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Constant, Predictable and Degenerate Directions of the Discrete-Time Riccati Equation*†

Directions Constantes, Prédictibles et Dégénérées de l'Equation de Riccati Discrète

Konstante, vorhersagbare und degenerierte Richtungen der Diskretzeit-Riccati-Gleichung

Постоянное, предсказуемое и вырождающееся направления для дискретного во времени уравнения Рикатти

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The solution of a Riccati equation may go to a constant or zero value in certain directions. The number of such directions is related to the relative order of the output covariance and the transfer function, respectively.

Summary—Several recent papers have dealt with the phenomenon that the solution of the Riccati equation for certain discrete-time linear stochastic systems can attain a constant value in certain directions after a limited number of iterations, thus enabling a reduction in the effective order of the Riccati equation. Curiously these results do not have exact continuous-time analogs. In this paper we explain the reasons for this, chiefly by introducing the concept of predictable directions along which the solution goes to zero rather than a nonzero constant. In continuous-time, the predictable and constant directions coincide and their number depends upon the relative order, a measure of smoothness, of the transfer function of the system or equivalently of its output covariance. This equivalence breaks down in discrete-time, where the number of constant directions is the relative order of the covariance while the number of predictable directions is the relative order of the transfer functions. The insight provided by our approach not only shows how to convert constant directions to predictable directions but also shows how the concept may be extended to time-variant systems, where the name degenerate directions is more descriptive than constant directions.

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1. INTRODUCTION AND OUTLINE OF RESULTS

CONTINUOUS-time linear estimation problems with colored noise are basically solved by means of repeated differentiations until a white-noise process appears in the observations. Furthermore, if this takes α differentiations it is known that the order of the Riccati equation can be reduced by α, an important computational consideration. It seems evident that similar results can be obtained in discrete-time, and in fact several authors have carried out such analyses: e.g. BRYSON and HENRIKSON [1], BROWN and SAGE [2]. It was therefore a surprise when BUCY *et al.* [3] pointed out that differencing in discrete-time did not determine the maximum possible reduction in the order of the Riccati equation. That is, even if the first q differences of the observations contained no white noise sequence, the order of the Riccati equation for the state–error matrix $P(i|i-1)$ could be reduced not just by q, but by a larger number, say α. In this paper we shall explain the reasons for this behaviour. As we shall show, the point is that in the q directions along which the differenced observations do not contain white noise, the projections of the error matrix $P(i|i-1)$ are identically zero after a finite number of steps corresponding to the order of the difference; in the remaining $\alpha-q$ directions, the projections of $P(i|i-1)$ do not go to zero but for time-invariant systems, Bucy *et al.* showed that they go to constant values in a number of steps less than the dimension of the state-vector. These directions were therefore called *invariant* or *constant directions* [3, 4]. Necessary and sufficient conditions on the system matrices for the existence of invariant directions have been given and will be quoted later. The directions in which the error matrices go to zero will be called *predictable* directions [5], because along these directions the state can be predicted without error. As we shall see below it will be convenient to use the term degenerate directions to cover both constant and predictable directions and also the extensions of these concepts to time-variant systems.

One major aim of this paper is to explore the reasons for this difference in behaviour of continuous-time and discrete-time systems. That is, more specifically why is it that in continuous-time problems the predictable and constant directions coincide and are equal to the number of process derivatives that do not contain white noise, while in discrete-time the latter number only determines the number of predictable directions and does not reveal the possible existence of further degenerate directions?

For scalar output constant systems we shall give a complete explanation of the above differences; moreover, our explanation will show how the concept of invariant directions can be naturally extended to time-variant systems. The multiple-output problem [4] is still not entirely clarified from our point of view.

Our results rely heavily on the observation that the number of degenerate directions for $P(i|i-1)$, or invariant directions for a time-invariant system, is equal to the relative order of the covariance function of the output of the state-variable system, a fact that apparently escaped the authors of [3, 4]. Once this connection is established, we can use the result of Ref. [6] that there is no unique relationship in discrete time between the relative order α of the covariance of a given process and the relative order q of its state-variable representation, except to say that $q \leq \alpha$. This is in contrast to the continuous-time situation where the relative order of the covariance of a process is equal to the relative order of all its state-variable representations (cf. [7], Lemma 1). It was shown in [6] that this difference is due to the required time-delay introduced in discrete-time by the differencing operation (cf. [6], Section 5).

As an introduction to the mathematical treatment that follows we shall briefly introduce the main definitions and heuristically explain the results of this paper.

For discrete-time systems:

(i) The number of degenerate, or invariant, directions, say α, of a state-variable model is equal to the relative order of the covariance function of the observed process. The latter is equal to the number of differencing operations required to produce a Kronecker delta function component in the differenced covariance; for stationary processes it is also equal to the difference between the degrees of the denominator and numerator polynomials of the power spectral density function, $S(z)$, the z-transform of the covariance function.

(ii) The number of predictable directions is equal to the relative order of the model for the observed process. The latter is one less than the number of differencing operations required to produce a term proportional to the input in the differenced output; for single-input single-output (SISO) constant systems it is also equal to one less than the difference between the degrees of the denominator and numerator polynomials of the transfer function, i.e. the z-transform of its impulse response.

(iii) The relative order of a covariance function has no unique relationship to the relative order of a model whose response to white noise has the given covariance. For constant SISO systems this follows from the decomposition

$$S(z)=H(z)H(z^{-1})$$
$$=z^{k}H(z)\cdot z^{-k}H(z^{-1}),\ k \text{ arbitrary.} \quad (1)$$

(iv) The preceding result also shows how in a given situation invariant directions of a given model can be converted to predictable directions of a closely related model, and one with the same covariance function.

For continuous-time systems: the spectral decomposition for stationary continuous-time processes is

$$S(s)=H(s)H(-s)$$
$$=e^{s\tau}H(s)\cdot e^{-s\tau}H(-s),\ \tau \text{ arbitrary} \quad (2)$$

where $S(s)$ is the power spectral density of the observed process, i.e. the bilateral transform of its covariance function, and $H(s)$ is the transfer function of a model whose response to white noise has the given spectral density. There is a non-uniqueness introduced by the factor $e^{s\tau}$, but if we wish to work only with lumped models, i.e. models with rational transfer functions, we must take $\tau=0$. With this restriction, the difference between the denominator and numerator polynomials of the transfer function of a model, i.e. its relative order, must always be one-half the difference between the corresponding polynomials of the spectral density of the response of the model to white noise, that is, the relative order of the covariance of the observed process. The first difference equals the number of differentiations required to produce a term proportional to the input in the differentiated output of a model, while the second difference equals the number of differentiations required to produce a delta function term in the differentiated covariance function. This strict relationship explains why the numbers of predictable and invariant directions must coincide in continuous-time and why differentiation always correctly identifies the maximum possible reduction in the order of the Riccati equations that arise in the problem.

Finally we note that the close relation we have identified between predictable and invariant directions shows that both notions can readily be extended to time-variant systems. There is no change in the definition of predictable directions, i.e. directions along which $P(i|i-1)$ goes to *zero* in a finite number of steps. But the definition of invariant directions has to be modified because in time-variant systems, $P(i|i-1)$ may not go to a *constant* value in any direction. The real point is one of "degeneracy" rather than "constancy"—for a covariance of a given relative order, there are directions along which $P(i|i-1)$ goes to a trivial or degenerate value, i.e. a value that can be computed from the model by inspection without having to solve the Riccati equation along those directions. Clearly this degeneracy helps to reduce the order of the Riccati equation that has to be solved and this is the appropriate generalization of the concept of invariant directions. We remark, for reasons that should be clear from the above discussion, that in continuous-time the degenerate directions of both time-invariant and time-variant systems are also predictable directions (cf. [7], Appendix I).

The major contribution of this paper is to present a unified treatment of degenerate directions via the connection with the relative order of the output covariance. Invariant and predictable directions appear as special cases of degenerate directions, and the number of predictable directions is shown to depend upon the particular factorization of the covariance. The differences between the discrete and continuous-time case result from the different properties of the factorization in discrete and continuous time.

We present the proofs of the above results in Sections 2 and 4. Section 3 develops some of the main properties of predictable systems. Some of these properties are used in Section 4, but they also have other applications, e.g. in stability studies along the lines of [8].

2. DEGENERATE DIRECTIONS AND THE RELATIVE ORDER PROPERTY

We shall consider a Markovian representation of a p-dimensional observation process $y(\cdot)$ of the form

$$x(i+1)=\phi(i+1, i)x(i) + G(i+1)u(i+1),\ x(0)=x_0 \quad (3a)$$

$$y(i)=H(i)x(i),\ 0\leq i\leq N \quad (3b)$$

where $\phi(\cdot, \cdot)$, $G(\cdot)$, $H(\cdot)$ are known functions of dimension $n\times n$, $n\times m$, $p\times n$ respectively, and $\{x_0, u(\cdot)\}$ are zero-mean random variables with

$$E[x_0u'(\cdot)]=0,\ E[x_0x_0']=\Pi_0 \quad (4a)$$

$$E[u(k)u'(l)]=I\delta(k-l) \quad (4b)$$

$$\delta(k-l)=1 \text{ for } k=l,\ \delta(k-l)=0,\ k\neq l. \quad (4c)$$

It is well known that the Kalman filter equations for the one-step prediction estimates of the state, $\hat{x}(i+1|i)$, require the solution of the following matrix Riccati equation

$$P(i+1|i)=\phi(i+1, i)P(i|i-1)\phi'(i+1, i) + G(i+1)G'(i+1)-\phi(i+1, i)P(i|i-1)H'(i)[H(i)P(i|i-1)H'(i)]^{-1}H(i)P(i|i-1)\phi'(i+1, i) \quad (5a)$$

$$P(0|-1)=\Pi_0 \quad (5b)$$

where Π_0 is a non-negative definite symmetric matrix. The matrix $P(i|i-1)$ is the error–covariance matrix of the Kalman filter estimates

$$P(i|i-1)=E[(x(i)-\hat{x}(i|i-1))(x(i)-\hat{x}(i|i-1))']. \quad (6)$$

The system (3a) often arises by state-augmentation from a system that has a correlated noise term in its output equation [1]. The Riccati equation may then be ill-conditioned because its dynamic rank is lower than its order, though this can, of course, be alleviated by proper partitioning of the augmented system equations. For constant systems, RAPPAPORT *et al.* [3, 4] have obtained the dynamic rank by exhibiting a space of what they called invariant directions for the Riccati equation

$$P(i+1|i)=\phi P(i|i-1)\phi'+GG'-\phi P(i|i-1)H'[HP(i|i-1)H]^{-1}HP(i|i-1)\phi' \quad (7a)$$

$$P(0|-1)=\Pi_0. \quad (7b)$$

Definition 1 [3, 4]

Let $P(i, \Pi_0)$ be the solution at time i of the Riccati equation (7) started with initial condition Π_0. The n-vector e is a k-invariant direction of (7) if and only if

$$e'\cdot P(i, \Pi_0)=e'\cdot P(k, 0)=constant \quad (8)$$

for all $i\geq k$ and for all symmetric non-negative definite Π_0.

For single-output constant systems the main results of BUCY *et al.* are as follows as described in [3]. We denote by I_k the vector space of all k-invariant directions and by I the vector space of all invariant directions

$$I=\bigcup_{k=1}^{\infty} I_k. \quad (9)$$

The first lemma exhibits a basis for I_k. Define row vectors

$$e_i=H\phi^{-i},\ i=1, 2, \ldots, n \quad (10)$$

and a $k\times n$ matrix

$$E_k'=[e_1', \ldots, e_k']. \quad (11)$$

We shall assume that the system is completely observable, although this is by no means essential. It can be shown [3] that any k-invariant direction, is a linear combination of the rows of E_k.

Next we define the $(i+1)-$vectro L_i by

$$L_i'=[0l_1, \ldots, l_i],\ i=0, 1, \ldots, n-1 \quad (12)$$

where

$$l_j=\sum_{s=1}^{j} H\phi^{-s}GG'\phi'^{j-s}H',\ j=1, \ldots, n-1. \quad (13)$$

With these notations we can now state the main result of [3].

Proposition 1 [3]

Suppose $\alpha\leq n$. Then e_i is i-invariant, $i=1, \ldots, \alpha$, $I=I_\alpha$ and $I_i=\text{span}(e_1, \ldots, e_i)$ for $i\leq\alpha$ if and only if

(i) for $\alpha<n$: $L_\alpha\neq 0$, $L_{\alpha-1}=0$,

or (14)

(ii) for $\alpha=n$: $L_{\alpha-1}=0$.

An important consequence of Proposition 1 is that the projections of $P(i, \Pi_0)$ on the subspace of invariant directions have known values that are independent of Π_0 and can be expressed in terms of H, ϕ and G. Indeed, if the Riccati equation (7) has α degenerate directions, it is easy to see that

$$e_jP(i, \Pi_0)=\sum_{s=1}^{j} H\phi^{-s}GG'\phi'^{j-s},\quad j=1, \ldots, \alpha \quad i\geq\alpha. \quad (15)$$

Using these known projections of $P(i, \Pi_0)$ on the α-dimensional invariant subspace I one can then reduce the order of the Riccati equation to $n-\alpha$, since, after an initial transient period, $P(i, \Pi_0)$ varies only on the orthogonal subspace $I^{\perp}$.

But as we shall show now the number of degenerate directions is more fundamentally a property of the covariance of the output signal; more specifically, it is equal to the relative order of this covariance. This observation, which was made to us by GEESEY [9], will enable us to extend the concept of invariant directions to that of degenerate directions for time-variant systems.

The covariance of the output of the system (3) is (cf. e.g. [6])

$$R_y(i,j)=H(i)\phi(i,j)N(j)1(i-j)+N'(i)\phi'(j, i)H'(j)1(j-i-1) \quad (16)$$

where

$$1(i-j)=i \text{ if } i\geq j$$
$$=0 \text{ if } i<j$$
$$N(i)=\Pi(i)H'(i) \quad (17)$$

and $\Pi(\cdot)$ is the state-variance and obeys the difference equation

$$\Pi(i+1)=\phi(i+1, i)\Pi(i)\phi'(i+1, i)+G(i+1)G'(i+1). \quad (18)$$

The covariance (16) will be said to have definite relative order α [6] if there exists a finite integer α, $\alpha>0$, such that

(1) $H(i-k)\phi(i-k, i)N(i)-N'(i-k)\phi'(i, i-k)H'(i)=0$, $k=1, 2, \ldots, \alpha-1$; $k\leq i\leq N$ (19a)

(2) $H(i-\alpha)\phi(i-\alpha, i)N(i)-N'(i-\alpha)\phi'(i, i-\alpha)H'(i)>0$, $\alpha\leq i\leq N$. (19b)

Let us now define the parameters

$$l_j(i) \triangleq \sum_{s=0}^{j-1} H(i-j)\phi(i-j,\ i-s)G(i-s)G'(i-s)\phi'(i,\ i-s)H'(i) \quad j=1, \ldots, \alpha-1; \quad j \le i \le N. \qquad (20)$$

For constant systems $l_j(i)$ is independent of i and specializes to l_j defined in (14).

Theorem 1

The Riccati equation (7) has α invariant directions if and only if the covariance of its output has definite relative order α.

Proof. Using (17) and (18) we establish, after some algebra, the identity

$$l_k(i) \triangleq \sum_{s=0}^{k-1} H(i-k)\phi(i-k,\ i-s)G(i-s)G'(i-s)\phi'(i,\ i-s)H'(i) = H(i-k)\phi(i-k,\ i)N(i) - N'(i-k)\phi'(i,\ i-k)H'(i), \quad k \le i \le N. \qquad (21)$$

This identity, specialized to the constant case, establishes the equivalence between the conditions of Proposition 1 and the conditions (19) for the relative order α of the output covariance. This completes the proof.

The equivalence established by Theorem 1 implies that the number of invariant directions is an invariant property of all state-variable models whose outputs have the same covariance. In Section 4 we shall show how, for a constant single-output system, this property can be used to replace a system with invariant directions by a "simpler" one with a full set of predictable directions. Moreover, the fact that the equivalence has been established by means of the time-variant identity (21) points to the fact that the fundamental concept is one of degeneracy, not constancy, as we show next.

Definition 2

Let $P(i,\ \Pi_0)$ be the solution at time i of the Riccati equation (5) started with initial condition Π_0. Then the row-vector $e(i)$ is a k-degenerate direction of (5) if and only if the projection of $P(i,\ \Pi_0)$ on $e(i)$ is a function of $H(\cdot)$, $\phi(\cdot,\ \cdot)$ and $G(\cdot)$, independent of Π_0, i.e.

$$e(i)P(i,\ \Pi_0) = f(H(\cdot),\ \phi(\cdot,\ \cdot),\ G(\cdot)) \qquad (22)$$

for all $i \ge k$ and for all Π_0.

Theorem 2

If the output covariance of the system (3) has definite relative order α, then the following are degenerate directions of the Riccati equation (5)

$$\begin{aligned} e_1(i) &= H(i-1)\phi(i-1,\ i),\ i \ge 1 \\ &\ \ \vdots \\ e_\alpha(i) &= H(i-\alpha)\phi(i-\alpha,\ i),\ i \ge \alpha. \end{aligned} \qquad (23)$$

Proof. From the identity (21) of Theorem 1 and the relative order conditions (19) it follows that

$$l_k(i) = 0,\ k=1, \ldots, \alpha-1;\ k \le i \le N. \qquad (24)$$

After some straightforward manipulations on the Riccati equation (5), the conditions (24) on $H(\cdot)$, $\phi(\cdot,\ \cdot)$ and $G(\cdot)$ imply that

$$e_k(i)P(i,\ \Pi_0) = \sum_{j=0}^{k} H(i-k)\phi(i-k,\ i-j)G(i-j)G'(i-j)\phi'(i,\ i-j),\ k=1, \ldots, \alpha; \quad k \le i \le N. \qquad (25)$$

Hence the directions $e_1(i), \ldots, e_\alpha(i)$ are degenerate.

The important property is that the projections of $P(i,\ \Pi_0)$ on the degenerate directions can be computed directly from the system parameters, without solving a Riccati equation. By a time-varying state-transformation, analogous to the one used by Bucy *et al.* in [3], one can again reduce by α the order of the Riccati equation.

3. PREDICTABLE DIRECTIONS OF A STATE-VARIABLE SYSTEM

The fact that the projection of $P(i,\ \Pi_0)$ on any degenerate direction is a known value that does not depend on Π_0—and a constant value for time-invariant systems—raises the question about whether this known value could not be made equal to zero, which would still further simplify the equations of the predictor. In such a case the projection on this direction of the error in the predicted state is zero with probability one, meaning that the corresponding costate is a known function of the past measurements only. For obvious reasons such a costate will be called predictable, and the corresponding direction a predictable direction of the system.

Definition 3a

A vector e will be called a k-predictable direction of the system (3) if and only if

$$e' \cdot x(i) = f\{y(0),\ y(1), \ldots, y(i-1)\} \qquad (26)$$

for all $i \geq k$, where f is a known function of the outputs $\{y(0), \ldots, y(i-1)\}$. The quantity $e' \cdot x(i)$ is then called a predictable costate.

It is essential to distinguish a predictable costate from the concept of a constructible costate, as introduced by KALMAN [10]. A costate of $x(i)$ is said to be constructible if it can be completely determined from past and present inputs, assuming that the past and present inputs are zero (or are known). Here however the inputs are assumed to be unknown. It should be clear that the vectors e_j for which (26) holds are in the nullspace of the Riccati matrix $P(i|i-1)$, since for such directions

$$e_j' \cdot \hat{x}(i|i-1) = e_j' \cdot x(i) \tag{27}$$

with probability one. This will be formally proved later in this section. First we shall derive conditions under which a state-variable system has a subspace of predictable directions. For reasons of space, we shall limit our analysis to a single-output constant parameter system. It contains all the ideas of the more general time-variant and multiple-output case, which is studied in [5], leads to closed form analytical conditions, and lends itself to an interesting transfer function analysis that will be presented in the next section. Thus we consider a single-output time-invariant system

$$x(i+1) = \phi x(i) + Gu(i+1), \; x(0) = x_0 \tag{28a}$$

$$y(i) = hx(i), \; 0 \leq i \leq N \tag{28b}$$

with the same assumptions on $u(\cdot)$ and x_0 as before. In addition, we shall assume that ϕ is non-singular. Note that [5] has extensions and more details of the derivation.

From the state equations (28) we derive the relation

$$\begin{bmatrix} h\phi^{-1} \\ h\phi^{-2} \\ \cdot \\ \cdot \\ \cdot \\ h\phi^{-i} \end{bmatrix} x(i) = \begin{bmatrix} y(i-1) \\ y(i-2) \\ \cdot \\ \cdot \\ \cdot \\ y(0) \end{bmatrix} + \begin{bmatrix} h\phi^{-1}G & 0 \ldots\ldots\ldots & 0 \\ h\phi^{-2}G & h\phi^{-1}G & 0 \;\ldots\ldots \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h\phi^{-i}G & \ldots\ldots\ldots\ldots & h\phi^{-1}G \end{bmatrix} \begin{bmatrix} u(i) \\ u(i-1) \\ \cdot \\ \cdot \\ \cdot \\ u(1) \end{bmatrix} \tag{29a}$$

or in matrix form

$$E_i^* x(i) = y^*(i-1) + D_i^* u^*(i), \; 1 \leq i \leq N \tag{29b}$$

with the obvious definitions. Since the system (28) is of order n, the last n outputs form a sufficient statistic; in addition it is clear from the linearity of (29) that Definition 3a can be replaced by the following equivalent definition.

Definition 3b

A vector e is a k-predictable direction of the system (28) if and only if for all $i \geq k$ and for some vector b_i

$$e' \cdot x(i) = b_i' \cdot \bar{y}(i-1), \; i \geq k, \tag{30}$$

where

$$\bar{y}(i-1) = [y'(i-1)y'(i-2), \ldots, y'(i-n \vee 0)]'. \tag{31}$$

Thus we replace (29) by the relation

$$E_i x(i) = \bar{y}(i-1) + D_i . \bar{u}(i), \; 1 \leq i \leq N \tag{32}$$

where

$$E_i = E^*_{i\wedge n}, \; D_i = D^*_{i\wedge n} \tag{33}$$

$$\bar{u}(i) = [u'(i)u'(i-1), \ldots, u'(i-n+1 \vee 1)] \tag{34}$$

i.e.

$$E_i = E_i^* \text{ if } i \leq n, \; E_i = E_n^* \text{ if } i \geq n. \tag{35}$$

* $i \vee k = \max\{i, k\}$. Notice that $\bar{y}(i-1)$ is the vector formed by either the first i or the last n outputs, whichever contains the least components.

It is easy to see that any k-predictable direction is a linear combination of the rows of E_k.

We shall say that a vector e in R^n is a predictable direction of system (28) if it is predictable for some k, $k>0$; the system has a q-dimensional predictable subspace if it has q linearly independent predictable directions.

Theorem 3

The system (28) has q k-predictable directions $(k \geq q)$ if and only if

$$h\phi^{-1}G = h\phi^{-2}G = \dots = h\phi^{-q}G = 0 \quad (36a)$$

$$\text{rank}\,(E_k) \geq q \quad (36b)$$

and either

$$h\phi^{-q-1}G \neq 0 \text{ or rank}\,(E_k) = q. \quad (36c)$$

Proof.

Sufficiency: Follows immediately from the relations (29). Indeed the conditions (36a, b) imply that the first q rows of E_k are k-predictable directions of the system (28).

Necessity: Suppose there exist q linearly independent k-predictable directions. Then there exist a $q \times n$ matrix E and a $q \times kp$ matrix B such that

$$Ex(k) = B\bar{y}(k-1) \quad (37)$$

with

$$\textbf{rank}\,(E) = q. \quad (38)$$

Comparing with (32) we have

$$BD_k = 0 \quad (39)$$

$$BE_k = E. \quad (40)$$

From (38) and (40) it follows that (36b) holds and that rank $(B) \geq q$. This last condition, together with (39), implies that the dimension of the left nullspace of D_k is at least q. But from the structure of D_k, this implies that the conditions (36a) hold. That (36c) must hold follows by contradiction, using the sufficiency part of the proof.

From the proof of Theorem 3 we can immediately derive the following corollaries.

COROLLARY 3. The system (28) has q k-predictable directions if and only if it has q q-predictable directions.

COROLLARY 4. The system (28) has a q-dimensional predictable subspace if and only if the conditions (36) hold with k replaced by n, the dimension of the system.

COROLLARY 5. If the system (28) has a q-dimensional predictable subspace, the vectors $\{e_1, \dots, e_q\}$, with

$$e_k' = h\phi^{-k},\ k=1, \dots, q, \quad (41)$$

form a basis for the predictable subspace. The projections of $x(i)$ on these directions are given by

$$e_k' \cdot x(i) = y(i-k),\ k=1, \dots, q \quad k \leq i \leq N. \quad (42)$$

COROLLARY 6. The number of predictable directions is a property of the impulse response, i.e. it is invariant under state transformation. This follows immediately from the conditions (36a).

Corollary 4 gives a very straightforward test for the number of predictable directions, if any, of the single-output system (28). Corollary 5 shows that the values at $x(i)$ of the costates $e_1, \dots, e_q$ are given directly as past outputs.

Theorem 4

The predictable subspace of the system (28) is in the null-space of $P(i, \Pi_0)$, the solution of the Riccati equation (7).

Proof. It is easy to show as indicated in [5] that the conditions (36a, c) are equivalent to

$$h\phi^{-k}P(i, \Pi_0) = 0,\ k=1, \dots, q;\ k \leq i \leq N \quad (43)$$

$$h\phi^{-q-1}P(i, \Pi_0) \neq 0,\ k+1 \leq i \leq N. \quad (44)$$

We show next that the nullspace of $P(i, \Pi_0)$ cannot be larger than the predictable subspace, i.e. we show that the error-covariance matrix cannot have a nullvector if the system does not have a predictable direction.

Theorem 5

Let Π_0 in the Riccati equation (7) be non-singular. Then $P(i, \Pi_0)$ is positive definite for all i in $[0, N]$ if and only if $h\phi^{-1}G \neq 0$.

Proof. We prove that if for some k, $P(k, \Pi_0)$ is positive definite and if $h\phi^{-1}G \neq 0$, then $P(i, \Pi_0)$ is positive definite for all i, $k \leq i \leq N$. The result will follow by iteration. The proof goes by contradiction. Suppose $P(k+1, \Pi_0)$ is singular, and let the row vector c be a null-vector of $P(k+1, \Pi_0)$. Then we can write (see, e.g. [3])

$$\left\| c \right\|^2_{P(k+1,\, \Pi_0)} = 0 = \min_r \left\| c\phi + rh \right\|^2_{P(k,\, \Pi_0)} + \left\| cG \right\|^2 \quad (45)$$

where

$$\left\|c\right\|_P^2 \triangleq cPc'. \tag{46}$$

Let r_k be the minimizing value of r. It follows from (45) that

$$(c\phi + r_k h)P(k, \Pi_0) = 0 \tag{47}$$

$$cG = 0. \tag{48}$$

Since $P(k, \Pi_0)$ is assumed positive definite, the first relation implies

$$c = r_k h \phi^{-1}. \tag{49}$$

This, together with (48) implies

$$h\phi^{-1}G = 0, \tag{50}$$

which is a contradiction. Repeating the argument for $k+2$, $k+3$, etc. proves the lemma.

Theorem 5 gives an easy test for the positive-definiteness of the error–covariance matrix;namely if the states are not known *a priori*, i.e. if Π_0 is positive definite, they cannot be perfectly predicted at some later time unless the system has a predictable subspace.

Finally, we show that a system with q predictable directions can be realized as a pure tapped-delay line with q delays together with a feed-back system of dimension $n-q$.

Theorem 6

The system (28) has a q-dimensional predictable subspace if and only if, for $q \le i \le N$, it is state-equivalent to the form

$$x(i+1) = \left[\begin{array}{cccc|ccc} -a_1 & -a_2 & \cdots & & \cdots & & -a_n \\ 1 & 0 & \cdots & & \cdots & & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ \hline \vdots & & & 1 & 0 & \cdots & 0 \\ \vdots & & & & 1 & & \vdots \\ \vdots & & & & 0 & & \vdots \\ \vdots & & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & 1\ 0 \end{array}\right] x(i) + \left[\begin{array}{ccc} 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline & \hat{G} & \end{array}\right] u(i+1) \tag{51}$$

$$y(i) = [-a_1 - a_2, \ldots -a_n]x(i), \quad q \le i \le N. \tag{52}$$

The parameters $\{a_1, \ldots, a_n\}$ are the coefficients of the characteristic polynomial of ϕ. The submatrix $\hat{G}$ of G is $(n-q) \times m$. The system is assumed to be completely observable.

Proof. The form (51)–(52) follows from the constraints (36a) and (42) after a state-transformation

$$x_{\text{new}}(i) = E_n^* x_{\text{old}}(i) \tag{53}$$

with E_n^* as defined in (29).

Comments.

(1) The continuous-time analog of this result is presented in [7, Appendix II].

(2) Notice from the tapped delay structure of ϕ that

$$y(i) = x^1(i+1) = x^2(i+2) = \ldots = x^q(i+q). \tag{54}$$

It is obvious therefore, that the predicted Kalman filter estimates of the first q components coincide with the state components themselves

$$\hat{x}^k(i|i-1) = x^k(i), \; k = 1, \ldots, q. \tag{55}$$

Thus the Kalman filter needs to be written only for the $n-q$ remaining components of $\hat{x}(i|i-1)$.

(3) From the remarks just made we know that the error–covariance matrix, $P(i|i-1)$, has the form

$$P(i|i-1) = \left[\begin{array}{ccc|c} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & 0 & \cdots & 0 \\ \hline \vdots & \vdots & & \\ \vdots & \vdots & & \hat{P}(i|i-1) \\ \vdots & \vdots & & \\ 0 & \cdots\ 0 & & \end{array}\right], \quad q < i < N \tag{56}$$

where $\hat{P}(i|i-1)$ has dimension $(n-q)x(n-q)$ and obeys the reduced-order Riccati equation

$$\hat{P}(i+1|i)=F\hat{P}(i|i-1)F'-F\hat{P}(i|i-1)b'[b\hat{P}(i|i-1)b']^{-1}b\hat{P}(i|i-1)F'+\hat{G}\hat{G}',\ q\leq i\leq N \quad (57)$$

$$\hat{P}(q|q-1)=\text{lower righthand submatrix of } P(q|q-1) \quad (58)$$

where F is the lower righthand submatrix of ϕ in (51) and

$$b=[-a_{q+1},\ldots,-a_n]. \quad (59)$$

The reader is referred to [5] for more details on the structural properties of the Kalman filter and the whitening filter of a system with a q-dimensional predictable subspace.

(4) Notice finally that a system that has q predictable directions has at least $q+1$ degenerate, or invariant, directions. Indeed it follows from (51), (52) and (54) that

$$\hat{x}^{q+1}(i|i-1)=y(i-q-1). \quad (60)$$

This last estimate is not perfect, but its error has constant variance, i.e.

$$E[(x^{q+1}(i)-\hat{x}^{q+1}(i|i-1))^2]=\text{constant}. \quad (61)$$

Finally we note that we have assumed throughout that ϕ is nonsingular, and in fact the basis for the predictable subspace actually involves ϕ^{-1}. But it is well known that in discrete time the state-transition matrix may be singular. It can be shown, [5], that in such case the system (28) can have a predictable direction only if there exists some n-vector a such that $a'\phi=h$. However, in such case the system is not completely observable. The reader is referred to [5] for more details.

4. TRANSFER FUNCTION ANALYSIS

The concepts of degenerate and predictable directions, and the relation between these two concepts, become intuitively obvious when they are examined in the z-transform domain for single-input single-output (SISO) systems. In this section we examine the relation between the number of predictable and degenerate directions of a constant SISO system and the form of its transfer function and the power spectral density of its output. Next we shall show how the Kalman filter equations for the one-step prediction state estimates of a system with α degenerate but nonpredictable directions can be replaced by filter equations for a related system that has $\alpha-1$ predictable directions.

Consider the SISO system (28) with G replaced by a n-vector g. The general form of its transfer function is

$$H(z)=\frac{b_0+b_1z^{-1}+\ldots+b_{n-1}z^{-n+1}}{1+a_1z^{-1}+\ldots+a_nz^{-n}}. \quad (62)$$

We assume that the system is minimal and that $a_n\neq 0$, so that ϕ is invertible. We have the following simple lemmas.

Lemma 1

(MORF [11]). The SISO system (28) with transfer function (62) has q predictable directions if and only if

$$b_{n-1}=b_{n-2}=\ldots=b_{n-q}=0,\ b_{n-q-1}\neq 0. \quad (63)$$

Proof. By Theorem 3 we need only show the equivalence between the conditions (63) and (36). From the realization of (62) in control canonical form, recalling that the number of predictable directions is the same for all realizations of a same transfer function, one finds

$$h\phi^{-1}g=-\frac{1}{a_n}b_{n-1} \quad (64)$$

$$h\phi^{-2}g=\frac{a_{n-1}}{a_n}b_{n-1}-\frac{1}{a_n}b_{n-2} \quad (65)$$

etc. The equivalence between (63) and (36) follows immediately.

Lemma 1 shows that the number of predictable directions is the difference between the degrees of the denominator and the numerator polynomials of its transfer function, minus one. This number has been called the relative order of the transfer function (cf. Section 1).

Lemma 2

The SISO system (28) with transfer function (62) has α degenerate, or invariant, directions, q of them predictable, if and only if (14) holds and

$$b_0=b_1=\ldots=b_{s-2}=0,\ b_{s-1}\neq 0 \quad (66)$$

for $s\triangleq\alpha-q>1$.

Proof. The first part is true by Lemma 1. Furthermore, we know by Comment 4 of Section 3 that the system (28) always has one degenerate but nonpredictable direction. Hence if $s=1$, no further condition is needed. Let $s>1$. By Proposition 1 the system has α degenerate directions if and only if

$$l_1=l_2=\ldots=l_{\alpha-1}=0 \quad (67)$$

with l_j defined by (13). Together with the conditions (36) for q predictable directions, (67) implies

$$hg = h\phi g = \dots = h\phi^{s-2}g = 0. \quad (68)$$

These are the first $s-1$ terms of the impulse response and they are zero if and only if (66) holds as can be easily verified by obtaining the impulse response from (62) by "long division".

If a system with transfer function (62) is driven by unit variance white noise, the power spectral density of its output has the form

$$S_y(z) = H(z)H(z^{-1})$$
$$= \frac{c_0 + c_1(z+z^{-1}) + \dots + c_{n-1}(z^{n-1}+z^{-n+1})}{d_0 + d_1(z+z^{-1}) + \dots + d_n(z^n+z^{-n})} \quad (69)$$

with $d_n = a_n \neq 0$. The parameters c_k are related to the parameters b_k as follows

$$c_{n-1} = b_0 b_{n-1}$$
$$c_{n-2} = b_0 b_{n-2} + b_1 b_{n-1}$$
$$c_{n-3} = b_0 b_{n-3} + b_1 b_{n-2} + b_2 b_{n-1} \quad (70)$$

etc.

Theorem 7

The number of degenerate directions of any realization with power spectral density (69), or the relative order of the signal $y(\cdot)$, is the difference between the degrees of the denominator and numerator polynomials.

Proof. The proof is obvious from Lemma 2 and the relations (70).

Comment. Let a signal $y(\cdot)$ of definite relative order α have power spectral density $S_y(z)$, and let $H_0(z)$ be a factorization of $S_y(z)$ such that

$$H_0(z) = \frac{b_{\alpha-1}z^{-\alpha+1} + b_\alpha z^{-\alpha} + \dots + b_{n-1}z^{-n+1}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}. \quad (71)$$

That is, $y(\cdot)$ is obtained by passing unit variance white noise through the transfer function $H_0(z)$. But if $H_0(z)$ is a proper factorization of $S_y(z)$, so are the following

$$H_k(z) = z^k H_0(z),\ 0 \le k \le \alpha - 1, \quad (72)$$

and in particular

$$H_{\alpha-1}(z) = \frac{b_{\alpha-1} + b_\alpha z^{-1} + \dots + b_{n-1}z^{-n+\alpha}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}. \quad (73)$$

This follows from the fact that

$$S_y(z) = H_0(z)H_0(z^{-1}) = z^k H_0(z)\cdot z^{-k}H_0(z^{-1}). \quad (74)$$

But notice from Lemma 1 that $H_0(z)$ has no predictable directions while the number of predictable directions of the transfer function (72) is equal to the number of zeroes that is added to the transfer function $H_0(z)$, i.e. the power of z by which $H_0(z)$ is multiplied. Stated otherwise, a transfer function with α degenerate and q predictable directions ($q < \alpha - 1$) can be transformed into a transfer function with $\alpha - 1$ predictable directions by removing $\alpha - q - 1$ delays from the feedforward loops, without affecting the relative order of the output signal.

These observations suggest a procedure to replace the Kalman filter equations of a SISO system with α degenerate directions, q of them predictable, by a Kalman filter of a related system with $\alpha-1$ predictable and one degenerate direction. Thus consider a SISO system with α degenerate directions, q of them predictable ($q < \alpha - 1$). In terms of h, ϕ and g the transfer function is

$$H(z) = h[I - z^{-1}\phi]^{-1}g. \quad (75)$$

The maximum number of predictable directions ($\alpha - 1$) is obtained by replacing $H(z)$ by

$$H^*(z) = hz^{\alpha-q-1}[I - z^{-1}\phi]^{-1}g. \quad (76)$$

This can be achieved by replacing h by

$$h^* = h\phi^{\alpha-q-1}, \quad (77)$$

which effectively removes $\alpha - q - 1$ delays from the feedforward loops. The system h^*, ϕ, g now has $\alpha - 1$ predictable directions, since by condition (68) of Lemma 2 and (77) we have

$$\begin{cases} h^*\phi^{-1}g = h\phi^{\alpha-q-2}g = 0 \\ \quad\vdots \\ h^*\phi^{-\alpha+1}g = h\phi^{-q}g = 0. \end{cases} \quad (78)$$

Let θ be defined as

$$\theta(i) = x(i - \alpha + q + 1),\ i \ge \alpha - q - 1 \quad (79)$$

where $x(\cdot)$ is the state of the original system. Then, for $i \geq \alpha - q - 1$, the original system can be replaced by

$$\begin{cases} \theta(i+1) = \phi\theta(i) + gu(i-\alpha+q+2) & \text{(80a)} \\ y(i) = h^*\theta(i). & \text{(80b)} \end{cases}$$

The predicted estimates $\hat{\theta}(i+1|i)$ can be computed using the reduced order Kalman filter mentioned in Section 3 and for which $\alpha - 1$ components are known without error from past outputs. These predicted states are related to the predicted states $\hat{x}(i+1|i)$ of the system (28) as follows.

Theorem 8

Let the SISO system (28) have α degenerate and q predictable directions. If $\theta(\cdot)$ is defined by (79) and if $P_\theta(i+1|i)$ is the error–covariance matrix for the Kalman filter estimates $\hat{\theta}(i+1|i)$ of $\theta(i+1)$, then

$$\hat{x}(i+1|i) = \phi^{\alpha-q-1}\hat{\theta}(i+1|i) \qquad (81a)$$

$$P(i+1|i) = \phi^{\alpha-q-1}P_\theta(i+1|i)\phi'^{\alpha-q-1} + \sum_{j=0}^{\alpha-q-2} \phi^j gg'\phi'^j \qquad (81b)$$

where $P(i+1|i)$ is the error–covariance matrix of the predicted estimates $\hat{x}(i+1|i)$.

Proof. By Lemma 2 the relations (68) hold for the parameters h, ϕ, g of the original system (28). It follows easily that

$$\hat{u}(i-k|i) = 0,\ k = 0, 1, \ldots, \alpha - q - 2. \qquad (82)$$

By the state equation for $x(\cdot)$ we have

$$x(i+1) = \phi^{\alpha-q-1}x(i-\alpha+q+2) + \sum_{j=0}^{\alpha-q-2} \phi^j gu(i+1-j). \qquad (83)$$

Conditioning on $\{y(0), \ldots, y(i)\}$ and taking the expected values gives, using (82) and (79),

$$\hat{x}(i+1|i) = \phi^{\alpha-q-1}\hat{x}(i-\alpha+q+2|i)$$
$$= \phi^{\alpha-q}\hat{x}(i-\alpha+q+1|i) \qquad (84a)$$
$$= \phi^{\alpha-q-1}\hat{\theta}(i+1|i) = \phi^{\alpha-q}\hat{\theta}(i|i) \qquad (84b)$$

which proves (81a). Subtracting (84b) from (83), using (79), and taking the variance gives (81b).

Comments.

(1) It should be clear from the proof of Theorem 8 that the predicted estimates $\hat{\theta}(i+1|i)$ of the transformed system (80) are the smoothed estimates of the original system. Thus, in a SISO system, the degenerate costates that are not predictable are costates for which a perfect smoothed estimate can be obtained. That the predicted estimates $\hat{x}(i+1|i)$ are obtained by simply multiplying the smoothed estimates $\alpha - q$ steps earlier by $\phi^{\alpha-q}$, as indicated in (84a), is a consequence of the relative order property that is responsible for the crucial relations (82). These imply that the outputs $\{y(i-\alpha+q+2), \ldots, y(i)\}$ do not bring any information about the inputs $\{u(i-\alpha+q+2), \ldots, u(i)\}$.

(2) As for all procedures involving degenerate or predictable directions in discrete-time, the procedure suggested here can start only after a transient period, i.e. for $i \geq \alpha - 1$. For $i \leq \alpha - 1$, the full-dimension filter has to be used. Actually the Riccati equation for $P_\theta(i+1|i)$ has to be started with initial condition

$$P_\theta(\alpha-1|\alpha-2) = P(q|\alpha-2) \qquad (85)$$

where $P(q|\alpha-2)$ is the error–covariance matrix for the smoothed estimate $\hat{x}(q|\alpha-2)$. It is obtained from $P(q|q-1)$ by setting equal to zero the projections of $P(q|q-1)$ on $h^*\phi^{-1}, \ldots, h^*\phi^{-\alpha+q+1}$. For more details the reader is referred to [5].

5. EXAMPLE

Consider the SISO system

$$x(i+1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(i) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(i+1),\ x(0) = x_0 \qquad (86a)$$

$$y(i) = [0\ \ 0\ \ 1]x(i) \qquad (86b)$$

where x_0 and $u(\cdot)$ obey the usual assumptions. The system is minimal, and we have

$$h\phi^{-1}g = h\phi^{-2}g = 0,\ h\phi^{-3}g \neq 0. \qquad (87)$$

By Theorem 3 we know that the system has two predictable directions. By Comment 4 of Section 3 we know that the third direction is degenerate. By Corollary 5 we have

$$h\phi^{-1}x(i) = x^1(i) = y(i-1),\ i \geq 1 \qquad (88a)$$

$$h\phi^{-2}x(i)=x^2(i)=y(i-2),\ i\geq 2. \tag{88b}$$

Notice that the system is already in the canonical form of (51)–(52). The Kalman filter is, trivially

$$\hat{x}(i|i-1)=\begin{bmatrix} y(i-1) \\ y(i-2) \\ y(i-3) \end{bmatrix},\quad i\geq 3. \tag{89}$$

The error–covariance matrix is

$$P(i|i-1)=GG'=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad i\geq 3. \tag{90}$$

Notice that the input–output relation is of the autoregressive form

$$y(i)-y(i-3)=u(i),\ i\geq 3 \tag{91}$$

and

$$H(z)=\frac{1}{1-z^{-3}}. \tag{92}$$

The power spectral density of $y(\cdot)$ is

$$S_y(z)=H(z)H(z^{-1})=\frac{1}{2-(z^3+z^{-3})}. \tag{93}$$

An alternate realization for $y(\cdot)$ is with

$$\bar{H}(z)=\frac{z^{-2}}{1-z^{-3}} \tag{94}$$

i.e. ϕ and g are as in (86a) and

$$\bar{h}=[0\quad 1\quad 0]. \tag{95}$$

This realization has, of course, also three degenerate directions, since this is a property of the power spectral density of $y(\cdot)$, but it has no predictable directions, i.e. $\alpha=3$, $q=0$. Notice that the first realization is obtained from the second by replacing $\bar{h}$ by $\bar{h}\phi^2$, as expressed in (77).

6. CONCLUSIONS

We have used the results of Ref. [6] to show how the much discussed problems of correlated noise filtering and invariant directions can be better understood in terms of the relative order of the covariance of the output signal. Once the connection is established between this relative order and the number of degenerate directions, it becomes clear that the differencing techniques [1, 2] do not always lead to the lowest-order Riccati equation. Indeed the results of [6] show that the relative orders of a signal covariance and its realizations do not usually coincide in discrete-time, although they always coincide for innovations models.

In Ref. [12] it has been shown that in certain problems the autoregressive moving average (ARMA) model is superior to the state-variable model. The results of the present paper reinforce this conclusion. Indeed the degenerate and predictable costates artificially augment the dimension of the Riccati equation in the state-estimation problem. The computation of a predictable costate through the Riccati equation leads to ill-conditioning and is actually unnecessary whenever $\hat{y}(i|i-1)$ rather than $\hat{x}(i|i-1)$ is needed. It should be clear from Lemma 2, for example, that if an ARMA model with n autoregressive and m moving average parameters $(m\leq n)$ is transformed into state-variable form, the n-th order Riccati equation to be solved for $P(i|i-1)$ will have a true dynamic rank of m, rather than n, it will be ill-conditioned whenever $n>m$. In particular the computation of $\hat{y}(i|i-1)$ in a pure n-th order autoregressive model, solved via a state-variable model, leads to a model that has $n-1$ predictable directions and hence a Riccati matrix that has $n-1$ singularities; this follows directly from Lemma 1. A direct solution via the ARMA model, using the innovations methods of [12], is preferable in all such cases. The matrix to be inverted is $R\varepsilon(\cdot)$, the covariance matrix of the innovations, and this will always be nonsingular if the signal $y(\cdot)$ has a positive definite covariance.

Even though some more work needs to be done in the multivariable case, where we are now developing [13] the appropriate matrix polynomial transfer function specifications, we believe that the results of this paper show conclusively that in the discrete-time filtering and prediction problems the properties of the signal covariance should dictate the choice of the model that is best suited for the particular problem. The presence of degenerate or predictable directions and the fact that some degenerate directions may not be predictable merely reflect the fact that in discrete time all models do not inherit the properties of the signal.

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Résumé—Plusieurs articles ont étudié le fait que l'équation de Riccati pour certains systèmes stochastiques linéaires discrest peut atteindre une valeur constante dans certaines directions après un nombre restreint d'itérations. La présence de ces directions constantes permet une réduction de l'ordre effectif de l'équation de Riccati. Curieusement ces résultats n'ont pas d'analogues exacts en temps continu. Dans cet article nous présentons les raisons qui expliquent ce comportement différent. Pour ce faire nous introduisons le concept de directions prédictibles. Ce sont des directions dans lesquelles la solution de l'équation de Riccati s'annule au lieu de tendre vers une constante non nulle. En temps continu, les directions prédictibles et constantes coincident et leur nombre dépend de l'ordre relatif (une mesure de la continuité) de la fonction de transfert du système ou - ce qui est équivalent - de la covariance de sa sortie. Cette équivalence n'existe pas en temps discret, ou nous montrons que le nombre de directions constantes est égal à l'ordre relatif de la covariance tandis que le nombre de directions prédictibles est égal à l'ordre relatif de la fonction de transfert. Notre approche montre non seulement comment convertir des directions constantes endi rections prédictibles, mais elle montre aussi que le concept peut être étendu à des systèmes à paramètres variables le temps.

Zusammenfassung—Mehrere frühere Arbeiten befaßten sich mit dem Phänomen, daß die Lösung der Riccati-Gleichung für gewisse lineare stochastische Diskretzeitsysteme nach einer begrenzten Zahl von Iterationen in gewissen Richtungen einen konstanten Wert erreichen kann und so eine Reduktion in der effektiven Ordnung der Riccati-Gleichung ermöglicht. Seltsamerweise haben diese Ergebnisse keine exakten Analoga bei kontinuierlicher Zeit. In dieser Arbeit setzen wir die Gründe hierfür auseinander, hauptsächlich durch Einführung des Konzepts vorhersagbarer Richtungen, längs denen die Lösung eher zu Null als zu einer von Null verschiedenen Konstanten geht. Bei kontinuierlicher Zeit fallen die vorhersagbaren und konstanten Richtungen zusammen und ihre Zahl hängt von der relativen Ordnung (einem Maß der Glättung) der Übertragungsfunktion des Systems oder äquivalent von seiner Ausgangskovarianz ab. Diese Äquivalenz bricht in Diskretzeit ab, wo die Zahl der konstanten Richtungen die relative Ordnung der Kovarianz ist, während die Zahl vorhersagbarer Richtungen die relative Ordnung der Übertragungsfunktion ist. Die durch unsere Näherung erlangte Einsicht zeigt nicht nur, wie die konstanten Richtungen in vorhersagbare Richtungen übergeführt werden, sondern zeigt auch, wie das Konzept auf zeitvariante Systeme ausgedehnt werden könnte, wo die Bezeichnung degenerierte Richtungen mehr erläutert als konstante Richtungen.

Резюме—Ряд последних статей посвящен явлению, когда решение уравнения Рикатти для определенных линейных дискретных во времени стохастических систем может достичь постоянного значения в определенных направлениях после ограниченного числа итераций, позволяя тем самым уменьшение эффктивного порядка уравнения Рикатти Курьезно, но эти результаты не имеют точного аналога для непрерывных случаев. В данной статье мы объясняем причину этого, главным образом введением понятия предсказуемого направления вдоль которого решение идет к нулю, а не к ненулевой постоянной В непрерывном по времени случае предсказуемое и постоянное направления совпадают, а цх число зависит от относительного порядка (мера гладкости) передаточной функции системы или эквивалентности ее выходного рассогласования Эта эквивалентность разбивается на временные интервалы, где число постоянных направлений соответствует относительному порядку рассогласования в то время как число предсказуемых направлений соответствует относительному порядку передаточной функции. Понимание, обеспечиваемое нашим подходом не толькопоказыа ет, как преобразовать постоянные направления в предсказуемые, но показывает также как подобн концепцию можно распространить на дискретные во времени системы, где вырождающиеся направления более информативны чем постоянные