y_{t-i} = \mathbf{G} Y^t \tag{2}$$

where \mathbf{G} and Y^t are defined similarly to \mathbf{H} and E^t above.

Assumption 2.2. We shall limit ourselves in this paper to causally invertible (i.e. innovations) representations.

We now define the *predictor space* \hat{Y}_t

$$\hat{Y}_t \triangleq \text{span}\{\hat{y}_{t+k}^i | i = 1, \dots, p; k = 1, 2, \dots\} \tag{3}$$

where \hat{y}_{t+k}^i is the linear least-squares prediction of the i th component of y_{t+k} given Y^t . Since $\{y_t\}$ has a rational spectrum, \hat{Y}_t has finite dimension. Let n be this dimension; then n is called *the order of the process* $\{y_t\}$. It is the order of any minimal realization of $\{y_t\}$. We shall be concerned in this paper with two classes of minimal representations for $\{y_t\}$.

Minimal state-space representation. The process $\{y_t\}$ of order n admits a minimal state-space representation

$$\begin{aligned} x_{t+1} &= Fx_t + Ke_t \\ y_t &= Hx_t + e_t \end{aligned} \tag{4}$$

with the following properties

- (a) x_t is n -dimensional and e_t is as in (1);
- (b) F, K, H are constant matrices of appropriate dimensions;
- (c) F and $F - KH$ have all their eigenvalues strictly inside the unit circle;
- (d) (H, F, K) is of minimal order n , i.e.

$$\text{rank} \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} = n, \quad \text{rank} [K \quad FK \dots F^{n-1}K] = n.$$

Definition 2.1a. The set of all state-space representations (4) with the properties (a)–(d) will be denoted by S_n .

Minimal ARMA representation. The process $\{y_t\}$ of order n admits a minimal ARMA representation

$$A(z)y_t = B(z)e_t \tag{5}$$

with the following properties

- (a) e_t is as in (1);
- (b) $A(z)$ and $B(z)$ are $p \times p$ polynomial matrices in z ;
- (c) $\det A(z) \neq 0$ and $\det B(z) \neq 0$ for $|z| \geq 1$;
- (d) $\lim_{z \rightarrow \infty} A^{-1}(z)B(z) = I_p$;
- (e) $\deg \det A(z) = n$;
- (f) the polynomial matrices $A(z)$ and $B(z)$ are left coprime, i.e. if $A(z) = M(z)A^*(z)$ and $B(z) = M(z)B^*(z)$ for some polynomial matrices $M(z), A^*(z), B^*(z)$, then $M(z)$ must be unimodular [$\det M(z) = \text{constant} \neq 0$].

Definition 2.1b. The set of all ARMA representations (5) with the properties (a)–(f) will be denoted by S_n^* .

Identifiability (uniqueness oriented)

We shall now define precisely what we mean by (uniqueness oriented) identifiability. First we introduce the concept of indistinguishability, following Picci (1982).

Definition 2.2. Let θ be the vector of parameters in either the triple (H, F, K) of representation (4) or the pair $[A(z), B(z)]$ of representation (5), and let Q be the covariance matrix of $\{e_t\}$. Two parameter pairs (θ_1, Q_1) and (θ_2, Q_2) are called *indistinguishable* if and only if

$$p(Y_0^N; \theta_1, Q_1) = p(Y_0^N; \theta_2, Q_2), \quad \forall Y_0^N \text{ and } \forall N > 0 \tag{6}$$

where Y_0^N is the vector made up of $\{y_0, y_1, \dots, y_N\}$.

Remark 2.1. For Gaussian processes, (6) can be replaced by

$$R_y(k; \theta_1, Q_1) = R_y(k; \theta_2, Q_2), \quad \forall k \geq 0 \tag{7}$$

where $R_y(k; \theta_i, Q_i)$ is the covariance function of the process $\{y_t\}$ generated by model i . Alternatively, all the results of this paper will hold for non-Gaussian processes, provided the notion of identifiability is replaced by 'second-order identifiability', by substituting (7) for (6) in Definition 2.2.

Definition 2.3. A set of parametric representations $\mathcal{M}(\theta)$ of the process $\{y_t\}$ is uniquely identifiable if it does not contain indistinguishable parameter pairs.

For state-space models, say, $\mathcal{M}(\theta)$ denotes a set of models (4) with a particular parametrization of H, F, K ; the set ranges over all values of θ for which property d holds. The identifiability problem arises in the following terms: if, say, a model (4) is sought, then θ can contain up to $n^2 + 2np$ parameters, while Q ranges over the family of $p \times p$ symmetric positive definite matrices, and it turns out that there are in

general many indistinguishable parameter pairs (θ_i, Q_i) .

With the Assumptions 2.1 and 2.2, it can be shown that (θ_1, Q_1) and (θ_2, Q_2) are indistinguishable if and only if

$$Q_1 = Q_2 \text{ and } H_i(\theta_1) = H_i(\theta_2), \quad i = 0, 1, 2, \dots (8)$$

where $H_i(\theta)$ is defined as follows: for a state-space representation: $\theta = (H, F, K)$ and

$$H_i(\theta) = HF^{i-1}K, \quad H_0 = I_p \quad (9)$$

for an ARMA representation: $\theta = [A(z), B(z)]$ and

$$H(z) = \sum_{i=0}^{\infty} H_i(\theta)z^{-i} = A^{-1}(z)B(z). \quad (10)$$

By the previous discussion, Definition 2.3 is equivalent with

Definition 2.3'. A set of parametric representations $\mathcal{M}(\theta)$ of the process $\{y_i\}$ is uniquely identifiable if two different values of θ correspond to two different sequences of Markov parameters $H_i(\cdot)$.

Because of (8), we shall drop the explicit dependence of $p(Y_0^N)$ on Q in the following. The indistinguishability concept induces an equivalence relation on the sets S_n and S_n^* , which we shall denote by the symbol \sim . Let (H, F, K) and $(\bar{H}, \bar{F}, \bar{K})$ be two elements of S_n parametrized by θ and $\bar{\theta}$, respectively and let $[A(z), B(z)]$ and $[\bar{A}(z), \bar{B}(z)]$ be corresponding elements of S_n^* . It follows from (8)–(10) that

$$\theta \sim \bar{\theta} \Leftrightarrow H_i(\theta) = H_i(\bar{\theta}), \quad i = 0, 1, \dots \quad (11)$$

$$\Leftrightarrow \bar{H} = HT, \bar{F} = T^{-1}FT, \bar{K} = T^{-1}K \quad (12)$$

$$\Leftrightarrow \bar{A}(z) = M(z)A(z), \bar{B}(z) = M(z)B(z) \quad (13)$$

for some nonsingular matrix T and for some unimodular matrix $M(z)$. The matrix triples (H, F, K) and $(\bar{H}, \bar{F}, \bar{K})$ [the polynomial matrix pairs $[A(z), B(z)]$ and $[\bar{A}(z), \bar{B}(z)]$] are called *equivalent* if the relations (12) [or (13)] hold.

In order to achieve identifiability, we have to find a *reparametrization* of the family $p(Y_0^N; \theta)$, or of the set of models (4) and (5), in such a way that this reparametrized family does not contain indistinguishable parameter pairs. We will express this reparametrization in terms of invariants (MacLane and Birkhoff, 1968; Rissanen, 1974; Guidorzi, 1981). Some basic facts about invariants are summarized in the appendix.

Canonical forms and overlapping forms

The reparametrization is achieved through a factorization of the map $p: \theta \rightarrow p(\cdot; \theta)$ in the following way:

$$\begin{array}{ccc}
 \mathcal{S}'_n & \xrightarrow{P} & P \\
 \downarrow f & & \uparrow \hat{p} \\
 & X_n &
 \end{array} \quad (14)$$

Here \mathcal{S}'_n is the set of all systems of order n , p is the map defined by the probability law, P is the image of p ; the set X_n contains the image of f . The functions $f: \mathcal{S}'_n \rightarrow X_n$ and $\hat{p}: X_n \rightarrow P$ must satisfy the following conditions:

$$(a) \quad \forall \theta \in \mathcal{S}'_n, \quad \xi = f(\theta) \text{ is finite dimensional} \quad (15a)$$

$$(b) \quad p(\cdot; \theta) = \hat{p}(\cdot; f(\theta)), \quad \forall \theta \in \mathcal{S}'_n \quad (15b)$$

$$(c) \quad \hat{p}(\cdot; \xi_1) = \hat{p}(\cdot; \xi_2) \Leftrightarrow \xi_1 = \xi_2. \quad (15c)$$

The function f consists of a finite number of scalar components, say f_1, \dots, f_k , which form a complete system of invariants (see appendix) for the equivalence relation (11), since by (15b) and (15c)

$$\theta_1 \sim \theta_2 \Leftrightarrow \hat{p}(\cdot; f(\theta_1)) = \hat{p}(\cdot; f(\theta_2)) \Leftrightarrow f(\theta_1) = f(\theta_2). \quad (16)$$

We would also like f to be surjective because in this case there is a bijection between X_n and the quotient spaces $S_{n/\sim}$ or $S_{n/\sim}^*$.

It can be shown that the class of all systems admitting a minimal realization of order n , is an analytic manifold of dimension $2np$ (Clark, 1976; Kalman, 1974; Dunsmuir and Hannan, 1976). Hence, every element of $S_{n/\sim}$ can be represented by a set of $2np$ numbers in an appropriate coordinate system. However a crucial point is that, when $p > 1$, no single parametrization is able to describe this manifold (Hazewinkel, 1977). In other words, in order to cover $S_{n/\sim}$, we need several local coordinate systems, which may or may not overlap, as we shall see later. Each point in one of these local coordinate systems is determined by specifying at most $2np$ coordinates (hence the dimension of the manifold), but in order to determine an element of $S_{n/\sim}$, we also need to specify in which coordinate system it is described. The local coordinate systems can be determined by specifying the values of p integer-valued indices which will be called *structure indices*. (Recall that p is the number of components of y_i .)

At this point, there are two different ways to proceed further, both leading to identifiable parametrizations. The first way is to cover $S_{n/\sim}$ by non-overlapping local coordinate systems; this leads to so-called *canonical forms*. In this case, the p integer-valued numbers which specify the local coordinate system constitute an *invariant* for the equivalence relation \sim . They have to be computed from the impulse response sequence before the parametrization can be defined (see next section). The advantage of these canonical forms is that sometimes less than $2np$ coordinates will be necessary to parametrize a process. On the other hand, a major drawback is that the determination of the canonical structure indices, which are invariants here, is difficult, and it is impossible to parametrize a process with wrongly estimated structure in-

variants. A procedure for the estimation of the structural invariants has been proposed by Guidorzi (1981).

The alternative is to cover $\mathcal{S}_{n/\sim}$ with overlapping coordinate systems; this leads to so-called *overlapping forms*. In this case, the structure indices are no more invariants for the equivalence relation \sim . A given process can be represented within several coordinate systems. In general each of these local coordinate systems covers a subset of $\mathcal{S}_{n/\sim}$ which is dense in $\mathcal{S}_{n/\sim}$. This means that almost any choice of structure indices will normally produce a parametrization that allows an exact description of the process. As can be guessed, advantages become disadvantages and vice versa. Overlapping parametrizations always need $2np$ parameters to describe a point in a local coordinate system, but if the chosen structure indices render the parametrization ill-conditioned, it will be easy to move to another overlapping parametrization by a coordinate transformation (van Overbeek and Ljung, 1982).

In the canonical form approach, we shall define a complete system of independent invariants as follows:

$$f: \mathcal{S}_n \rightarrow \mathbb{N}^p \times \mathbb{R}^k \quad (17)$$

where n , the order of the process, will be equal to the sum of the p integer-valued invariants n_1, \dots, n_p , and k , the number of real-valued invariants, depends on the values of the structure invariants, but in any case is less than or equal to $2np$. In the overlapping form approach, we first have to specify p integer valued structure indices n_1, \dots, n_p , which are *not* invariants, and then we can define a complete system of independent invariants as follows:

$$f_{n_1, \dots, n_p}: \mathcal{S}_{n; n_1, \dots, n_p} \rightarrow \mathbb{R}^{2np} \quad (18)$$

where $n = n_1 + \dots + n_p$ as before, and $\mathcal{S}_{n; n_1, \dots, n_p}$ is the subset of \mathcal{S}_n which can be parametrized within a local coordinate system indexed by n_1, \dots, n_p . (A precise definition will be given in Section 3.)

In the next section we shall see how to define structure indices, whether they be invariants or not, and show how a choice of structure indices n_1, \dots, n_p defines a set of $2np$ (or less in the case of canonical forms) invariants.

3. STRUCTURE INDICES AND INTRINSIC INVARIANTS

From (11) it is clear that the Markov parameters are a complete system of invariants for the equivalence relation \sim . The problem is that the impulse response sequence is infinite. But since we have assumed that the process $\{y_t\}$ has a rational spectrum, the Hankel matrix of impulse responses

$$\mathcal{H}_{\infty, \infty} = \begin{bmatrix} H_1 H_2 H_3 \dots \\ H_2 H_3 H_4 \dots \\ H_3 H_4 H_5 \dots \\ \vdots \\ \vdots \end{bmatrix} \quad (19)$$

has a finite rank, n , which is the order of the process. Hence one can extract n independent rows of this infinite matrix and reconstruct all Markov parameters from a finite portion of these rows.

From (4) and (7) we can write the linear least-squares k -step ahead predictor $\hat{y}_{t+k/t}$ as follows:

$$\hat{y}_{t+k/t} = \sum_{i=k}^{\infty} H_i e_{t-i+k}. \quad (20)$$

Therefore

$$\hat{Y}_t = \begin{bmatrix} \hat{y}_{t+1/t} \\ \hat{y}_{t+2/t} \\ \hat{y}_{t+3/t} \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} H_1 H_2 H_3 \dots \\ H_2 H_3 H_4 \dots \\ H_3 H_4 H_5 \dots \\ \vdots \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} e_t \\ e_{t-1} \\ e_{t-2} \\ \vdots \\ \vdots \end{bmatrix} = \mathcal{H}_{\infty, \infty} E'. \quad (21)$$

Thus the structure selection problem can be thought of as the selection of n independent rows of the matrix $\mathcal{H}_{\infty, \infty}$ or, equivalently, of n independent components of the vector \hat{Y}_t , which form a basis of the prediction space spanned by \hat{Y}_t . To any such selection we shall associate a *multi-index* $i = (i_1, \dots, i_n)$, where the numbers i_1, \dots, i_n , arranged in increasing order, are the indices of the rows of $\mathcal{H}_{\infty, \infty}$ (or the components of \hat{Y}_t) that form the basis. Of course there is an infinite number of such selections. Most often, one introduces restrictions on these selections in order to obtain a sparse parametrization of the state-space or ARMA equations for the process (see below). These restrictions will reduce the number of possible selections. An obvious choice of restrictions results from the full rank assumption on the predictor $\hat{y}_{t/t-1}$ and the particular structure of the Hankel matrix.

By the full rank assumption, the first p components of the prediction vector \hat{Y}_t , and hence the first rows of $\mathcal{H}_{\infty, \infty}$, are linearly independent. From the Hankel structure of $\mathcal{H}_{\infty, \infty}$ it is also clear that if row i is in the linear span of the preceding ones, then so is row $i+p$. Hence the following definition.

Definition 3.1. A multi-index $i = (i_1, \dots, i_n)$ defining n independent rows of $\mathcal{H}_{\infty, \infty}$ is called *nice* if it fulfils the following conditions:

Condition 1. $1, 2, \dots, p \in i$

Condition 2. If $j \in i$, $j > p$, then $j-p \in i$. The corresponding selection is called a *nice selection*.

All nice multi-indices correspond to a choice of the basis inside the first $n - p + 1$ block rows of $\mathcal{H}_{\infty, \infty}$. For given n and p , there are only a finite number of possible nice multi-indices. For example, if $n = 3$ and $p = 2$, then there are only two nice multi-indices: $\mathbf{i}_1 = (1, 2, 3)$ and $\mathbf{i}_2 = (1, 2, 4)$. In general, there are $\binom{n-1}{p-1}$ nice multi-indices for a process of dimension p and of order n . One could be even more restrictive and select the first n independent rows of the Hankel matrix $\mathcal{H}_{\infty, \infty}$. Let us state that as Condition 3:

Condition 3. $\mathbf{i} = (i_1, \dots, i_n)$, where i_1, \dots, i_n are the first linearly independent rows of $\mathcal{H}_{\infty, \infty}$.

From the assumptions we have made, Condition 3 obviously implies Conditions 1 and 2. In the sequel Condition 3 will not be imposed, unless otherwise specified. We now show how to define structure indices from these multi-indices.

Definition 3.2. Let $\mathbf{i} = (i_1, \dots, i_n)$ be a nice multi-index defining a basis for the rows of $\mathcal{H}_{\infty, \infty}$. For $k = 1, \dots, p$, let n_k be the least natural number such that $(k + n_k p) \notin \mathbf{i}$. Then n_1, \dots, n_p are called the 'structure indices' corresponding to that basis; they specify which rows of $\mathcal{H}_{\infty, \infty}$ are taken in the basis. Note that $\sum_{i=1}^p n_i = n$ and that $n_i \geq 1, i = 1, \dots, p$.

Definition 3.3. The set of structure indices $\{n_1, \dots, n_p\}$ with $\sum_1^p n_i = n$ is called *generic* if

$$n_1 = n_2 = \dots = n_r = \left\lfloor \frac{n}{p} \right\rfloor + 1$$

and

$$n_{r+1} = \dots = n_p = \left\lfloor \frac{n}{p} \right\rfloor$$

for some r , where $\left\lfloor \frac{n}{p} \right\rfloor$ is the integer part of $\frac{n}{p}$.

To any process, there corresponds only one multi-index obeying Condition 3. The corresponding structure indices are invariants and are the same as the output Kronecker invariants obtained by searching for the first n independent rows of the observability matrix.

Definition 3.4. $\mathcal{S}_{n, n_1, \dots, n_p}$ is the set of all n th order systems for which the rows of $\mathcal{H}_{\infty, \infty}$ indexed by $1, 1 + p, \dots, 1 + (n_1 - 1)p; 2, \dots, 2 + (n_2 - 1)p; \dots; p, \dots, p + (n_p - 1)p$ are linearly independent.

$\mathcal{S}_{n, n_1, \dots, n_p}$ is a proper subset of \mathcal{S}_n . Now for most n th order processes of dimension p , all selections obeying Conditions 1 and 2 will correspond to

independent rows. Therefore $\mathcal{S}_{n, n_1, \dots, n_p}$ is also dense in \mathcal{S}_n . This leads to the following definition.

Definition 3.5. A process of dimension p and order n is called *generic* if all nice selections of the rows of $\mathcal{H}_{\infty, \infty}$ (and of the components of \hat{Y}_t) yield a basis of the row space of $\mathcal{H}_{\infty, \infty}$ (and of the prediction space). Such a process belongs to

$$\sum_n = \bigcap_{(n_1, \dots, n_p) \in J_{p, n}} \mathcal{S}_{n, n_1, \dots, n_p}$$

where

$$J_{p, n} = \left\{ (n_1, \dots, n_p) \mid n_i \geq 1, \sum_1^p n_i = n \right\}.$$

As an example of a non-generic process, take a process for which the H and F matrices of a state-space representation have the following form

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \times & \times & 0 \\ 0 & 0 & 1 \\ \times & \times & \times \end{bmatrix}.$$

It is easy to see that the selection $\mathbf{i} = (1, 2, 3)$ (i.e. $n_1 = 2, n_2 = 1$) does not yield a basis.

Consider now an element of $\mathcal{S}_{n, n_1, \dots, n_p}$ specified by its Hankel matrix $\mathcal{H}_{\infty, \infty}$. We shall construct a complete system of $2np$ independent invariants for this process, i.e. a reparametrization of this process using $2np$ parameters.

Let H^i be the i th block of p rows of $\mathcal{H}_{\infty, \infty}$ (e.g. $H^2 = [H_2 H_3 H_4 \dots]$) and let

$$H^i = \begin{bmatrix} h_{1i} \\ h_{2i} \\ \vdots \\ h_{pi} \end{bmatrix} \tag{22}$$

where h_{ki} are rows of infinite length. Since $\mathcal{H}_{\infty, \infty}$ is an element of $\mathcal{S}_{n, n_1, \dots, n_p}$, the rows $h_{11}, \dots, h_{1n_1}; h_{21}, \dots, h_{2n_2}; \dots; h_{p1}, \dots, h_{pn_p}$ form a basis. Therefore the rows $h_{1(n_1+1)}, \dots, h_{p(n_p+1)}$ can be expressed in a unique way as linear combinations of these basis rows

$$h_{i(n_i+1)} = \sum_{j=1}^p \sum_{k=1}^{n_j} \alpha_{ijk} h_{jk}, \quad i = 1, \dots, p. \tag{23}$$

These relations define np scalar numbers α_{ijk} . Now denote by $h_{ij}(k)$ the k th element of row h_{ij} . Then the $2np$ numbers

$$\{\alpha_{ijk}, k = 1, \dots, n_j, i, j = 1, \dots, p; h_{ij}(k), i = 1, \dots, p; j = 1, \dots, n_i; k = 1, \dots, p\} \tag{24}$$

completely coordinatize $\mathcal{S}_{n, n_1, \dots, n_p}$, i.e. they map that set in a one to one manner on Euclidean space of dimension $2np$. The impulse response sequence $H_1,$

H_2, H_3, \dots is completely specified by the p structure indices and these $2np$ numbers. The $2np$ numbers constitute a complete system of independent invariants which will be called *intrinsic invariants* of the process. In the notation of Section 2

$$\int_{n_1, \dots, n_p} \mathcal{S}_{n_1, \dots, n_p} \rightarrow \mathbb{R}^{2np} \quad (25)$$

$$:(H_1, H_2, \dots) \rightarrow \{\alpha_{ijk}, h_{ij}(k)\}.$$

The word *intrinsic* is used because the invariants are defined from the unique infinite impulse response representation, and not from a finite state space or ARMA model. [Recall (11)–(13).]

Suppose now that we take a canonical form approach, i.e. the local coordinate system in which the system is parametrized is defined by the structure invariants of the system rather than by an arbitrary set of structure indices. In this case, because the basis is formed by the *first* n independent rows, formula (23) is replaced by

$$h_{i(n_i+1)} = \sum_{j=1}^p \sum_{k=1}^{n_{ij}} \alpha_{ijk} h_{jk}, \quad i = 1, \dots, p \quad (26)$$

where $n_{ij} = n_i$ for $i = j$

$$n_{ij} = \min(n_i + 1, n_j), \quad \text{for } i > j \quad (27)$$

$$n_{ij} = \min(n_i, n_j), \quad \text{for } i < j.$$

The effect of this change of indices is that in general, there will be less than np parameters α_{ijk} . We illustrate this with an example. Consider a process $\{y_t\}$ of dimension 2 ($p = 2$) and of order 3 ($n = 3$). We know that there are only two possible nice multi-indices: $\mathbf{i}_1 = (1, 2, 3)$ and $\mathbf{i}_2 = (1, 2, 4)$. If \mathbf{i}_2 has been obtained by a procedure selecting any three independent rows under Conditions 1 and 2, we will write row h_{12} as

$$h_{12} = \alpha_{111}h_{11} + \alpha_{121}h_{21} + \alpha_{122}h_{22}. \quad (28)$$

But if \mathbf{i}_2 has been obtained by a procedure selecting the *first* 3 independent rows, i.e. if \mathbf{i}_2 obeys Condition 3, then

$$h_{12} = \alpha_{111}h_{11} + \alpha_{121}h_{21}. \quad (29)$$

There is one parameter less in (29) than in (28).

Remark: There are other ways of defining invariants from the Markov parameters. When studying ARMA representations, we will consider two other systems of invariants. Note also that in Bosgra and van der Weiden (1980) a complete system of independent invariants has been defined with $2np$ entries of the Markov parameters H_k . However, their definition requires the selection of nice multi-indices for both the rows and the columns of the Hankel matrix $\mathcal{H}_{\infty, \infty}$.

The use of overlapping representations raises an important question: since a given process can be

represented within several different representations (corresponding to different choices of structure indices), is any one of these representations ‘better’ than the others? The word ‘better’ can of course have different meanings, depending on the objective of the identification. This question has been studied by different authors (Ljung and Rissanen, 1976; Wertz, Gevers and Hannan, 1982) and we refer to Wertz (1982) for a discussion on this subject. In Wertz, Gevers and Hannan (1982), the following result is proved, which can be regarded as a partial answer to the above question.

Theorem 3.1. Let $\{y_t\}$ be a process of dimension p and order n which belongs to both $\mathcal{S}_{n, n_1, \dots, n_p}$ and $\mathcal{S}_{n, n_1^*, \dots, n_p^*}$, where (n_1, \dots, n_p) and (n_1^*, \dots, n_p^*) are two different sets of structure indices. Let $[\alpha_{ijk}, h_{ij}(k)]$ and $[\alpha_{ijk}^*, h_{ij}^*(k)]$ be the corresponding intrinsic invariants. Then, the determinants of the information matrices corresponding to these two parametrizations are identical.

Recall that the information matrix M_θ is defined as

$$M_\theta = E_{Y/\theta} \left\{ \left(\frac{\partial \log p(Y/\theta)}{\partial \theta} \right)^T \left(\frac{\partial \log p(Y/\theta)}{\partial \theta} \right) \right\}$$

where $\theta = [\alpha_{ijk}, h_{ij}(k)]$, and that under certain assumptions the inverse of this matrix is equal to the asymptotic value of the covariance matrix of the estimation errors when using prediction error methods. This justifies the use of some scalar measure of this information matrix to discriminate between various structures.

In the next section, we will relate the intrinsic invariants to invariants of state-space and ARMA models. It follows from the construction of the factorization (14) and the developments of this section that we can now give a more precise formulation of Definition 2.3 for state-space or ARMA models.

Definition 3.6. A representation such as (4) or (5) is a uniquely identifiable parametrization for a process belonging to $\mathcal{S}_{n, n_1, \dots, n_p}$ if each element of $\mathcal{S}_{n, n_1, \dots, n_p}$ can be represented in this parametrization by a unique set of parameters.

We are now ready to define uniquely identifiable state-space and ARMA models.

4. OVERLAPPING STATE-SPACE AND ARMA REPRESENTATIONS

In this section, we shall first give an overlapping state space representation whose free parameters are directly obtained from the intrinsic invariants (24). Then we derive an ARMA representation which is related to this state-space model.

(c) Using again the nice multi-index i_1 we now obtain the following representation 5.2

$$\bar{A}(z) = \begin{bmatrix} z^3 - z^2 + 1 & 2 \\ 6z^2 - z + 2 & z^3 - 4 \end{bmatrix},$$

$$\bar{B}(z) = \begin{bmatrix} z^3 - 3z + 4 & z^2 + z \\ 6z^2 + 6z - 6 & z^3 - z^2 + 2z - 2 \end{bmatrix}.$$

The corresponding time-domain model is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y_{t+3} + \begin{bmatrix} -1 & 0 \\ 6 & 0 \end{bmatrix} y_{t+2} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} y_{t+1} + \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} y_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e_{t+3} + \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} e_{t+2} + \begin{bmatrix} -3 & 0 \\ 6 & 2 \end{bmatrix} e_{t+1} + \begin{bmatrix} 4 & 2 \\ -6 & -2 \end{bmatrix} e_t.$$

Again, the free parameters are bold. The relation between $[\bar{A}(z), \bar{B}(z)]$ and $[A(z), B(z)]$ is easily established. Again, the transformation matrix is not unimodular

$$\bar{A}(z) = \begin{bmatrix} 1 & 0 \\ -z - 2 & z^2 + 2 \end{bmatrix} A(z),$$

$$\bar{B}(z) = \begin{bmatrix} 1 & 0 \\ -z - 2 & z^2 + 2 \end{bmatrix} B(z).$$

It is left to the reader to check that these three representations verify all the properties described above. Notice in particular that, with the first model, the second component of y_t, y_t^2 , is the output of an apparently noncausal relationship

$$y_{t+1}^2 = -y_{t+2}^1 - y_{t+1}^1 + e_{t+2}^1 + 2e_{t+1}^1 + 2e_{t+1}^2 + e_t^1 + e_t^2.$$

This apparent noncausality disappears with Representations 5.1 and 5.2.

6. CONCLUSIONS

To obtain uniquely identifiable models for multivariable systems, one must construct a factorization of the input-output map (or of the probability map in the case of stochastic processes) in such a way that each system is represented, in an appropriate quotient space, by a unique set of local coordinates called invariants. We have shown that there are basically two ways to obtain these invariants. In the canonical form approach, the process (or the system) determines which local coordinate system is used, via a set of structural invariants which must therefore be estimated. In the overlapping form approach, the choice of a local coordinate system is determined by a set of structure indices which can be chosen almost at will, provided they obey some 'nice' properties.

In this paper we have constructed uniquely identifiable overlapping state-space and ARMA

models. We have shown that, once an appropriate set of structure indices has been chosen, uniquely identifiable parametrizations in either state-space or ARMA form are all related to a set of $2np$ 'intrinsic invariants', which are determined directly from the Hankel matrix of Markov parameters. In the state-space representations, the free parameters are exactly the $2np$ intrinsic invariants, put in appropriate positions. The ARMA forms on the other hand contain more than $2np$ parameters: they are, however, functions of the $2np$ intrinsic invariants. We have presented three alternative uniquely identifiable overlapping ARMA forms, and we have discussed their properties. Asymptotically, the overlapping forms corresponding to different sets of structures indices will give the same value to the determinant of the Fisher information matrix. This means that the determinant of the covariance matrix of the parameter estimates will asymptotically be identical, whichever overlapping parametrization is used.

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APPENDIX: SOME BASIC PROPERTIES OF INVARIANTS

In this appendix we briefly review some basic definitions and results about invariants. These are all taken from Guidorzi (1981).

Definition A.1

Let X be a set and let \sim be an equivalence relation on X . Let S be a second set. A function $f: X \rightarrow S$ is called an *invariant* for \sim when

$$x \sim y \Rightarrow f(x) = f(y). \tag{A1}$$

It is called a *complete invariant* for \sim when

$$x \sim y \Leftrightarrow f(x) = f(y). \tag{A2}$$

Hence the invariant f is complete if the map $f: X \rightarrow S$ is injective.

Definition A.2.

A set f_1, \dots, f_k of functions $f_i: X \rightarrow S_i$ is called a *complete system of invariants* for \sim when the function $f: (f_1, \dots, f_k): X \rightarrow S_1 \times S_2 \times \dots \times S_k$ is a complete invariant for \sim .

Definition A.3

A set of invariants f_1, \dots, f_k is called *independent* when the associated invariant $f = (f_1, \dots, f_k)$ is surjective. Definition A.3 implies in particular that no invariant f_i can be expressed as a function of the other f_j , but the definition is stronger than that. Two important results about independent invariants are as follows. The proof can be found in Guidorzi (1981).

Property A.1

Let $f = (f_1, \dots, f_k): X \rightarrow S = S_1 \times \dots \times S_k$ be a complete set of independent invariants for an equivalence relation \sim on X . Then every other invariant for \sim can be uniquely computed from f .

Property A.2

Let $f: X \rightarrow S$ be a complete set of independent invariants for \sim . If $h: S \rightarrow T$ is a bijection, then $g = h \circ f: X \rightarrow T$ is also a complete set of independent invariants for \sim . Finally, we give the definition of canonical forms and another result which is used in the paper.

Definition A.4

Let \sim be an equivalence relation on X . A set of canonical forms for \sim is a subset C of X such that to each $x \in X$ there corresponds exactly one $c \in C$ such that $x \sim c$. This element c is the canonical form of x . The function $g: X \rightarrow C$ thus defined is a complete surjective invariant for \sim .

Property A.3

Let C be a set of canonical forms for an equivalence relation \sim on X , and let f be a complete system of independent invariants for \sim . Then there exists a unique bijection between C and the image of f .

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