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# On the minimality of feedback realizations†

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Given minimal dimension realizations for two linear systems with exogenous and feedback inputs, we derive necessary and sufficient conditions under which the feedback interconnection of these two realizations is itself minimal. The conditions involve poles and zeros of certain transfer function matrices, and are relevant in problems of identification of feedback systems.

## 1. Introduction

Consider the discrete time feedback system illustrated in Fig. 1 which results from the feedback interconnection of two linear models

$$M1: \ y_i = F(z)u_i + G(z)w_i \tag{1}$$

$$M2: \ u_i = H(z)y_i + K(z)v_i \tag{2}$$

with $y_i \in \mathbb{R}^p$ and $u_i \in \mathbb{R}^m$. $F(z)$, $G(z)$, $H(z)$, $K(z)$ are proper real rational transfer function matrices. This arrangement occurs commonly in problems of

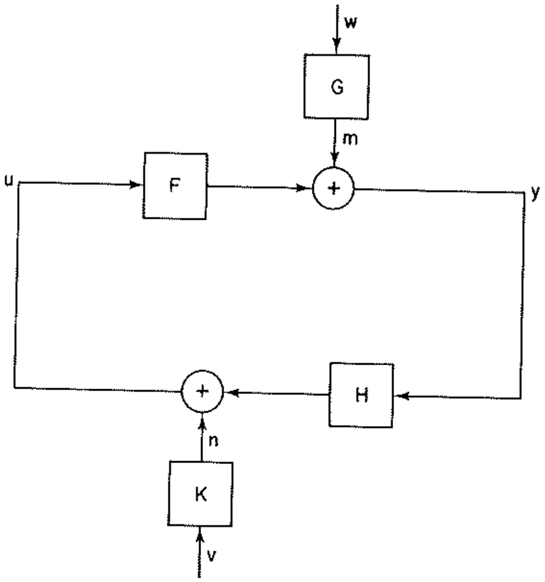


Figure 1. Basic feedback system.

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identifying feedback processes: the processes $\{y_i\}$ and $\{u_i\}$ can be thought of as the output and the input of a linear feedback process, and the processes $\{m_i\}$ and $\{n_i\}$ can be thought of as noises acting on $\{y_i\}$ and $\{u_i\}$, in which case $\{w_i\}$ and $\{v_i\}$ are white noise sources and $G(z)$ and $K(z)$ shaping filters. Of course, $\{w_i\}$ and $\{v_i\}$ can also be deterministic inputs to the processes modelled by $M1$ and $M2$.

We shall be concerned with the following problems. Assuming that $M1$ and $M2$ are described by two minimal (state space or matrix fraction description) realizations, under what conditions is the state variable realization naturally induced in forming the feedback interconnection of these two realizations itself minimal? Equivalently, when is the McMillan degree of the interconnection the sum of the McMillan degrees of $M1$ and M2?

The conditions will be on the poles and zeros of $F(z)$, $G(z)$, $H(z)$, $K(z)$. Most papers dealing with feedback interconnected models as shown in Fig. 1 have been concerned with the stability of such feedback models (see, for example, Desoer and Chan 1975, 1976, Callier *et al.* 1978, Anderson and Gevers 1981) or with the properties of jointly stationary stochastic feedback processes (see, for example, Gevers and Anderson 1981, Caines and Chan 1975) but questions of minimality have turned out to be relevant (Gevers and Anderson 1981). In Callier and Nahum (1975) the observability and controllability (and hence the minimality) of feedback interconnected systems were studied for the special case where $m=0$ and $K(z)=I$ in Fig. 1.

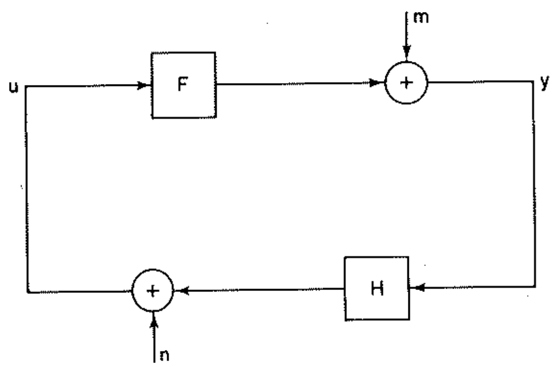


Figure 2. Specialized version of scheme of Fig. 1.

We shall consider here three situations, which will be studied in §§ 2, 3 and 4, respectively. First we present a very simple result for the special case represented by Fig. 2. In this case we show that the McMillan degree of the transfer function matrix that links $(m, n)$ to $(y, u)$ is the sum of the McMillan degrees of $F(z)$ and $H(z)$. In § 3, we study the configuration of Fig. 1 for the special case where all the signals are scalar. We derive a set of necessary and sufficient conditions for the McMillan degree of the transfer function matrix linking $(w, v)$ to $(y, u)$ to be the sum of the McMillan degrees of $M1$ and $M2$. In § 4, we derive necessary and sufficient conditions for this to hold for the same arrangement, save that $\{y_i\}$ and $\{u_i\}$ are vector processes.

## 2. Notations and a preliminary result

Consider the set up of Fig. 1, with $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$ with $r \geqslant p$, and $v \in \mathbb{R}^s$ with $s \geqslant m$. We shall consider polynomial matrix fraction descriptions (MFDs) for $F(z)$, $G(z)$, $H(z)$, $K(z)$

(1) left coprime MFDs denoted

$$F = D_F^{-1} N_F, \quad G = D_G^{-1} N_G, \quad H = D_H^{-1} N_H, \quad K = D_K^{-1} N_K \tag{3}$$

(2) and right coprime MFDs denoted

$$F = \bar{N}_F \bar{D}_F^{-1}, \quad G = \bar{N}_G \bar{D}_G^{-1}, \quad H = \bar{N}_H \bar{D}_H^{-1}, \quad K = \bar{N}_K \bar{D}_K^{-1} \tag{4}$$

Here the matrices $D_F, N_F, \ldots, \bar{N}_K, \bar{D}_K$ are polynomial matrices of appropriate dimensions in the indeterminate $z$, with the denominator matrices $D_F, D_G, \ldots, \bar{D}_K$ square.

We use the usual definitions for the poles and zeros of a transfer matrix (see, for example, Desoer and Schulman 1974): $z_0$ is a pole of $F(z)$ if and only if $\det D_F(z_0) = 0$; $z_0$ is a zero of $F(z)$ if and only if rank $N_F(z_0) <$ normal rank $N_F(z)$, where normal rank $N_F(z) = \max_{z \in C} \{\text{rank } N_F(z)\}$. We recall also that if $F = D_F^{-1} N_F$ with $D_F, N_F$ left coprime, then the McMillan degree of $F$, denoted $\delta(F)$, is equal to the polynomial degree of $\det D_F(z)$, where $\det D_F(z)$ denotes the determinant of $D_F(z)$. The McMillan degree is also the order of any minimal state variable realization of $F(z)$.

Having defined left coprime MFDs for the individual transfer function matrices $F$, $G$, $H$ and $K$, we also define left coprime MFDs for the matrices $[F \vdots G]$ and $[H \vdots K]$. Let $A(z)$ be the least common left multiple of $D_F(z)$ and $D_G(z)$; then $A = E_G D_F = E_F D_G$ for some polynomial matrices $E_G(z)$ and $E_F(z)$, with $E_G$ and $E_F$ left coprime. Then, defining $B = E_G N_F$ and $C = E_F N_G$, we have the following left coprime MFD for the transfer function matrix $[F \vdots G]$

$$[F \vdots G] = A^{-1}[B \vdots C] \tag{5}$$

with

$$A = E_G D_F = E_F D_G, \quad B = E_G N_F, \quad C = E_F N_G \tag{6}$$

Similarly, a left coprime MFD for the transfer matrix $[H \vdots K]$ is defined as follows

$$[H \vdots K] = L^{-1}[M \vdots N] \tag{7}$$

where

$$L = E_K D_H = E_H D_K, \quad M = E_K N_H, \quad N = E_H N_K \tag{8}$$

with $E_K$ and $E_H$ left coprime.

From the coprimeness of the MFDs (6) and (8) it follows that the order of any minimal realization of the model $M1$ is $n_1 \triangleq \deg \det A(z)$, while the order of a minimal realization of $M2$ is $n_2 \triangleq \deg \det L(z)$. Now consider the feedback connection of Fig. 1. From eqns. (1) and (2) it is easy to derive an expression for the transfer matrix $W(z)$ linking the $z$-transform of the process $\begin{bmatrix} w_i \\ v_i \end{bmatrix}$ to the $z$-transform of the process $\begin{bmatrix} y_i \\ u_i \end{bmatrix}$

$$W(z)=\begin{bmatrix} (I-FH)^{-1}G & (I-FH)^{-1}FK \\ (I-HF)^{-1}HG & (I-HF)^{-1}K \end{bmatrix} \tag{9}$$

Equivalently, using eqns. (1) and (2), and the coprime MFDs (6) and (8), we can write $W(z)$ using a natural MFD form as

$$W(z)=\begin{bmatrix} A & -B \\ -M & L \end{bmatrix}^{-1}\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} \triangleq P^{-1}(z)Q(z) \tag{10}$$

With this natural MFD we associate the following polynomial matrix $X(z)$ which will be used frequently later on

$$X(z)\triangleq\begin{bmatrix} A & -B & C & 0 \\ -M & L & 0 & N \end{bmatrix} \tag{11}$$

Since

$$\det P(z)\triangleq\det\begin{bmatrix} A & -B \\ -M & L \end{bmatrix}=\det A\det[L-MA^{-1}B] \tag{12}$$

and since $H=A^{-1}B$ and $K=L^{-1}M$ are proper, it follows that $\deg\det P(z)=n_1+n_2$. Therefore, $\delta[W]=\delta[F\vdots G]+\delta[H\vdots K]$ if and only if the factorization (11) of $W(z)$ is left coprime. We shall thus be concerned with specifying conditions under which the factorization (11) for $W(z)$ is coprime.

We first prove a preliminary result for a simpler feedback configuration, namely that of Fig. 2, i.e. $G(z)=I$ and $K(z)=I$.

*Theorem* 1

Let $F=D_F^{-1}N_F$ and $H=D_H^{-1}N_H$ be left MFDs for $F(z)$ and $H(z)$, and let $W=P^{-1}Q$ be a left MFD for $W(z)$ with

$$P(z)=\begin{bmatrix} D_F & -N_F \\ -N_H & D_H \end{bmatrix},\quad Q(z)=\begin{bmatrix} D_F & 0 \\ 0 & D_H \end{bmatrix} \tag{13}$$

Then $P^{-1}Q$ is left coprime if and only if $D_F^{-1}N_F$ and $D_H^{-1}N_H$ are left coprime.

*Proof*

Recall that a factorization $D^{-1}N$ is coprime if and only if the matrix $[D\vdots N]$ has full row rank, and consider the matrix

$$Y(z)=\begin{bmatrix} D_F & -N_F & D_F & 0 \\ -N_H & D_H & 0 & D_H \end{bmatrix}$$

Then the result follows immediately from

$$\operatorname{rank} Y(z) = \operatorname{rank}\begin{bmatrix} 0 & -N_F & D_F & 0 \\ -N_H & 0 & 0 & D_H \end{bmatrix}$$
$$= \operatorname{rank}\,[N_F \;\; D_F] + \operatorname{rank}\,[N_H \;\; D_H]$$

### 3. Scalar feedback systems

In this section we consider feedback configurations as shown in Fig. 1 where $\{y_i\}$ and $\{u_i\}$ are scalar processes. We introduce the following additional notation. Let $F$ be a transfer function and let $z_0$ be a pole of $F$; then we denote by $m_F(z_0)$ the multiplicity of the pole $z_0$ of $F$. Our main result concerning scalar feedback systems is as follows.

*Theorem* 2

Let $(E_G D_F)^{-1}[E_G N_F \,\vdots\, E_F N_G]$ be a coprime factorization of $[F(z) \,\vdots\, G(z)]$ and let $(E_K D_H)^{-1}[E_K N_H \,\vdots\, E_H N_K]$ be a coprime factorization of $[H(z) \,\vdots\, K(z)]$ with $E_G, D_F, E_G, \ldots, N_K$ scalar but otherwise defined as before. Then the rank of the matrix

$$X(z) = \begin{bmatrix} E_G D_F & -E_G N_F & E_F N_G & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix}$$

is strictly less than two at a point $z_0$, i.e. the MFD in (11) for $W(z)$ is not coprime, if and only if one of the following three conditions holds:

(i) $z_0$ is a zero of $H$ and $K$ and a pole of $F$, if in addition $z_0$ is a pole of $G$, then $m_G(z_0) < m_F(z_0)$;

(ii) $z_0$ is a zero of $F$ and $G$, and a pole of $H$, if in addition $z_0$ is a pole of $K$, then $m_K(z_0) < m_H(z_0)$;

(iii) $z_0$ is a zero of $G$ and $K$, and a zero of $[D_F D_H - N_F N_H]$.

Equivalently, $\delta[W] = \delta[F \;\; G] + \delta[H \;\; K]$ if and only if none of the above conditions hold.

*Proof*

*(a) Necessity*

Suppose rank $X(z_0) < 2$ for some $z_0$. Then $E_F N_G = 0$ or $E_H N_K = 0$ at $z_0$. We shall analyse the case where $E_F N_G = 0$; the other case follows directly by the symmetry between $F$, $G$ and $H$, $K$. $E_F N_G = 0$ implies $E_F = 0$ or $N_G = 0$ or both.

(1) *Case* 1. Assume $E_F = 0$. Then $D_F = 0$ since $E_G D_F = E_F D_G = 0$ with $E_G$ and $E_F$ coprime. Therefore $N_F \neq 0$ and $E_G \neq 0$ by coprimeness. But then rank $X(z_0) < 2$ implies $E_K N_H = E_H N_K = 0$. Now $E_K(z_0) \neq 0$, because if $E_K = 0$, then $D_K = 0$ and hence $N_K \neq 0$ and $E_H \neq 0$ by coprimeness, which is a contradiction. By the same argument $E_H(z_0) \neq 0$. It follows that $N_H(z_0) = N_K(z_0) = 0$. Therefore case 1 implies that a pole of $F$ is also a zero of both $H$ and $K$. In addition, since $E_F D_G$ is the least common multiple of the denominators of $F$ and $G$, $E_F(z_0) = 0$ implies that if $G$ also has a pole at $z_0$, then $m_G(z_0) < m_F(z_0)$.

(2) *Case* 2. Assume $E_F \neq 0$ at $z_0$. Then $N_G = 0$, and hence $D_G \neq 0$ by coprimeness of $D_G$ and $N_G$. Therefore, $E_G D_F \neq 0$, and rank $X(z_0) \neq 2$ only if $E_H N_K = 0$.

*Case* 2 *a*. Assume first that $E_H = 0$. Then by symmetry with the case when $E_F = 0$, it follows from case 1 that $z_0$ is a pole of $H$ which is also a zero of both $F$ and $G$. If in addition $z_0$ is a pole of $K$, then $m_K(z_0) < m_H(z_0)$.

*Case* 2 *b*. Assume now that $E_H \neq 0$ at $z_0$. Then $N_K = 0$ and $D_H \neq 0$. By coprimeness of $N_K$ and $D_K$ it follows that $D_K \neq 0$, and therefore $E_K \neq 0$. It then follows that rank $X(z_0) < 2$ only if the $2 \times 2$ matrix

$$\begin{bmatrix} E_G D_F & -E_G N_F \\ -E_K N_H & E_K D_H \end{bmatrix}$$

is singular. But since $E_G \neq 0$ and $E_K \neq 0$, it follows that $D_F D_H - N_F N_H = 0$ at $z_0$. Therefore, Case 2 *b* implies that a zero of $G$ and $K$ is also a zero of $D_F D_H - N_F N_H$. Note that $D_F D_H - N_F N_H$ is the denominator of the transfer function of the closed-loop formed by $F$ and $H$ (see Fig. 2).

(*b*) *Sufficiency*

We show now that if any one of the three situations (i), (ii) or (iii) arises at a point $z_0$, then rank $X < 2$ at that point. Since (i) and (ii) are completely symmetrical (replace $F$ by $H$ and $G$ by $K$), we consider only cases (i) and (iii).

*Case* (i). Let $D_F = N_H = N_K = 0$ at some $z_0$. Then $N_F \neq 0$, $D_H \neq 0$ by coprimeness of $N_F$, $D_F$ and $N_H$, $D_H$, respectively. Also $E_G D_F = E_F D_G$ implies $E_F D_G = 0$. By the condition $m_G(z_0) < m_F(z_0)$ it follows that $E_F(z_0) = 0$, because otherwise $D_G = 0$ and $E_F \neq 0$ at $z_0$ would imply that $G$ has a pole at $z_0$ with $m_G(z_0) \geqslant m_F(z_0)$. It follows immediately that rank $X(z_0) < 2$, because all columns except the second are zero.

*Case* (iii). Let $N_G = N_K = 0$ and $D_F D_H - N_F N_H = 0$ at $z_0$. Then clearly every $2 \times 2$ minor of $X(z_0)$ is zero. □

The conditions on pole zero cancellations between a pole of $F$ and a common zero of $H$ and $K$, a pole of $H$ and a common zero of $F$ and $G$, or a pole of the closed-loop transfer function and a common zero of $G$ and $K$ are fairly straightforward and easy to interpret. The additional conditions on the multiplicities of the poles of $G$ and $F$ (case (i)) or of the poles of $K$ and $H$ (case (ii)) are not so easy to interpret. We show now by a simple example why these multiplicity conditions are relevant.

*Example* 1

Let

$$F = \frac{1}{(z-1)^2}, \quad G = \frac{z-3}{z-1}, \quad H = \frac{z-1}{z-2}, \quad K = \frac{z-1}{z-4}.$$

Then

$$X(z)=\begin{bmatrix} (z-1)^2 & -1 & (z-3)(z-1) & 0 \\ -(z-1)(z-4) & (z-2)(z-4) & 0 & (z-1)(z-2) \end{bmatrix}$$

All the conditions of Case (i) of theorem 2 are satisfied at $z=1$; we find rank $X(1)=1$, and the feedback connection of $M1$ and $M2$ will not be minimal. Suppose however that $F=1/(z-1)$ rather than $1/(z-1)^2$, i.e. $m_F(1)=m_G(1)$. Then $X(1)$ is non-singular, and the feedback connection of $M1$ and $M2$ will still be minimal despite the fact that $F$ has a pole in common with a zero of both $H$ and $K$.

## 4. Minimality conditions for multivariable feedback systems

We now consider the case where $\{y_i\}$ and $\{u_i\}$ are multivariable processes, and we present a set of necessary and sufficient conditions on the poles and zeros of $F, G, H, K$ that will guarantee that the McMillan degree of the feedback interconnection will be the sum of the McMillan degrees of $M1$ and $M2$.

In the scalar case, the first condition under which coprimeness failed required that $z_0$ be a zero of $H$ and $K$ and a pole of $F$; also, if $z_0$ is a pole of $G$, it must be a pole of lesser degree. We now explain how to express these conditions in a form suitable for the multivariable case.

We shall say that *the transfer function matrices $A$ followed by $B$ have a zero of $A$ cancelling a pole of $B$* if and only if the matrix product $BA$ can be formed, and with $A=\bar{N}_A\bar{D}_A^{-1}$ and $B=\bar{N}_B\bar{D}_B^{-1}$ constituting coprime matrix fraction descriptions, $\bar{N}_A$ and $\bar{D}_B$ are not left coprime. (Thus in the expression $BA=\bar{N}_B\bar{D}_B^{-1}\,\bar{N}_A\bar{D}_A^{-1}$, there is an internal cancellation.)

To capture a multivariable version of the second part of the first scalar condition, suppose that

$$\begin{bmatrix} D_F \\ D_G \end{bmatrix}=\begin{bmatrix} \tilde{D}_F \\ \tilde{D}_G \end{bmatrix} S \tag{14}$$

where $S$ is a greatest common right divisor of $D_F$ and $D_G$. We shall define $\tilde{F}$ as *the part of $F$ excluding poles of $G$* by

$$\tilde{F}=\tilde{D}_F^{-1}\,N_F \tag{15}$$

It is trivial that the MFD here is coprime when $D_F^{-1}\,N_F$ is coprime.

These two definitions allow us to restate the first condition in the scalar problem as: $[H\ \ K]$ followed by $\tilde{F}$ has a zero of $[H\ \ K]$ cancelling a pole of $\tilde{F}$.

The second condition of the scalar problem is restated for the multivariable problem in an obvious way. The third condition needs rather more restating. In the scalar case, a zero of $G$ and $K$ cannot be simultaneously a pole of $G$ or $K$, or, *a fortiori*, a pole of $F$ and $G$ or of $H$ and $K$; in contrast, this situation can happen in the multivariable case, and we have to take the possibility into account.

Consider the redrawing of Fig. 1 in the form of Fig. 3 ($a$), where $T$ is the greatest common right divisor of $D_H$ and $D_K$, and $\tilde{G}, \tilde{H}, \tilde{K}$ are defined in the obvious way. The arrangement of Fig. 3 ($a$) is further redrawn as Fig. 3 ($b$).

For the scalar problem, we were concerned with a zero in both entries of the diagonal transfer function linking $[w \;\; v]'$ to $[m \;\; n]'$ cancelling a pole in the transfer function linking $[m \;\; n]'$ to $[y \;\; u]$. Such cancellations in the scalar problem are identical with the cancellations which occur if $[m \;\; n]$ is replaced by $[\tilde{m} \;\; \tilde{n}]$. This is not so in the vector problem, where it proves necessary to work with $[\tilde{m} \;\; \tilde{n}]$.

The relevant transfer function matrices in the vector case are definable by

$$\begin{bmatrix} \tilde{m} \\ \tilde{n} \end{bmatrix} = \begin{bmatrix} \tilde{G} & 0 \\ 0 & \tilde{K} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{D}_G^{-1} & 0 \\ 0 & \tilde{D}_K^{-1} \end{bmatrix} \begin{bmatrix} N_G & 0 \\ 0 & N_K \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} \begin{bmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} \tilde{m} \\ \tilde{n} \end{bmatrix}$$
$$= \begin{bmatrix} D_F & -N_F \\ -N_H & D_H \end{bmatrix}^{-1} \begin{bmatrix} \tilde{D}_F & 0 \\ 0 & \tilde{D}_H \end{bmatrix} \begin{bmatrix} \tilde{m} \\ \tilde{n} \end{bmatrix} \tag{17}$$

The MFDs in (16) and (17) are easily checked to be coprime.

Before proving the main theorem, we shall establish an equivalent statement for the zero pole cancelling ideas, involving left rather than right MFDs.

*Lemma* 1

Let $A = D_A^{-1} N_A$, $B = D_B^{-1} N_B$ be coprime realizations. Then $A$ followed by $B$ has a zero of $A$ cancelling a pole of $B$ if and only if

$$V(z) = \begin{bmatrix} D_B & N_B & 0 \\ 0 & D_A & N_A \end{bmatrix} \tag{18}$$

is not left coprime.

*Proof*

(Only if) with $\bar{N}_A$ etc. as in the definition of zero pole cancelling, there exists $\alpha \neq 0$ and $z_0$ such that $\alpha^{\mathrm{T}} \bar{N}_A(z_0) = 0$, $\alpha^{\mathrm{T}} \bar{D}_B(z_0) = 0$. But also, because the factorizations $A = D_A^{-1} N_A = \bar{N}_A \bar{D}_A^{-1}$ and $B = D_B^{-1} N_B = \bar{N}_B \bar{D}_B^{-1}$ are all coprime, the left nullspaces of

$$\begin{bmatrix} \bar{N}_A(z_0) \\ \bar{D}_A(z_0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{N}_B(z_0) \\ \bar{D}_B(z_0) \end{bmatrix}$$

are spanned by the rows of $[D_A(z_0) \;\; -N_A(z_0)]$ and $[D_B(z_0) \;\; -N_B(z_0)]$, respectively. Hence there exists $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ with

$$\lambda_2^{\mathrm{T}} [D_A(z_0) \;\; -N_A(z_0)] = [\alpha^{\mathrm{T}} \;\; 0]$$
$$\lambda_1^{\mathrm{T}} [D_B(z_0) \;\; -N_B(z_0)] = [0 \;\; \alpha^{\mathrm{T}}]$$

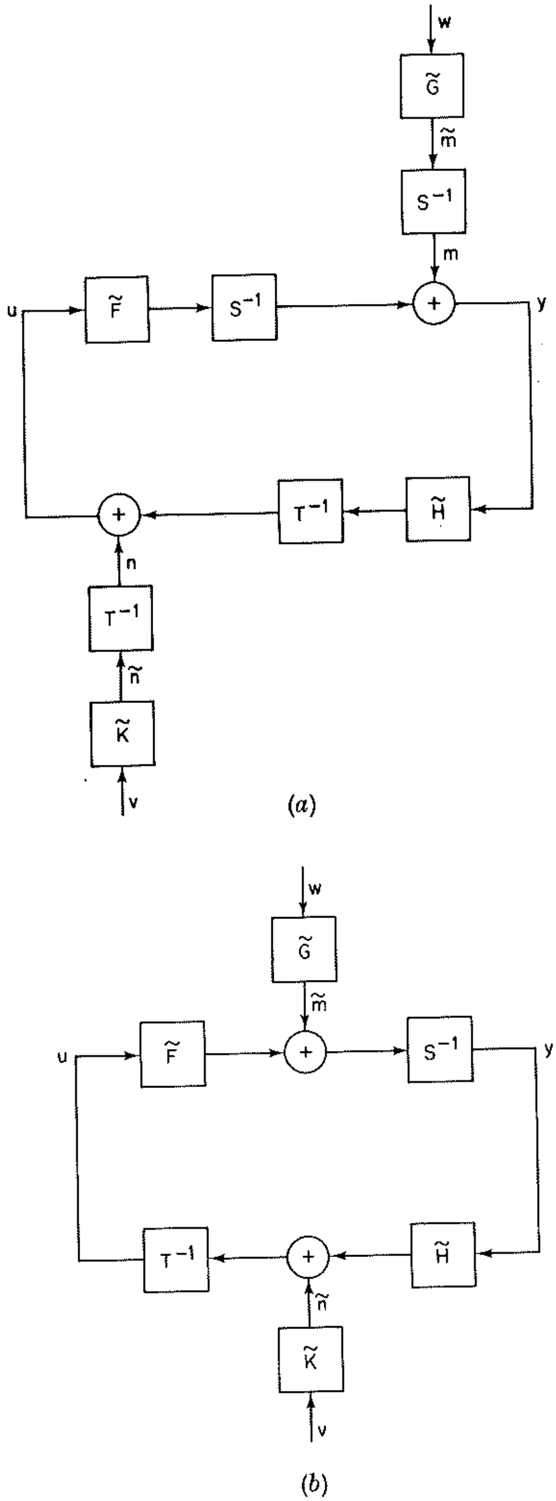


Figure 3. (*a*) Redrawing of Fig. 1 when $F$, $G$ or $H$, $K$ have common poles. (*b*) Rearrangement of scheme of Fig. 3 (*a*).

whence

$$[\lambda_1^{\mathrm{T}} \quad -\lambda_2^{\mathrm{T}}]\begin{bmatrix} D_B(z_0) & -N_B(z_0) & 0 \\ 0 & D_A(z_0) & -N_A(z_0) \end{bmatrix} = 0$$

This establishes the claim. The converse follows by simply reversing the above argument.

We now can state the main theorem, which parallels the earlier scalar result.

*Theorem* 3

Let $[E_G D_F]^{-1}[E_G N_F \;\; E_F N_G]$ be a left coprime MFD of $[F \vdots G]$ and $[E_K D_H]^{-1}[E_K N_H \;\; E_H N_K]$ a left coprime MFD of $[H \vdots K]$, with $E_G$, $D_F$ etc. as defined previously. Let $\tilde{F}$ and $\tilde{H}$ be defined as above. Then the matrix

$$X(z) = \begin{bmatrix} E_G D_F & -E_G N_F & E_F N_G & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix} \triangleq \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} \tag{19}$$

has less than full normal row rank at $z_0$ if and only if one of the following conditions holds :

(i) the cascade connection of $[H \vdots K]$ followed by $\tilde{F}$ has a zero pole cancellation (i.e. a zero of $[H \vdots K]$ cancels a pole of $\tilde{F}$) ;

(ii) the cascade connection of $[F \vdots G]$ followed by $\tilde{H}$ has a zero pole cancellation ;

(iii) the cascade connection of

$$\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix}$$

followed by

$$\begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1}\begin{bmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}$$

has a zero pole cancellation ; assuming neither condition (i) nor condition (ii) holds, any such zero is also a zero of $G$ and $K$, and a pole of

$$\begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1}$$

If none of these three conditions hold, then

$$\delta[W] = \delta[F \vdots G] + \delta[H \vdots K]$$

*Proof*

The final conclusion is obvious if the first part of the theorem can be proved. Accordingly, assume $X(z_0)$ has less than full rank and that there exist constant vectors $\lambda_1$, $\lambda_2$ not both zero such that $\lambda_1^{\mathrm{T}} X_1(z_0) + \lambda_2^{\mathrm{T}} X_2(z_0) = 0$.

Observe immediately that both $\lambda_1$ and $\lambda_2$ are non-zero. For if say $\lambda_1=0$, $\lambda_2\neq 0$, we have at $z_0$

$$\lambda_2^{\mathrm{T}}[E_K N_H \quad E_K D_H \quad E_H N_K]=0 \tag{20}$$

which violates the coprimeness of $(E_K D_H)^{-1}[E_K N_H \quad E_H N_K]$. Now we have at once that

$$\lambda_1^{\mathrm{T}}\, E_F(z_0) N_G(z_0)=0, \quad \lambda_2^{\mathrm{T}}\, E_H(z_0) N_K(z_0)=0$$

*Case* 1. $\lambda_1^{\mathrm{T}}\, E_F(z_0)=0$. Then $\lambda_1^{\mathrm{T}}\, E_G(z_0)\neq 0$ and

$$\begin{aligned}[\lambda_1^{\mathrm{T}} \quad \lambda_2^{\mathrm{T}}]X(z_0) &= [\lambda_1^{\mathrm{T}} \quad \lambda_2^{\mathrm{T}}]\begin{bmatrix} 0 & -E_G N_F & 0 & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix} \\ &= [\lambda_1^{\mathrm{T}}\, E_G \quad \lambda_2^{\mathrm{T}}]\begin{bmatrix} 0 & -N_F & \tilde{D}_F & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix}\end{aligned} \tag{21}$$

(Note that $E_G\tilde{D}_F=E_F\tilde{D}_G$.) This means that condition (i) holds by lemma 1.

*Case* 2. $\lambda_2^{\mathrm{T}}\, E_H(z_0)=0$. This yields condition (ii), by a parallel argument to that for case 1.

*Case* 3. $\alpha_1=\lambda_1^{\mathrm{T}}\, E_F(z_0)\neq 0$ and $\beta_1=\lambda_2^{\mathrm{T}}\, E_H(z_0)\neq 0$. Then we verify easily that at $z_0$

$$[-\lambda_1^{\mathrm{T}}\, E_G \quad -\lambda_2^{\mathrm{T}}\, E_K \quad \lambda_1^{\mathrm{T}}\, E_F \quad \lambda_2^{\mathrm{T}}\, E_H] \times \begin{bmatrix} D_F & -N_F & \tilde{D}_F & 0 & 0 & 0 \\ -N_H & D_H & 0 & \tilde{D}_H & 0 & 0 \\ 0 & 0 & \tilde{D}_G & 0 & N_G & 0 \\ 0 & 0 & 0 & \tilde{D}_K & 0 & N_K \end{bmatrix}=0 \tag{22}$$

This yields condition (iii) using lemma 1.

We now consider the converse. Consider first case 1 and suppose that at $z_0$

$$\begin{bmatrix} 0 & -N_F & \tilde{D}_F & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix}$$

fails to have full rank. Then there exist $\alpha_1$, $\alpha_2$, not both zero (and it is easily seen that both must be non-zero) with $[\alpha_1^{\mathrm{T}} \quad \alpha_2^{\mathrm{T}}]$ a left nullvector. Now at $z_0$

$$[\alpha_1^{\mathrm{T}} \quad 0]\begin{bmatrix} \tilde{D}_F \\ \tilde{D}_G \end{bmatrix}=0$$

and

$$[E_G \quad -E_F]\begin{bmatrix} \tilde{D}_F \\ \tilde{D}_G \end{bmatrix}=0 \tag{23}$$

where $[E_G \ \ -E_F]$ has full row rank and $\begin{bmatrix} \tilde{D}_F \\ \tilde{D}_G \end{bmatrix}$ has full column rank. Therefore, $\alpha_1^{\mathrm{T}} = \lambda_1^{\mathrm{T}} E_G(z_0)$ and $0 = \lambda_1^{\mathrm{T}} E_F(z_0)$ for some $\lambda_1$. So at $z_0$ with $\lambda_2 = \alpha_2$

$$[\lambda_1^{\mathrm{T}} \ \ \lambda_2^{\mathrm{T}}] \begin{bmatrix} 0 & -E_G N_F & E_G \tilde{D}_F & 0 \\ -E_K N_H & E_K D_H & 0 & E_H N_K \end{bmatrix} = 0 \quad \text{and} \quad \lambda_1^{\mathrm{T}} E_F = 0$$

from which it is immediate that $[\lambda_1^{\mathrm{T}} \ \ \lambda_2^{\mathrm{T}}] X(z_0) = 0$.

Case 2 proceeds similarly. Finally, suppose that

$$\begin{bmatrix} D_F & -N_F & \tilde{D}_F & 0 & 0 & 0 \\ -N_H & D_H & 0 & \tilde{D}_H & 0 & 0 \\ 0 & 0 & \tilde{D}_G & 0 & N_G & 0 \\ 0 & 0 & 0 & \tilde{D}_K & 0 & N_K \end{bmatrix}$$

fails to have full row rank, and let $[\alpha_1^{\mathrm{T}} \ \ \alpha_2^{\mathrm{T}} \ \ \alpha_3^{\mathrm{T}} \ \ \alpha_4^{\mathrm{T}}]$ be a non-zero left null-vector. Since $\alpha_1^{\mathrm{T}} \tilde{D}_F + \alpha_3^{\mathrm{T}} \tilde{D}_G = 0$, it follows using (23) again that

$$\alpha_1^{\mathrm{T}} = -\lambda_1^{\mathrm{T}} E_G, \quad \alpha_3^{\mathrm{T}} = \lambda_1^{\mathrm{T}} E_F$$

for some $\lambda_1$. Similarly

$$\alpha_2^{\mathrm{T}} = -\lambda_2^{\mathrm{T}} E_K, \quad \alpha_4^{\mathrm{T}} = \lambda_2^{\mathrm{T}} E_H$$

It is then immediate that $[\lambda_1^{\mathrm{T}} \ \ \lambda_2^{\mathrm{T}}] X(z_0) = 0$.

Last, we must show that if condition (iii) holds, but not condition (i) nor condition (ii), $G(z)$ and $K(z)$ have a zero at $z_0$ and $\begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1}$ has a pole there. Let $\lambda_1$, $\lambda_2$ be as in the previous paragraph. We have $\lambda_1^{\mathrm{T}} E_F \neq 0$, $\lambda_2^{\mathrm{T}} E_H \neq 0$ since conditions (i) and (ii) do not hold. Hence at $z_0$, $\alpha_3^{\mathrm{T}} N_G = 0$, $\alpha_4^{\mathrm{T}} N_K = 0$ for non-zero $\alpha_3$, $\alpha_4$. Also

$$\begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1} = \begin{bmatrix} D_F & -N_F \\ -N_H & D_H \end{bmatrix}^{-1} \begin{bmatrix} D_F & 0 \\ 0 & D_H \end{bmatrix}$$

and at $z_0$

$$[\alpha_1^{\mathrm{T}} \ \ \alpha_2^{\mathrm{T}}] \begin{bmatrix} D_F & -N_F \\ -N_H & D_H \end{bmatrix} = 0 \tag{24}$$

If $\alpha_1$ and $\alpha_2$ were zero, we should have $\alpha_3^{\mathrm{T}}[\tilde{D}_G \ \ N_G] = 0$ and $\alpha_4^{\mathrm{T}}[\tilde{D}_K \ \ N_K]$ at $z_0$ with $\alpha_3$, $\alpha_4$ non-zero, which is a contradiction. Hence (24) is non-trivial, and $\begin{bmatrix} I & -F \\ -H & I \end{bmatrix}^{-1}$ has a pole at $z_0$. $\square$

We conclude this section by showing that the class of feedback models which we have called 'generic' in the context of feedback models for jointly stationary stochastic $\begin{bmatrix} y_i \\ u_i \end{bmatrix}$ processes (Gevers and Anderson 1981) always have the property that the feedback representation of the joint model obtained from individual models $M1$ and $M2$ is minimal if the individual realizations are minimal. We first recall the definition of generic feedback models (see Gevers and Anderson 1981).

*Definition*

Let $F$, $G$, $H$, $K$ be the transfer function matrices of a feedback model as in Fig. 1 and let $[F \,\vdots\, G] = A^{-1}[B \,\vdots\, C]$ and $[H \,\vdots\, K] = L^{-1}[M \,\vdots\, N]$ be left coprime MFDs. Let $R(z) = \operatorname{diag}(z^{r_1}, \ldots, z^{r_p})$ where $r_k$ is the row degree of the $k$th row of $A(z)$, and $S(z) = \operatorname{diag}(z^{s_1}, \ldots, z^{s_m})$ where $s_k$ is the row degree of the $k$th row of $L(z)$. The model $F$, $G$, $H$, $K$ is called generic if for every $z$:

(1) the polynomials $\det[C(z)C^{\mathrm{T}}(z^{-1})R(z)]$ and $\det[N(z)N^{\mathrm{T}}(z^{-1})S(z)]$ are coprime;

(2) the matrix $\Omega(z)$ has full rank

$$\Omega(z) = \begin{bmatrix} A(z) & -B(z) & C(z)C^{\mathrm{T}}(z^{-1})R(z) & 0 \\ -M(z) & L(z) & 0 & N(z)N^{\mathrm{T}}(z^{-1})S(z) \end{bmatrix} \tag{25}$$

We shall not develop here the significance of generic feedback models, or the interest of this notion in the context of identification of feedback systems. We refer the reader to (Gevers and Anderson 1981, 1982, Anderson and Gevers 1982) for details. Let it be sufficient to say that almost all systems are generic.

*Theorem* 4

Consider a feedback model obtained from the feedback connection of a minimal realization of $M1$ and a minimal realization of $M2$, and let $F$, $G$, $H$, $K$ be generic. Then $\delta[W] = \delta[F \,\vdots\, G] + \delta[H \,\vdots\, K]$.

*Proof*

With $X(z)$ as defined in (14), one can check that

$$\Omega(z) = X(z) \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & C^{\mathrm{T}}(z^{-1})R(z) & 0 \\ 0 & 0 & 0 & N^{\mathrm{T}}(z^{-1})S(z) \end{bmatrix}$$

so that if $\Omega(z)$ has full row rank, so must $X(z)$. □

## 5. Conclusions

We have derived conditions under which the McMillan degree of a linear time invariant feedback model is the sum of the degrees of the two component models. Stated otherwise, we have answered the question : under what condition is the realization obtained in the natural way from the feedback interconnection of two minimal realizations itself a minimal realization ? Finally we have shown that generic feedback processes, as defined in earlier work, always have the desirable property that the McMillan degree of the feedback realization is the sum of the degrees of the two component models.

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