Correlation-Based Tuning of Decoupling Multivariable Controllers

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Abstract

The iterative method labelled Correlation-based Tuning (CbT) is considered in this paper for tuning linear time-invariant multivariable controllers. The approach allows one to tune some elements of the controller transfer function matrix to satisfy the desired closed-loop performance, while the other elements are tuned to mutually decouple the closed-loop outputs. Decoupling is achieved by decorrelating a given reference with the non-corresponding outputs. The controller parameters are calculated either by solving a correlation equation (decorrelation procedure) or by minimizing a cross-correlation function (correlation reduction). In addition, the preferred way of exciting a 2×2 system for CbT is investigated via the accuracy of the estimated controller parameters. It is shown that simultaneous excitation of both reference signals does not improve the accuracy of the estimated controller parameters compared to the case of sequential excitation. In fact, one must choose between low experimental cost (simultaneous excitation) and better accuracy of the estimated parameters (sequential excitation). The theoretical results are illustrated via three simulation studies.

Key words: Correlation-based tuning, data-based controller design, multivariable control, instrumental variables, decoupling, asymptotic variance expressions.

1 Introduction

The essential ingredients of any control design procedure include the acquisition of process knowledge and its efficient integration into the controller. Reliable models of industrial plants are often difficult or impossible to obtain due to the high complexity of the plants and/or the excessive cost of modeling. On the other hand, the controllers designed on the basis of reduced-order models might well lead to unsatisfactory performance when applied to real plants due to model mismatch.

An alternative to model-based control design is to use the information collected on the plant *directly* for controller design or controller update, i.e. without the intermediate step of estimating a plant model. This idea stems from the area of direct adaptive control, in particular from self-tuning regulation (STR) and model reference adaptive control (MRAC) [2]. Recently, several methods have appeared in the field of data-driven controller tuning; these methods include controller unfalsification [20,1], Simultaneous Perturbation Stochastic Approximation control (SPSA) [24], Iterative Feedback Tuning (IFT) [8] and Virtual Reference Feedback Tuning (VRFT) [4]. An important issue in this research area regards the way to cope with the noise that necessarily corrupts the measurements and therefore also affects closed-loop performance.

Recently, another controller tuning approach was proposed in [11]. It is labelled Correlation-based Tuning (CbT) since its underlying idea is inspired by the well-known correlation approach in system identification [22]. The controller parameters are tuned to decorrelate the closed-loop output error between the designed and achieved closed-loop sys-

tems from the external reference signal. Ideally, the resulting closed-loop output error contains only the contribution of the noise and thus perfect model-following can be achieved. It is shown in [12] that the sensitivity functions of the achieved closed-loop system tend asymptotically to those of the desired closed-loop system. The analysis performed in [15] shows that the CbT controller is asymptotically insensitive to noise. One may argue that this insensitivity to noise implies that CbT is not appropriate for disturbance rejection. However, since the output sensitivity function of the resulting closed-loop system approaches that of the desired closed-loop system, it is possible to handle the disturbances indirectly by imposing disturbance rejection specifications on the desired output sensitivity function. An adaptation of this approach to the disturbance rejection problem has been treated in [15] in the situation where the disturbances can be measured or where it is possible to inject a test signal in closed-loop operation. A theoretical presentation of CbT is given in [12].

CbT belongs to gradient-based methods and uses an identified model for computing the gradient of the underlying criterion. Thus, it can-not be classified as a "model-free" method. However, since a model is not used explicitly to compute the controller, it is not "model-based" either. Considering that the controller parameters are computed directly using closed-loop data, the approach can be considered as "data-driven".

A common approach to multivariable control is to proceed in two steps: first a "decoupler" is designed to eliminate process interactions, then a set of controllers is tuned for the "diagonalized" plant to achieve specific performance objectives [21]. The design of decouplers and controllers using standard model-based methods may be very sensitive to modeling errors and uncertainties. On the other hand, since data-driven methods use data collected in closed-loop operation, these data reflect the local behavior of the plant in the vicinity of the present operating point (with the acting controller). The use of the present closed loop system as an "implicit model" for computing decouplers and controllers is likely to improve the performance. However, the difficulty with data-driven gradient-based methods is the computation of the gradient of the criterion. Typically, the number of experiments needed to estimate the gradient increases with the number of plant inputs n_u and outputs n_y. For example, the IFT approach typically requires $n_y n_u + 1$ experiments for each iteration of the controller parameter vector [7], even though some efforts have been made recently to reduce the number of experiments with this approach (for more details see [5,6,10]).

The main contribution of this paper is to extend the CbT method developed in [11,12,15] to the design of multivariable LTI decoupling controllers. For simplicity of presentation, an equal number of inputs and outputs is assumed. The off-diagonal elements of the controller transfer function matrix are tuned to eliminate interaction between the controlled outputs (in the sequel this will be called "diagonal-

ization of the closed-loop system"), while the elements on the main diagonal are tuned to provide the desired closed-loop performance. The fact that the decoupling is done in a natural way by decorrelating a given reference from the non-corresponding outputs without the need for additional experiments makes CbT particularly appealing for tuning MIMO controllers. The controllers on the main diagonal feature the same characteristics as those for SISO systems. The parameters of the decouplers and controllers are asymptotically not affected by noise. A single experiment per iteration is sufficient for tuning all controllers and decouplers regardless of the number of inputs and outputs since all reference inputs can be excited simultaneously.

Two ways of computing the controller parameters can be distinguished, depending on the complexity of the controller with respect to the plant. When the controller structure is able to perfectly decorrelate the closed-loop output error from the reference signal, the controller parameters can be calculated as the solution to a system of correlation equations. This approach is called "decorrelation procedure". When this is not possible, the controller parameters are computed so as to minimize a correlation criterion; this approach is labelled "correlation reduction". The features of these two variants are discussed and compared in terms of applicability to practical control situations.

Our work shows that CbT offers a considerable advantage in terms of decoupling compared to other data-driven control design methods in which the controller parameters are calculated by minimizing the norm of some error signal, such as is done with the IFT or VRFT methods. Using the latter methods, perfect decoupling cannot be achieved due to the nature of the underlying criterion that introduces a trade-off between the decoupling objective and the noise rejection objective. In contrast, the CbT criterion is asymptotically independent of the noise so that the resulting controller satisfies the decoupling specifications exactly, provided that it is of appropriate order.

An additional contribution of this paper is to provide an analysis of the variance of the estimated controller parameters for two cases of excitation. The case where all reference signals are excited simultaneously is compared to the case where the elements of the reference signal vector are excited one-by-one; the latter type of excitation will be called "sequential excitation". Our analysis shows that, as a result of the decoupling specifications, the simultaneous excitation of all references increases the variance of the estimated controller parameters compared to the case of sequential excitation.

The remainder of the paper is organized as follows. Some notations and the basic idea of the multivariable CbT approach are introduced in Section 2. Section 3 deals with the tuning of multivariable LTI controllers by both the decorrelation procedure and the correlation reduction procedure. Simulation results are presented in Section 4, and concluding remarks are given in Section 5.

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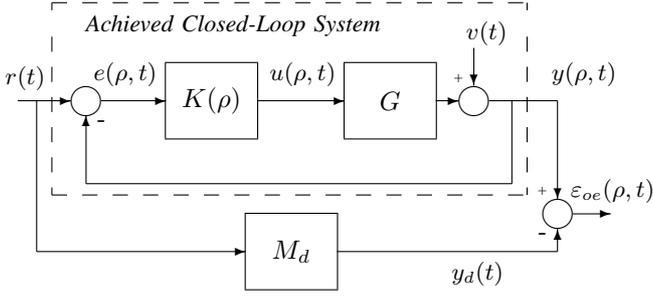
Fig. 1. Achieved multivariable closed-loop system and its reference model M_d

2 Preliminaries

2.1 System description and notations

For the sake of simplicity, and without loss of generality, it is assumed that the plant has two inputs and two outputs. Consider the block diagram of the model-following problem presented in Fig. 1. The upper part shows the achieved closed-loop system with the unknown true plant whose outputs can be described by the following LTI multivariable discrete-time model:

$$y(t) = G(q^{-1})u(t) + v(t) \qquad (1)$$

where $y(t) \in \mathcal{R}^2$ denotes the outputs of the true plant at time t, $u(t) \in \mathcal{R}^2$ the control signals, $v(t) \in \mathcal{R}^2$ the disturbances on the outputs, and $G(q^{-1}) \in \mathcal{R}^{2 \times 2}$ a transfer function matrix with q^{-1} being the backward-shift operator. It is assumed that $v(t)$ is a zero-mean quasi-stationary stochastic process.

The 2×2 controller transfer function matrix $K(q^{-1}, \rho)$ is parameterized using the parameter vector $\rho \in \mathcal{R}^{n_\rho}$, and $r(t) \in \mathcal{R}^2$ represents external reference signals.

The (j, k) element of the controller transfer function matrix is:

$$K_{jk}(q^{-1}, \rho) = \frac{S_{jk}(q^{-1}, \rho)}{R_{jk}(q^{-1}, \rho)} \quad j, k = 1, 2 \qquad (2)$$

where

$$R_{jk}(q^{-1}, \rho) = 1 + r_{jk}^{(1)} q^{-1} + \cdots + r_{jk}^{(n_r)} q^{-n_r}$$
$$S_{jk}(q^{-1}, \rho) = s_{jk}^{(0)} + s_{jk}^{(1)} q^{-1} + \cdots + s_{jk}^{(n_s-1)} q^{-n_s+1} \qquad (3)$$

It is assumed, for simplicity of notation, that the four controllers $K_{jk}(q^{-1}, \rho)$, $j = 1, 2$, $k = 1, 2$ have the same number of poles and the same number of zeros. Moreover, it is assumed that the controllers have no common parameters. The controller parameter vector ρ can be written as follows:

$$\rho^T = [\rho_{K_{11}}^T, \rho_{K_{12}}^T, \rho_{K_{21}}^T, \rho_{K_{22}}^T] \qquad (4)$$

where

$$\rho_{K_{jk}}^T = [r_{jk}^{(1)}, r_{jk}^{(2)}, \dots, r_{jk}^{(n_r)}, s_{jk}^{(0)}, s_{jk}^{(1)}, \dots, s_{jk}^{(n_s-1)}] \qquad (5)$$

Thus, $n_\rho = 4(n_r + n_s)$.

The lower part of Fig. 1 shows the reference model M_d defining the desired response of the closed-loop outputs, $y_d(t)$ to the reference signals $r(t)$. In this paper, we consider that this desired reference model is diagonal. It can be constructed, for example, as the closed-loop system obtained from the feedback interconnection of a model G_0 of the plant G and a controller K_0:

$$M_d \triangleq \begin{pmatrix} M_{d11} & 0 \\ 0 & M_{d22} \end{pmatrix} = (I + G_0 K_0)^{-1} G_0 K_0. \qquad (6)$$

with $I \in \mathcal{R}^{2 \times 2}$ being the identity matrix. It is therefore assumed that G_0 and G are diagonalizable by output feedback. Necessary and sufficient conditions for a linear multivariable system to be diagonalizable by output feedback can be found in [25] and references therein.

The closed-loop response can be written as:

$$y(\rho, t) = \mathcal{T} r(t) + \mathcal{S} v(t), \qquad (7)$$

and the control error is:

$$e(\rho, t) = r(t) - y(\rho, t) = \mathcal{S}\left(r(t) - v(t)\right), \qquad (8)$$

where $\mathcal{S}$ denotes the output sensitivity function:

$$\mathcal{S} = (I + GK)^{-1} \qquad (9)$$

and $\mathcal{T}$ the complementary sensitivity function:

$$\mathcal{T} = (I + GK)^{-1} GK. \qquad (10)$$

The closed-loop output error is defined as:

$$\varepsilon_{oe}(\rho, t) = y(\rho, t) - y_d(t). \qquad (11)$$

Notational Remarks

The signals collected under closed-loop operation using the controller $K(\rho)$ will carry the argument ρ. The elements of signal vectors and transfer function matrices will carry the position as a subscript. For example, $y_k(\rho, t)$ will denote the kth component of the output vector $y(\rho, t)$. In contrast, the coefficients in numerator and denominator polynomials of the controllers $K_{jk}(q^{-1}, \rho)$ will carry the position as a superscript: see (5). Furthermore, the backward-shift operator q^{-1} will be omitted whenever appropriate.

2.2 Multivariable correlation-based tuning

Consider the controller structure presented in Fig. 2. Since the desired sensitivity functions $\mathcal{S}_d$ and $\mathcal{T}_d$ are in diagonal form, it is required that GK be diagonal. Hence, the following design specifications will be imposed on the elements of K:

- The diagonal elements $K_{jj}(\rho)$, $j = 1, 2$ of the controller transfer matrix $K(\rho)$ are tuned to provide satisfactory tracking of $y_{d1}(t)$ by $y_1(\rho, t)$ and $y_{d2}(t)$ by $y_2(\rho, t)$, respectively.
- The off-diagonal elements $K_{jk}(\rho)$, $j \neq k$, $j, k = 1, 2$ are tuned to be decouplers. That is, the controller $K_{12}(\rho)$ is tuned to eliminate the influence of the reference signal $r_2(t)$ on the output $y_1(\rho, t)$. Hence, if the decoupler $K_{21}(\rho)$ is tuned similarly, the mutual influences of $y_1(\rho, t)$ and $y_2(\rho, t)$ are suppressed.

Consider first the tuning of the decoupler $K_{12}(\rho)$. When applying the controller K_0, defined in (6), to the true plant excited by the reference signal $r(t)$, the output $y_1(\rho, t)$ contains the contributions due to the reference signals $r_1(t)$ and $r_2(t)$ and the disturbance $v(t)$. The reference signals $r_1(t)$ and $r_2(t)$ are mutually independent and uncorrelated with $v(t)$. Hence, the idea is to adjust the parameters of $K_{12}(\rho)$ to make the output $y_1(\rho, t)$ uncorrelated with the reference signal $r_2(t)$. The resulting decoupler provides $y_1(\rho, t)$ that contains only the contributions due to $v_1(t)$ and $r_1(t)$, i.e. the influence of $v_2(t)$ and $r_2(t)$ on $y_1(\rho, t)$ is eliminated.

Consider next the tuning of $K_{11}(\rho)$. Again, with K_0 operating in the loop, the observed closed-loop output error $\varepsilon_{oe1}(\rho, t)$ contains a contribution due to the disturbance $v(t)$ and another contribution stemming from the difference between G and G_0 that, in turn, has two parts originating from $r_1(t)$ and $r_2(t)$. The idea is to adjust the parameters of $K_{11}(\rho)$ so as to make $\varepsilon_{oe1}(\rho, t)$ uncorrelated with $r_1(t)$. Since the parameters of $K_{11}(\rho)$ and $K_{12}(\rho)$ are tuned simultaneously, the effect of modeling errors excited by $r_2(t)$ is eliminated by the decoupler $K_{12}(\rho)$. Hence, the resulting controller compensates the effect of modeling errors to the extent that the closed-loop error $\varepsilon_{oe1}(\rho, t)$ contains only the disturbance filtered by the closed-loop system. This way, the output $y_1(\rho, t)$ will achieve the desired output $y_{d1}(t)$.

A similar reasoning follows for $K_{21}(\rho)$ and $K_{22}(\rho)$ that are related to the output $y_2(\rho, t)$.

3 CbT for the decoupling of MIMO systems

Let the cross-correlation function be defined as follows:

$$F(\rho) \triangleq E\left\{ \bar{F}(\rho) \right\} \qquad (12)$$
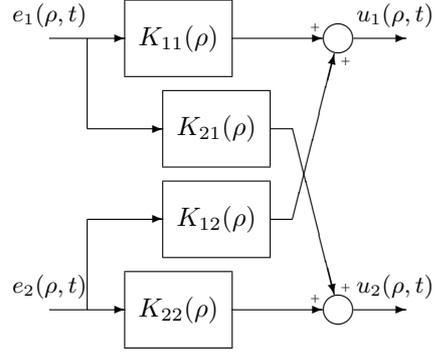
Fig. 2. Multivariable 2×2 controller. Signals $e_1(\rho, t)$ and $e_2(\rho, t)$ are defined in (8).

where $E\{\cdot\}$ is the mathematical expectation, and the vector $\bar{F}(\rho) \in \mathcal{R}^{n_\rho}$ reads:

$$\bar{F}^T(\rho) = \left[\bar{f}_{K_{11}}^T(\rho), \bar{f}_{K_{12}}^T(\rho), \bar{f}_{K_{21}}^T(\rho), \bar{f}_{K_{22}}^T(\rho) \right] \qquad (13)$$

with

$$\bar{f}_{K_{jk}}(\rho) = \frac{1}{N} \sum_{t=1}^{N} \zeta_{K_{jk}}(\rho, t) \mathcal{E}_{K_{jk}}(\rho, t) \qquad (14)$$

where N is the number of data and $\zeta_{K_{jk}}(\rho, t) \in \mathcal{R}^{n_\zeta}$ the vector of instrumental variables associated with the controller $K_{jk}(\rho)$. Note that $n_\zeta = n_r + n_s$, and $n_\rho = 4n_\zeta$. The component $\bar{f}_{K_{jk}}(\rho) \in \mathcal{R}^{n_\zeta}$ corresponds to the controller $K_{jk}(\rho)$. The way $\zeta_{K_{jk}}(\rho, t)$ and the variable $\mathcal{E}_{K_{jk}}(\rho, t) \in \mathcal{R}$ are constructed depends on whether the controller $K_{jk}(\rho)$ is a diagonal controller or an off-diagonal decoupler:

- $\underline{j = k}$: $K_{jj}(\rho)$ is tuned so as to reduce the correlation between $\varepsilon_{oej}(\rho, t)$ and $r_j(t)$. Taking into account the fact that the tuning of the controllers $K_{jj}(\rho)$ and the decouplers $K_{jk}(\rho)$ is done simultaneously, the output $y_j(\rho, t)$ will, in the case of perfect decorrelation, follow $y_{dj}(t)$ up to the effect of the disturbance. Thus, the vector of instrumental variables $\zeta_{K_{jj}}(\rho, t)$ should be chosen to be correlated with the reference signal $r_j(t)$ and independent of the disturbance $v_j(t)$. The variable $\mathcal{E}_{K_{jj}}(\rho, t)$ is chosen as $\varepsilon_{oej}(\rho, t)$.
- $\underline{j \neq k}$: To eliminate the influence of $r_k(t)$ on $y_j(t)$, it is sufficient to decorrelate these two signals, i.e. $\zeta_{K_{jk}}(\rho, t)$ should be correlated with $r_k(t)$ and $\mathcal{E}_{K_{jk}}(\rho, t) = y_j(\rho, t)$.

Hence, the variable $\mathcal{E}_{K_{jk}}(\rho, t)$ is constructed as follows:

$$\mathcal{E}_{K_{jk}}(\rho, t) = \begin{cases} \varepsilon_{oej}(\rho, t), & j = k \\ y_j(\rho, t), & j \neq k \end{cases} \qquad (15)$$

and the vectors of instrumental variables $\zeta_{K_{jk}}(\rho, t)$ for $j = k$ and $j \neq k$ are both a function of $r_k(t)$.

The parameters of the controller are computed either as the values that make the cross-correlation function (12) equal to

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zero or by minimization of a norm of this function. Observe that, in either case, the underlying criterion is nonlinear with respect to the controller parameters. Certain assumptions that are needed for the convergence and variance analysis are introduced next.

Assumptions

(i) The signals in the loop are bounded, i.e. the controllers stabilize the closed-loop system at each iteration.
(ii) The reference signals $r(t)$ are persistently exciting of sufficiently high order with respect to the number of controller parameters and are uncorrelated with the disturbances $v(t)$. Furthermore, the elements of the reference signal vector $r(t)$ are assumed to be mutually independent.
(iii) The decorrelating controller K^* exists and belongs to the set of parameterized controllers (the corresponding parameters will be denoted as ρ^*).

3.1 Decorrelation procedure for MIMO systems

Under Assumption (iii), the parameters of the controller that perfectly decorrelates $\zeta_{K_{jk}}(\rho, t)$ from $\mathcal{E}_{K_{jk}}(\rho, t)$ are computed as the solution to the following system of correlation equations:

$$F(\rho) = 0 \tag{16}$$

where $F(\rho)$ is defined by (12)-(14). A solution to (16) can be found using an iterative stochastic approximation procedure, for example the Robbins-Monro algorithm [19]:

$$\rho_{i+1} = \rho_i - \gamma_i \bar{F}(\rho_i). \tag{17}$$

where the sequence of positive numbers $\gamma_i \in \mathcal{R}$ is selected to satisfy the following convergence conditions: $\sum_{i=1}^{\infty} \gamma_i = \infty$ and $\sum_{i=1}^{\infty} \gamma_i^2 < \infty$.

It is shown in [12] that, under Assumptions (i)-(iii) and assuming that $F(\rho)$ possesses continuous partial derivatives of first and second order with respect to ρ, this scheme converges to a solution of the correlation equations (12), provided that $F(\rho)$ is monotonically increasing in the vicinity of the solution ρ^*, i.e. the following condition holds:

$$Q(\rho^*) = E \left\{ \frac{\partial \bar{F}(\rho)}{\partial \rho} \bigg|_{\rho = \rho^*} \right\} > 0. \tag{18}$$

The condition (18) is crucial for the convergence of the algorithm (17) to a solution of (16). One way to satisfy this condition is detailed in the next subsection.

3.1.1 Positive-definiteness of $Q(\rho^*)$

We now investigate the structure of $Q(\rho)$. It follows from (4), (13), (14) and (18) that $Q(\rho) \in \mathcal{R}^{n_\rho \times n_\rho}$ can be expressed as:

$$Q(\rho) = \begin{pmatrix} Q_{K_{11}}^{K_{11}} & Q_{K_{11}}^{K_{12}} & Q_{K_{11}}^{K_{21}} & Q_{K_{11}}^{K_{22}} \\ Q_{K_{12}}^{K_{11}} & Q_{K_{12}}^{K_{12}} & Q_{K_{12}}^{K_{21}} & Q_{K_{12}}^{K_{22}} \\ Q_{K_{21}}^{K_{11}} & Q_{K_{21}}^{K_{12}} & Q_{K_{21}}^{K_{21}} & Q_{K_{21}}^{K_{22}} \\ Q_{K_{22}}^{K_{11}} & Q_{K_{22}}^{K_{12}} & Q_{K_{22}}^{K_{21}} & Q_{K_{22}}^{K_{22}} \end{pmatrix} \tag{19}$$

where $Q_{K_{mn}}^{K_{jk}} \triangleq E\{\partial \bar{f}_{K_{jk}}(\rho)/\partial \rho_{K_{mn}}\}$ can be expressed as:

$$Q_{K_{mn}}^{K_{jk}} = E \left\{ \frac{1}{N} \sum_{t=1}^{N} \frac{\partial \zeta_{K_{jk}}(\rho, t)}{\partial \rho_{K_{mn}}} \mathcal{E}_{K_{jk}}(\rho, t) \right.$$
$$\left. + \frac{\partial \mathcal{E}_{K_{jk}}(\rho, t)}{\partial \rho_{K_{mn}}} \zeta_{K_{jk}}^{T}(\rho, t) \right\} \tag{20}$$

In the vicinity of the solution, the first term in (20) vanishes since the derivative of the instrumental variable vector $\zeta_{K_{jk}}(\rho, t)$ is not correlated with $\mathcal{E}_{K_{jk}}(\rho, t)$. Note also that:

$$\frac{\partial \mathcal{E}_{K_{jk}}(\rho, t)}{\partial \rho_{K_{mn}}} = \frac{\partial y_j(\rho, t)}{\partial \rho_{K_{mn}}} \quad \forall j, k. \tag{21}$$

At the solution ρ^*, (7)-(10) lead to:

$$\frac{\partial y(\rho, t)}{\partial \rho_{K_{mn}}^{(p)}} \bigg|_{\rho_{K_{mn}}^*} = \mathcal{S}(\rho^*) G \frac{\partial K(\rho)}{\partial \rho_{K_{mn}}^{(p)}} \bigg|_{\rho_{K_{mn}}^*} e(\rho^*, t) \tag{22}$$

where $p = 1, \ldots, n_r + n_s$. Considering that the subvectors $\rho_{K_{mn}}$ are independent and that $\mathcal{S}$ is diagonal at ρ^*, it follows from (22) and the second equality in (8) that:

$$\frac{\partial y(\rho, t)}{\partial \rho_{K_{mn}}^{(p)}} \bigg|_{\rho_{K_{mn}}^*} \overset{f}{\sim} e_n(\rho^*, t) \overset{f}{\sim} r_n(t), v_n(t) \tag{23}$$

where $\overset{f}{\sim}$ denotes that the signal on the left-hand side of this operator is a function of the right-hand side signal. Furthermore, according to the discussion leading to (15), one can write:

$$\zeta_{K_{jk}}(\rho, t) \overset{f}{\sim} r_k(t). \tag{24}$$

Using the relationships (23) and (24), the expression (20), and the fact that $r_1(t)$, $r_2(t)$, $v_1(t)$ and $v_2(t)$ are uncorrelated, it follows that

$$Q_{K_{mn}}^{K_{jk}} = 0, \quad k \neq n \tag{25}$$

i.e. the matrix $Q(\rho^*)$ takes the following form:

$$Q(\rho^*) = \begin{pmatrix} Q_{K_{11}}^{K_{11}} & 0 & Q_{K_{11}}^{K_{21}} & 0 \\ 0 & Q_{K_{12}}^{K_{12}} & 0 & Q_{K_{12}}^{K_{22}} \\ Q_{K_{21}}^{K_{11}} & 0 & Q_{K_{21}}^{K_{21}} & 0 \\ 0 & Q_{K_{22}}^{K_{12}} & 0 & Q_{K_{22}}^{K_{22}} \end{pmatrix} \qquad (26)$$

where

$$Q_{K_{mk}}^{K_{jk}} \overset{f}{\sim} r_k(t), v_k(t). \qquad (27)$$

From (20) and (26), it is obvious that the choice of the instrumental variables $\zeta_{K_{jk}}$ affects the positive definiteness of $Q(\rho^*)$. Hence, it is important to take this fact into account when constructing the instrumental variables. This will be investigated next.

Choice of Instrumental Variables

In the case of SISO systems, the typical choice for the instrumental variable vector is a noise-free estimate of the gradient $\partial y_j / \partial \rho_{K_{mn}}$, which guarantees the positive definiteness of $Q(\rho^*)$. However, in the case of MIMO systems, the choice of $\zeta_{K_{jk}}(\rho, t)$ is less trivial since the aforementioned choice ensures only the positive definiteness of the elements on the principal diagonal of $Q(\rho^*)$.

It is clear from (24) that the instrumental variables should be chosen as signals obtained by filtering $r_k(t)$. Therefore, the instrumental variables can be generated using the following model structure:

$$\zeta_{K_{jk}}(\rho, t) = \sum_{l=0}^{n_h - 1} \left(\mathcal{F}_l(q^{-1}) r_k(t) \right) h_{K_{jk}}^l \qquad (28)$$

where n_h denotes the model order, $\mathcal{F}_l(q^{-1})$ is the lth element of a set of stable basis transfer functions and $h_{K_{jk}}^l$ the corresponding weighting coefficient. The simplest choice for these functions is $\mathcal{F}_l(q^{-1}) = q^{-l}$. However, in order to reduce the model order, one can adopt

$$\mathcal{F}_l(q^{-1}) = \frac{1}{1 - q^{-1}\xi_l} \qquad (29)$$

where the poles ξ_l are chosen so as to incorporate some a priori information regarding the underlying dynamics [18]. Another possibility is to use orthonormal basis functions such as Laguerre or Kautz functions. Since $\zeta_{K_{jk}}(\rho, t)$ and consequently $Q(\rho^*)$ are linear with respect to the parameters $h_{K_{jk}}^l$, the problem of obtaining a positive definite matrix $Q(\rho^*)$ can be formulated as a convex feasibility problem and solved using Linear Matrix Inequalities (LMIs) [3]. Hence, once the basis functions $\mathcal{F}_l(q^{-1})$ have been selected, the

choice of the instrumental variables specializes to finding $h_{K_{jk}}^l$ that makes

$$Q(\rho^*, h_{K_{jk}}^l) > 0. \qquad (30)$$

In this feasibility problem, it is necessary to evaluate $Q(\rho^*, h_{K_{jk}}^l)$ for different values of $h_{K_{jk}}^l$. Plugging (20), (21), (22) and (28) in (26), it is obvious that $Q(\rho^*, h_{K_{jk}}^l)$ depends on the derivative $\dfrac{\partial y_j(\rho, t)}{\partial \rho_{K_{mn}}}$ that is unknown. However, this derivative can be estimated using (22), where: (i) the transfer function matrix $\mathcal{S}(\rho^*)$ is replaced by $\mathcal{S}_d$; (ii) the unknown plant G is replaced by either G_0 or an identified model $\hat{G}$; (iii) the signal $e(\rho^*, t)$ is replaced by its estimate $\hat{e}(\rho^*, t) = \mathcal{S}_d r(t)$.

In performing the experiments leading to the computation of $\zeta_{K_{jk}}(\rho, t)$, $\mathcal{E}_{K_{jk}}(\rho, t)$, $\bar{f}_{K_{jk}}(\rho)$ and $\bar{F}(\rho)$ required for the application of the Robbins-Monro algorithm (17) (see (13), (14)), the following question then arises: Is it better to excite all reference signals simultaneously or in a sequential manner? This question is addressed in the next subsection, where the accuracy of the parameter estimates around ρ^* is investigated as a function of the external reference signals $r_1(t)$ and $r_2(t)$.

3.1.2 Variance Analysis

The following theorem will be used.

Theorem 3.1 *Assume that*

(1) The iterative algorithm (17) converges to ρ^* almost surely as $i \to \infty$.
(2) The step size $\gamma_i = \frac{\alpha}{i}$, where α is a positive constant.
(3) All eigenvalues of the matrix $D = \frac{1}{2}I - \alpha Q(\rho^*)$ have a negative real part.

Then, the sequence $\sqrt{i}(\rho_i - \rho^*)$ converges asymptotically in distribution to a zero-mean normal distribution with covariance

$$V = \alpha^2 \int_0^\infty e^{Dx} P \, e^{D^T x} dx \qquad (31)$$

where

$$P = \lim_{i \to \infty} E\left\{ \bar{F}(\rho^*) \bar{F}^T(\rho^*) \right\} \qquad (32)$$

Proof. The proof can be found in [17]. □

We now examine the structure of the matrices P, D and V for our decorrelation procedure. Note that, at $\rho = \rho^*$, (15) reduces to:

$$\mathcal{E}_{K_{jk}}(\rho, t) = \begin{cases} \mathcal{S}_{jj}(\rho^*)v_j(t), & j = k \\ \{\mathcal{T}_{jj}(\rho^*)r_j(t) \\ \quad + \mathcal{S}_{jj}(\rho^*)v_j(t)\}, & j \neq k \end{cases} \qquad (33)$$

Considering that $r_1(t)$, $r_2(t)$, $v_1(t)$ and $v_2(t)$ are independent, and using (13), (14), (32) and (33), leads after straightforward but tedious calculations to:

$$P(\rho^*) = \begin{pmatrix} P_{K_{11}}^{K_{11}} & 0 & 0 & 0 \\ 0 & P_{K_{12}}^{K_{12}} & P_{K_{12}}^{K_{21}} & 0 \\ 0 & P_{K_{21}}^{K_{12}} & P_{K_{21}}^{K_{21}} & 0 \\ 0 & 0 & 0 & P_{K_{22}}^{K_{22}} \end{pmatrix} \quad (34)$$

where

$$P_{K_{mn}}^{K_{jk}} = E\left\{ \frac{1}{N^2} \sum_{t=1}^{N} \zeta_{K_{jk}}(\rho, t)\mathcal{E}_{K_{jk}}(\rho, t) \right.$$
$$\left. \times \sum_{s=1}^{N} \zeta_{K_{mn}}^{T}(\rho, s)\mathcal{E}_{K_{mn}}(\rho, s) \right\}. \quad (35)$$

Observe that the matrix $D(\rho^*) = \frac{1}{2}I - \alpha Q(\rho^*)$ has the same structure as $Q(\rho^*)$ in (26). Due to (27), its elements satisfy:

$$D_{K_{mk}}^{K_{jk}} \overset{f}{\sim} r_k(t), v_k(t). \quad (36)$$

Finally, note that the covariance matrix V can be partitioned as:

$$V = \begin{pmatrix} V_{K_{11}}^{K_{11}} & V_{K_{11}}^{K_{12}} & V_{K_{11}}^{K_{21}} & V_{K_{11}}^{K_{22}} \\ V_{K_{12}}^{K_{11}} & V_{K_{12}}^{K_{12}} & V_{K_{12}}^{K_{21}} & V_{K_{12}}^{K_{22}} \\ V_{K_{21}}^{K_{11}} & V_{K_{21}}^{K_{12}} & V_{K_{21}}^{K_{21}} & V_{K_{21}}^{K_{22}} \\ V_{K_{22}}^{K_{11}} & V_{K_{22}}^{K_{12}} & V_{K_{22}}^{K_{21}} & V_{K_{22}}^{K_{22}} \end{pmatrix}. \quad (37)$$

Sequential vs. Simultaneous Excitation

We now consider two different cases of excitation: sequential and simultaneous excitation. In the case of sequential excitation, the elements of $K(q^{-1}, \rho)$ are tuned in two phases. In the first phase, the closed-loop system is excited by $r_1(t)$ while $r_2(t)$ is kept constant, and the parameters of $K_{11}(\rho)$ and $K_{21}(\rho)$ are tuned. In the second phase, $r_2(t)$ is excited while $r_1(t)$ is kept constant, and the parameters of $K_{21}(\rho)$ and $K_{22}(\rho)$ are tuned. In the case of simultaneous excitation by $r_1(t)$ and $r_2(t)$, all elements of $K(q^{-1}, \rho)$ are tuned together. In the analysis that follows, these two cases are compared in terms of the asymptotic accuracy of the parameters of $K_{11}(\rho)$ and $K_{21}(\rho)$.

For clarity of notation, we distinguish between the following two cases:

a) The closed-loop system is excited by a single reference signal, say $r_1(t)$; for simplicity, it is assumed that the other reference signal, $r_2(t)$, is equal to zero; the corresponding matrices and their elements will carry the subscript "a", for example $V_{a, K_{mn}}^{K_{jk}}$, $j, k, m, n = 1, 2$, or D_a.

b) The closed-loop system is excited by $r_1(t)$ and $r_2(t)$ simultaneously; the corresponding matrices and their elements will carry the subscript "b".

When only $r_1(t)$ is excited, it follows from (14) and (24) that only the controllers $K_{11}(\rho)$ and $K_{21}(\rho)$ can be tuned. Hence, our variance analysis will compare the variances $V_{K_{11}}^{K_{11}}$ and $V_{K_{21}}^{K_{21}}$ for that case of excitation by $r_1(t)$ only, with the corresponding variances obtained for the case of simultaneous excitation. Furthermore, in order to enforce no signal path from $e_2(t)$ to $u_1(t)$ and $u_2(t)$, $K_{12}(\rho)$ and $K_{22}(\rho)$ are set to zero. This way, similarly to the case where the optimal controllers $K_{12}(\rho^*)$ and $K_{22}(\rho^*)$ are used, there is no influence of $y_2(t)$ and $v_2(t)$ on $y_1(t)$. Note, however, that it is not possible to set $K_{12}(\rho)$ and $K_{22}(\rho)$ to zero when the MIMO plant to be controlled is unstable. In this case, the tuning of the controller $K(\rho)$ can only be performed by exciting both components of $r(t)$ simultaneously.

Next, the following result can be established.

Theorem 3.2 *Consider the tuning of the parameters $\rho_{K_{11}}$ and $\rho_{K_{21}}$ of the controllers $K_{11}(\rho)$ and $K_{21}(\rho)$. Let the components $r_1(t)$ and $r_2(t)$ be persistently exciting of sufficient order and independent in the case of simultaneous excitation. Then, the covariance matrices of the parameter estimates $\hat{\rho}_{K_{11}}$ and $\hat{\rho}_{K_{21}}$ can only increase by addition of the second excitation $r_2(t)$, i.e.*

$$V_{b, K_{11}}^{K_{11}} \geq V_{a, K_{11}}^{K_{11}} \quad \text{and} \quad V_{b, K_{21}}^{K_{21}} \geq V_{a, K_{21}}^{K_{21}} \quad (38)$$

Proof. For simplicity, let α in (31) be set to 1. Now, observe that the matrices V, D and P are related by the following Lyapunov equation [9]:

$$P + DV + VD^T = 0. \quad (39)$$

Due to the specific form of D and P, this expression can be partitioned into two separate block-equations, one of which includes the variances $V_{K_{11}}^{K_{11}}$ and $V_{K_{21}}^{K_{21}}$ as follows:

$$\tilde{P} + \tilde{D}\tilde{V} + \tilde{V}\tilde{D}^T = 0, \quad (40)$$

where

$$\tilde{P} = \begin{pmatrix} P_{K_{11}}^{K_{11}} & 0 \\ 0 & P_{K_{21}}^{K_{21}} \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} D_{K_{11}}^{K_{11}} & D_{K_{11}}^{K_{21}} \\ D_{K_{21}}^{K_{11}} & D_{K_{21}}^{K_{21}} \end{pmatrix} \quad (41)$$

and

$$\tilde{V} = \begin{pmatrix} V_{K_{11}}^{K_{11}} & V_{K_{11}}^{K_{21}} \\ V_{K_{21}}^{K_{11}} & V_{K_{21}}^{K_{21}} \end{pmatrix}. \quad (42)$$

Equation (36) indicates that $\tilde{D}$ depends on $r_1(t)$ but not on $r_2(t)$. Therefore, $\tilde{D}$ is identical for both cases of excitation, i.e. $\tilde{D}_a = \tilde{D}_b$. Furthermore, since at the solution of (16) the closed-loop system is perfectly decoupled, it follows from (24), (33) and (35) that $P_{K_{11}}^{K_{11}}$ is also identical for both cases of excitation. Let us now consider $P_{K_{21}}^{K_{21}}$. Replacing (24) and (33) in (35) gives:

$$
\begin{aligned}
P_{K_{21}}^{K_{21}} = E \Bigg\{ & \frac{1}{N^2} \sum_{t=1}^{N} \zeta_{K_{21}}(r_1(t), \rho, t) \left\{ \mathcal{T}_{22}(\rho^*) r_2(t) \right. \\
& + \left. \mathcal{S}_{22}(\rho^*) v_2(t) \right\} \times \sum_{s=1}^{N} \zeta_{K_{21}}^T(r_1(t), \rho, s) \\
& \left\{ \mathcal{T}_{22}(\rho^*) r_2(s) + \mathcal{S}_{22}(\rho^*) v_2(s) \right\} \Bigg\}.
\end{aligned} \tag{43}
$$

It can be concluded from this expression that the contribution of $r_2(t)$ to $P_{K_{21}}^{K_{21}}$ is positive definite. This contribution will be denoted as $\Delta P_{K_{21}}^{K_{21}}$. Therefore, one can write:

$$
\tilde{P}_b = \tilde{P}_a + \begin{pmatrix} 0 & 0 \\ 0 & \Delta P_{K_{21}}^{K_{21}} \end{pmatrix} \triangleq \tilde{P}_a + \Delta \tilde{P} \tag{44}
$$

where $\Delta \tilde{P} \geq 0$. Similarly, for the covariance matrices $\tilde{V}_a$ and $\tilde{V}_b$, we have $\tilde{V}_b = \tilde{V}_a + \Delta \tilde{V}$, which leads to:

$$
\begin{aligned}
& \tilde{P}_b + \tilde{D}_b \tilde{V}_b + \tilde{V}_b \tilde{D}_b^T = \\
& \left(\tilde{P}_a + \Delta \tilde{P} \right) + \tilde{D}_b \left(\tilde{V}_a + \Delta \tilde{V} \right) + \left(\tilde{V}_a + \Delta \tilde{V} \right) \tilde{D}_b^T = \\
& \Delta \tilde{P} + \tilde{D}_b \Delta \tilde{V} + \Delta \tilde{V} \tilde{D}_b^T = 0
\end{aligned} \tag{45}
$$

The last equality can be written more illustratively as:

$$
\Delta \tilde{V} = \int_0^\infty e^{\tilde{D}_b x} \Delta \tilde{P} \, e^{\tilde{D}_b^T x} dx. \tag{46}
$$

It is obvious that if $\Delta \tilde{P} \geq 0$ then $\Delta \tilde{V} \geq 0$ [27]. The inequalities (38) follow from the fact that any principal submatrix of a positive semi-definite matrix is positive semi-definite. $\square$

Theorem 3.2 states that the presence of the component $r_2(t)$ does not improve the accuracy of the parameters related to the controllers $K_{11}(\rho)$ and $K_{21}(\rho)$. In fact, the accuracy can only be reduced. This result is rather interesting taking into account the work of the same authors where, in the case of direct closed-loop identification using prediction error methods, it was shown that the addition of $r_2(t)$ could only improve the variance of the estimated parameters [16].

Remarks:

- This result can be explained intuitively as follows. Consider the instrumental variable method in the field of system identification. The expression for the variance of the parameter estimates reads [23]:

$$
P_{IV} = R_{IV}^{-1} P_C R_{IV}^{-T} \tag{47}
$$

with

$$
P_C = \sigma^2 E \left\{ [H(q^{-1}) \zeta(t)][H(q^{-1}) \zeta(t)^T] \right\} \tag{48}
$$

and

$$
R_{IV} = E \left\{ \zeta(t) \tilde{\varphi}^T(t) \right\}.
$$

Here $\tilde{\varphi}(t)$ denotes the noise-free estimate of the regressor vector $\varphi(t)$, $H(q^{-1})$ a noise model, and σ^2 the variance of the zero-mean noise. Considering that the instrumental variables are filtered versions of the excitation signal, one can conclude that the power of the excitation signal has two opposite effects on the variance of the parameter estimates. An increase in the power of the excitation signal implies (i) an increase in P_C via $\zeta(t)$, which in turn increases the variance P_{IV}, and (ii) an increase of the modulus of R_{IV} via $\zeta(t)$ and $\tilde{\varphi}(t)$, which decreases the variance P_{IV} since R_{IV} enters as an inverse quadratic form in the expression of P_{IV}. In the case of a zero-mean white noise excitation, the overall effect is that the variance P_{IV} decreases as the variance of the excitation signal increases. For more details, the reader is referred to [23]. On the other hand, one can observe from (43) and (44) that $r_2(t)$ affects $\Delta \tilde{P}$ the same way it affects P_C in (48) via $\zeta(t)$. It is also clear that the effect of $\Delta \tilde{P}$ on the variance of the controller parameters $\Delta \tilde{V}$ in (46) is similar to that of P_C on P_{IV} in (47). However, due to the decoupling of the outputs $y_1(t)$ and $y_2(t)$, $\tilde{D}_b$ in (46) is insensitive to the changes in $r_2(t)$. Hence, only the first effect mentioned above is present, i.e. the presence of $r_2(t)$ increases the variance of the estimated controller parameters.
- A comparison of simultaneous and sequential excitations shows that $r_2(t)$ acts as an additional disturbance for the tuning of the decoupler $K_{21}(\rho)$. That is, the addition of this reference deteriorates the signal-to-noise ratio for the estimation of $K_{21}(\rho)$ and, by cross-correlation, influences negatively the variances of the other elements of the controller transfer function.
- For systems where M_d is not diagonal, i.e. decoupling is not part of the control design specifications, both effects of the power of the excitation signals on the variance of the parameter estimates are present.

This section has presented a variance analysis for the parameters of a multivariable controller tuned using the CbT approach. Two cases of excitation have been considered for 2×2 systems. This analysis indicates that the addition of the second reference signal can worsen the variance of the estimated controller parameters. Of course, it is a user's choice

to excite the elements of $\mathbf{r}(t)$ simultaneously or sequentially. Simultaneous excitation provides significantly smaller experimental cost. Sequential excitation, on the other hand, implies two experiments per iteration but provides more accurate controller parameters.

3.2 Correlation reduction for MIMO systems

When a restricted complexity controller is used, it is not guaranteed that, at the solution ρ^* of (16), the variable $\mathcal{E}_{K_{jk}}(\rho^*, t)$ is completely decorrelated from all past reference signals. For example, when only one parameter per controller K_{jk} is tuned, $F(\rho)$ represents the cross-correlation between $\mathcal{E}_{K_{jk}}(\rho, t)$ and $r(t-1)$. That is, at the solution of (16) these signals are decorrelated only for the delay of 1, and not necessarily for other delays. In other words, the resulting controller does not really decorrelate $\mathcal{E}_{K_{jk}}(\rho^*, t)$ and $r(t)$. One possibility around this problem is to increase the order of the controller. However, if the controller order is too high, this will induce pole-zero cancellations in the controller transfer functions, which, in turn, will lead to numerical problems in the algorithm. A more elegant solution would be to adopt a FIR controller structure. This way, even in the case when the controller order is overestimated, there are no numerical problems.

To circumvent these difficulties, the controller parameters can be computed by minimizing the following correlation criterion:

$$J(\rho) = F^T(\rho)F(\rho) \qquad (49)$$

with the cross-correlation function $F(\rho)$ defined by (12)-(14). The variables $\mathcal{E}_{K_{jk}}(\rho, t)$ are chosen as in (15). Assuming that the reference signals are known in advance, the instrumental variables vector is chosen as a shifted version of the reference signal $r_k(t)$:

$$\zeta_{K_{jk}}^T(t) = [r_k(t+n_z), \ldots, r_k(t), \ldots, r_k(t-n_z)] \quad (50)$$

with n_z sufficiently large with respect to the number of controller parameters, i.e. $2n_z + 1 \geq n_r + n_s$. Observe that, with this choice of instrumental variables, the number of equations is larger than the number of controller parameters, i.e. the cross-correlation between $\mathcal{E}_{K_{jk}}(\rho, t)$ and $r(t)$ is computed for $2n_z + 1$ delays. This way, the underlying system of cross-correlation equations is a better measure of the cross-correlation between $\mathcal{E}_{K_{jk}}(\rho, t)$ and $r(t)$ and, at the same time, it is independent of the controller order.

Remark: For the decorrelation procedure, the instrumental variables were chosen so as to ensure positive-definiteness of $Q(\rho^*)$. In contrast here, the required condition on positive-definiteness of $\left.\frac{\partial J(\rho)}{\partial \rho}\right|_{\rho=\rho^*}$ is automatically satisfied because $J(\rho)$ is a quadratic criterion. Hence, the task of choosing the instrumental variables with this method is much simpler.

3.2.1 Frequency-domain analysis

In this section, the properties of the achieved closed-loop system are investigated by frequency-domain analysis of the criterion (49). It follows from (12)-(14) and (49) that this criterion can be expressed as:

$$J(\rho) = \sum_{j=1}^{2} \sum_{k=1}^{2} E\left\{\bar{f}_{K_{jk}}^T\right\} E\left\{\bar{f}_{K_{jk}}\right\} \qquad (51)$$

By substituting (50) in (14), (51) becomes

$$J(\rho) = \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{\tau=-n_z}^{\tau=n_z} R_{K_{jk}}^2(\tau) \qquad (52)$$

where the cross-correlation $R_{K_{jk}}(\tau)$ is defined as:

$$R_{K_{jk}}(\tau) = E\left\{\mathcal{E}_{K_{jk}}(\rho, t)r_k(t-\tau)\right\}. \qquad (53)$$

Applying Parseval's formula to (52) and letting n_z go to infinity leads to:

$$\lim_{n_z \to \infty} J(\rho) = \sum_{j=1}^{2} \sum_{k=1}^{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\mathcal{B}_{jk}(e^{j\omega})\right|^2 \Phi_{r_j}^2(\omega)d\omega$$

$$(54)$$

where

$$\mathcal{B}_{jk}(e^{j\omega}) = \mathcal{T}_{jk}(e^{j\omega}, \rho) - M_{djk}(e^{j\omega}). \qquad (55)$$

Now, from (54) and (55), the following observations can be made:

- Criterion (49) is asymptotically unaffected by noise.
- The weighted discrepancy between the achieved $\mathcal{T}$ and the desired M_d sensitivity functions is minimized, with the weight being the square of the reference signal power. The discrepancy will be small at frequencies where the reference signal power is large.
- Ideally, $\mathcal{B}_{jk}(e^{j\omega}) = 0$. Since the desired sensitivity function M_d is diagonal, diagonal controllers provide $\mathcal{T}_{jj}(e^{j\omega}, \rho) = M_{djj}(e^{j\omega})$ and the off-diagonal decouplers provide $\mathcal{T}_{jk}(e^{j\omega}, \rho) = 0$, $j \neq k$.

Having analyzed the basic properties of (49), the next subsection presents a method for minimizing this criterion.

3.2.2 Minimization of an upper bound of the criterion

Minimization of criterion (49) is intractable since it involves the product of expectations that are unknown. Therefore, we define the following criterion:

$$J_u(\rho) = E\left\{\bar{F}^T(\rho)\bar{F}(\rho)\right\} \qquad (56)$$

9

which can be minimized using the stochastic approximation method. It can be shown that $J(\rho) \leq J_u(\rho)$, i.e. by minimizing (56) one minimizes an upper bound of (49) [12].

A local minimum of (56) can be found as the solution of:

$$J'_u(\rho) = E\left\{\frac{\partial \bar{F}(\rho)}{\partial \rho}\bar{F}(\rho)\right\} = 0 \qquad (57)$$

which can be obtained using the following iterative formula [19]:

$$\rho_{i+1} = \rho_i - \gamma_i \left.\frac{\partial \bar{F}(\rho)}{\partial \rho}\right|_{\rho_i} \bar{F}(\rho_i) \qquad (58)$$

Under Assumptions (i)-(ii), this scheme converges to a local minimum of the criterion as the number of iterations goes to infinity, provided that an unbiased estimate of the gradient $\left.\frac{\partial \bar{F}(\rho)}{\partial \rho}\right|_{\rho_i}$ is available. However, obtaining an unbiased estimate of this gradient for MIMO systems can be very costly [7]. In this work, the gradient is computed using an identified MIMO model, which requires only one closed-loop experiment regardless of the number of inputs and outputs. It is clear that the existence of an unbiased model of the plant $\hat{G}$ would guarantee the convergence of the CbT algorithm. However, an unbiased model $\hat{G}$ is very difficult to obtain. Instead of identifying a complex unbiased model, it is proposed here to identify, at each iteration, a low-order model that accurately reflects the local behavior of the plant in the vicinity of the current operating point (with the current parameter value ρ_i).

3.3 Implementation aspects

We have presented both the decorrelation procedure and the correlation reduction for the tuning of decoupling controllers for MIMO systems. Here we compare these two approaches in terms of their applicability for solving practical control situations.

The stochastic approximation algorithm used in the decorrelation procedure presented in Section 3.1 converges to ρ^*, under fairly weak conditions. However, the convergence rate could be too slow for industrial applications. If one can collect a large number of data, the influence of the noise on $\bar{F}(\rho_i)$ is reduced considerably, and the Newton-Raphson algorithm can be used to compute the controller parameters:

$$\rho_{i+1} = \rho_i - \hat{\bar{Q}}^{-1}(\rho_i)\bar{F}(\rho_i) \qquad (59)$$

where the elements of the matrix $\hat{\bar{Q}}(\rho_i) = \left(\left.\frac{\partial \hat{\bar{F}}(\rho)}{\partial \rho}\right|_{\rho_i}\right)$ are:

$$\hat{\bar{Q}}^{K_{jk}}_{K_{mn}}(\rho_i) = \frac{1}{N}\sum_{t=1}^{N}\frac{\partial \hat{y}_j(\rho_i,t)}{\partial \rho_{K_{mn}}}\zeta^T_{K_{jk}}(t). \qquad (60)$$

The derivatives $\frac{\partial \hat{y}_j(\rho_i,t)}{\partial \rho_{K_{mn}}}$ can be estimated using (22), where the transfer function matrix G is typically unknown but can be identified and replaced by its estimate $\hat{G}$. Finally, the estimate $\hat{S}(\rho_i)$ is calculated using $\hat{G}$ and the current value of the controller $K(\rho_i)$.

Similarly, for the correlation reduction method with N sufficiently large, the criterion (56) can be considered as deterministic and minimized using the much faster Gauss-Newton algorithm:

$$\rho_{i+1} = \rho_i - H_F(\rho_i)^{-1}\hat{\bar{Q}}(\rho_i)\bar{F}(\rho_i) \qquad (61)$$

where $H_F(\rho_i)$ is chosen as:

$$H_F(\rho_i) = \hat{\bar{Q}}(\rho_i)\left(\hat{\bar{Q}}(\rho_i)\right)^T. \qquad (62)$$

Observe that the Jacobian estimate $\hat{\bar{Q}}(\rho_i)$ is asymptotically unaffected by noise since the noisy part of $\frac{\partial \hat{y}_j(\rho_i,t)}{\partial \rho_{K_{mn}}}$ is not correlated with $\zeta^T_{K_{jk}}(t)$. However, it is sensitive to modelling errors.

For the decorrelation procedure, an inaccurate estimate of the Jacobian will not prevent the controller parameter vector from converging to ρ^* but will affect the convergence speed. In contrast, in the case of correlation reduction, the iteration will stagnate once the residual $||\rho_i - \rho^*||$ of (61) is roughly of the same size as the error in the gradient. For more details on these two iterative methods, the reader is referred to Chapter 5.4 in [13] and Chapter 2.3.1 in [14], respectively.

The above discussion suggests that it is preferable to use the decorrelation procedure for controller tuning. However, the generation of the instrumental variables is much more involved with the decorrelation procedure than with correlation reduction. Another advantage of the correlation-reduction method is that the criterion better reflects the amount of correlation between two signals and, furthermore, it can be applied to the tuning of restricted-complexity controllers. The overall conclusion is that the correlation-reduction method is more suited to industrial applications due to its simplicity of implementation and the fact that it is applicable to a broader range of applications.

4 Simulation Studies

Three simulation studies are presented in this section. In the first study, the sequential and simultaneous excitations are compared in terms of accuracy of both the estimated controller parameters and the estimated controller transfer functions. The second study investigates the basic features of the decorrelation procedure and correlation-reduction method, while the third study compares the correlation-reduction method to IFT for MIMO systems.

4.1 Sequential vs. simultaneous excitation

Consider the following discrete-time multivariable plant:

$$G(q^{-1}) = \begin{pmatrix} \frac{0.09516q^{-1}}{1-0.9048q^{-1}} & \frac{0.03807q^{-1}}{1-0.9048q^{-1}} \\ \frac{-0.02974q^{-1}}{1-0.9048q^{-1}} & \frac{0.04758q^{-1}}{1-0.9048q^{-1}} \end{pmatrix} \quad (63)$$

and let the initial controller for this plant be:

$$K_0 = \begin{pmatrix} \frac{1-0.99q^{-1}}{1-q^{-1}} & \frac{0.1-0.099q^{-1}}{1-q^{-1}} \\ -\frac{1-0.99q^{-1}}{1-q^{-1}} & \frac{1-0.99q^{-1}}{1-q^{-1}} \end{pmatrix} \quad (64)$$

The parametrized controller $K(\rho)$ has the same structure as K_0; each of the elements K_{ij} has an integrator and a first degree numerator with 2 free parameters. The vector of tuned parameters, defined in (2)-(5), is $\rho = [s_{11}^{(0)}, s_{11}^{(1)}, s_{12}^{(0)}, s_{12}^{(1)}, s_{21}^{(0)}, s_{21}^{(1)}, s_{22}^{(0)}, s_{22}^{(1)}]^T$. The numerator coefficients of the controllers $K_{jj}, j = 1, 2$ are tuned to provide the desired closed-loop response with a natural frequency of $3 rad/s$ and damping factor of 0.7, while the numerator coefficients of $K_{jk}, j \neq k$ are tuned for decoupling. The corresponding reference model reads:

$$M_{d_1} = \begin{pmatrix} \frac{0.1148q^{-1}-0.0942q^{-2}}{1-1.79q^{-1}+0.8106q^{-2}} & 0 \\ 0 & \frac{0.1148q^{-1}-0.0942q^{-2}}{1-1.79q^{-1}+0.8106q^{-2}} \end{pmatrix} \quad (65)$$

The instrumental variables for the decorrelation procedure are computed using the following Laguerre basis functions:

$$\mathcal{F}_l(q^{-1}) = \left(\frac{\sqrt{1-\xi^2}}{1-\xi q^{-1}} \right) \left(\frac{q^{-1}-\xi}{1-\xi q^{-1}} \right)^l \quad (66)$$

where $\xi = 0.895$ is chosen to approximately reflect the closed-loop dynamics, and the model order is $l = 2$. The weighting coefficients $h_{K_{jk}}^l$ are obtained by solving the feasibility problem

$$\hat{Q}(\rho^*, h_{K_{jk}}^l) > I \quad (67)$$

where $\hat{Q}(\rho^*, h_{K_{jk}}^l)$ is estimated as explained in Section 3.1.1. The identity matrix I on the right-hand side of inequality (67) is used (instead of 0) to ensure the positive definiteness of $Q(\rho^*, h_{K_{jk}}^l)$ with a safety margin, to compensate for possible differences between $Q(\rho^*, h_{K_{jk}}^l)$ and its estimate $\hat{Q}(\rho^*, h_{K_{jk}}^l)$. The following values for the coefficients $h_{K_{jk}}^l$ are obtained using Matlab LMI Control Toolbox: $h_{K_{11}}^0 = 200.0328$, $h_{K_{11}}^1 = -5.5225$, $h_{K_{12}}^0 = 107.4611$, $h_{K_{12}}^1 = -352.1068$, $h_{K_{21}}^0 = 226.8947$, $h_{K_{21}}^1 = -85.2721$, $h_{K_{22}}^0 = 234.8072$ and $h_{K_{22}}^1 = -371.1253$.

As stated in Section 3.3, the convergence rate of the Robbins-Monro scheme is too slow to be applicable in many practical control problems. Therefore, a comparison of the two

excitation methods by a Monte-Carlo simulation would necessitate an unreasonably large number of iterations. To circumvent this problem, the iterative scheme (17) has been modified as follows:

$$\rho_{i+1} = \rho_i - \gamma_i H_c \bar{F}(\rho_i). \quad (68)$$

where $\gamma_i = 1/i$ and H_c is a matrix computed in the following way. In the few first (say 5) iterations, H_c is computed as the inverse of the Jacobian estimate, i.e. $H_c = \bar{Q}^{-1}(\rho_i)$. Then, H_c is fixed to $H_c = \bar{Q}^{-1}(\rho_5)$. This modification allows the iterative scheme (68) to converge rapidly to ρ^*. Observe that with this modification, the expression (31) changes to

$$V = \alpha^2 \int_0^\infty e^{Dx} H_c P H_c^T e^{D^T x} dx \quad (69)$$

with

$$D = \frac{1}{2}I - \alpha H_c Q(\rho^*) \quad (70)$$

Considering that H_c is close to $Q^{-1}(\rho^*)$ after the few first iterations, i.e. $H_c \approx Q^{-1}(\rho^*)$, (70) can be rewritten as $D \approx (\frac{1}{2} - \alpha)I$. Since H_c and D are constant in the remaining iterations, and thus not dependent on the excitation signals, the results of Theorem 3.2 apply.

To compare the two excitation methods 100 Monte-Carlo simulation runs are performed. For each simulation run, the tuning is carried out in 9 iterations. The output of the plant is perturbed in each run by a different realization of a zero-mean, stationary, white Gaussian sequence $v(t)$ with the variance $0.01I$. The signals $r_1(t)$, $r_2(t)$, $v_1(t)$ and $v_2(t)$ are chosen mutually independent so as to satisfy the assumption of Theorem 3.2.

When the closed-loop system is excited sequentially, the asymptotic variances of the elements of ρ are:

$$var(\rho_a) = (0.001156 \ 0.001180 \ 0.004627 \ 0.004810 \\ 0.003225 \ 0.003340 \ 0.019222 \ 0.020984)$$

When $r_1(t)$ and $r_2(t)$ are excited simultaneously, the asymptotic variances of ρ are:

$$var(\rho_b) = (0.004941 \ 0.005203 \ 0.007360 \ 0.006754 \\ 0.014510 \ 0.014935 \ 0.030024 \ 0.031664)$$

The Monte-Carlo simulations confirm the results of Section 3.1.2. Indeed, the variances of the controller parameters are larger in the case of simultaneous excitation than in the case of sequential excitation. However, note that for sequential excitation two experiments per iteration are required, while a single experiment per iteration is sufficient in the case of simultaneous excitation.

4.2 Decorrelation procedure vs. correlation reduction

The same simulation example is used to compare the two CbT methods. The controller parameters are computed using
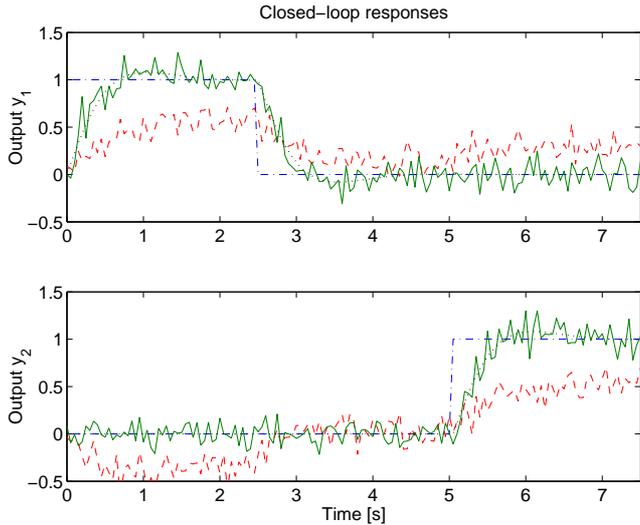
Fig. 3. Decorrelation procedure. Reference signals (dash-dot) and desired responses (dotted). Achieved responses: the initial controller (dashed) and final controller obtained by decorrelation procedure (solid). The references are changed in a step-like manner at 0 and 2.5s (for r_1) and 5s (for r_2).

Table 1
Comparison of CbT variants. $|F(\rho_i)|_2$ denotes the 2-norm of $F(\rho_i)$, while SSOE indicates the sum of squares of the output error.

Iter-	Decorrelation Procedure		Correlation Reduction	
ation	$|F(\rho_i)|_2$	$SSOE$	$J_u(\rho_i)$	$SSOE$
$i = 0$	7817.09	38.2165	141.3941	38.2165
$i = 6$	88.8512	3.3985	1.8310	3.4188

(59)-(60) for the decorrelation procedure, and (61)-(62) for correlation reduction. The tuning is carried out in 6 iterations, with one experiment per iteration where each experiment is performed with a different noise realization. The resulting responses obtained with the initial controller (dashed line) and the controller tuned using the decorrelation procedure (solid line) are shown in Fig. 3. With the tuned controller, the desired response (dotted line) can be followed up to the effect of the noise. Note also that the resulting closed-loop system is diagonalized. The responses obtained with the controller tuned by correlation reduction are very similar to those in Fig. 3 and are not shown here. The results are summarized in Table 1. Note that the correlation measures are reduced in both cases by 99%, while the sum of squares of the output error is reduced by 91%.

4.3 Correlation-reduction CbT vs. IFT

For this comparison, our simulation considers the tuning of a multivariable PI controller for a LV100 gas turbine engine [26]. The simulation conditions are taken from [7]. The plant is represented by a continuous-time state-space model with five states, two inputs and two outputs. The model is discretized using Tustin's approximation with the sampling
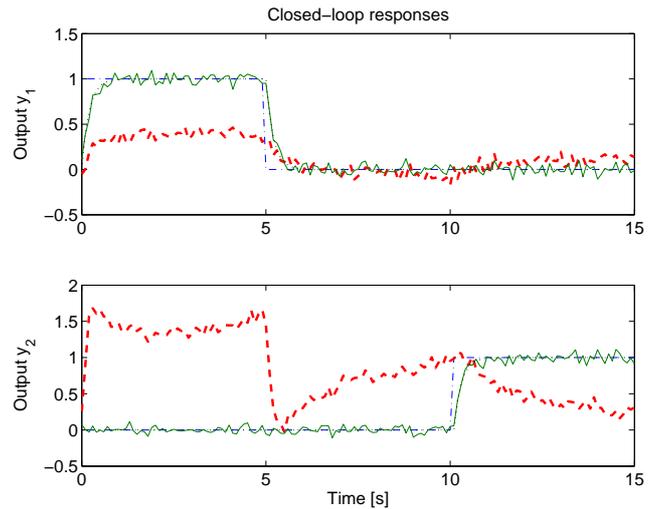
Fig. 4. Correlation-reduction CbT: Closed-loop responses in a noisy environment. Reference signals (dash-dot), desired responses (dotted), achieved responses with the initial controller (dashed) and final controller (solid). The references are changed in a step–like manner at 0 and 5s (for r_1) and 10s (for r_2).
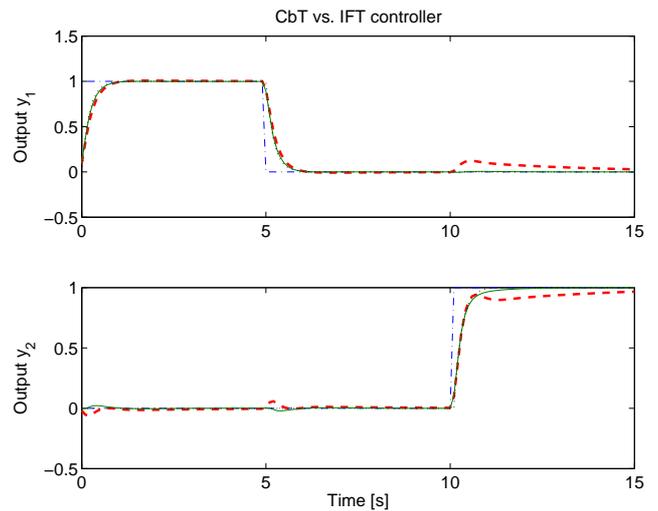
Fig. 5. Correlation-reduction CbT vs. IFT controller in a noise-free context. Reference signals (dash-dot), desired responses (dotted), closed-loop response with the CbT controller (solid) and IFT controller (dashed).

period $T_s = 0.1s$. Each experiment is performed with a different realization of the measurement noise $v(t)$, which is generated as a zero-mean, stationary, white Gaussian sequence with variance $0.0025I$.

The initial controller K_0 given in (64) is used. The responses obtained with this controller are plotted in Fig. 4 (dashed line). Eight numerator coefficients are tuned (two for each transfer function element), while the denominators are again kept fixed at $1 - q^{-1}$. The following reference model is

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specified:

$$M_{d_2} = \begin{pmatrix} \frac{0.4q^{-1}}{1-0.6q^{-1}} & 0 \\ 0 & \frac{0.4q^{-1}}{1-0.6q^{-1}} \end{pmatrix} \quad (71)$$

and the controller parameters are calculated according to (61).

A discrete-time state-space model with three states is identified in closed loop, using Matlab System Identification Toolbox, to compute the estimate $\hat{G}$. After eight iterations, this procedure provides the closed-loop response shown in Fig. 4. A comparison with the desired response (dotted line) shows that the two curves are nearly superposed except for the effect of the noise. In addition, changes in the reference signals $r_1(t)$ and $r_2(t)$ do not induce any visible change on the outputs $y_2(t)$ and $y_1(t)$, respectively. In other words, the closed-loop system is almost fully diagonalized. The value of the tuning criterion is reduced by more than 99%. The resulting CbT controller is:

$$K_{CbT} = \begin{pmatrix} \frac{0.3636-0.09866q^{-1}}{1-q^{-1}} & \frac{0.3653-0.2691q^{-1}}{1-q^{-1}} \\ \frac{18.69-18.16q^{-1}}{1-q^{-1}} & \frac{-3.453+2.652q^{-1}}{1-q^{-1}} \end{pmatrix} \quad (72)$$

For this problem, [7] provided the IFT controller

$$K_{IFT} = \begin{pmatrix} \frac{0.248-0.03q^{-1}}{1-q^{-1}} & \frac{0.38-0.199q^{-1}}{1-q^{-1}} \\ \frac{16.47-15.91q^{-1}}{1-q^{-1}} & \frac{0.063+0.054q^{-1}}{1-q^{-1}} \end{pmatrix}. \quad (73)$$

To compare it with the CbT controller designed above, an experiment is performed with the same simulation conditions. The observed $SSOE$ with the CbT controller is 0.0050, while that with the IFT controller is 0.0082. Since IFT contains a noise-rejection objective [1] while CbT does not, one would expect IFT to perform better in a noisy situation. However, the $SSOE$ obtained with CbT is smaller. The IFT controller did not succeed in (i) fully decoupling the closed-loop system, and (ii) completely satisfying the model-following specification. This indicates that the IFT algorithm may have converged to a local minimum. To illustrate this, an additional experiment without noise is performed. The results are shown in Fig. 5. The closed-loop response obtained with the CbT controller follows almost perfectly the desired response. In contrast, the closed-loop response obtained with the IFT controller shows some discrepancy in the last 5 seconds of the response. In addition, the influence of the change

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[1] Note that IFT minimizes the sum of squares of the output error:

$$SSOE = \frac{1}{N} \sum_{t=1}^{N} \varepsilon_{oe}^{T}(\rho,t)\varepsilon_{oe}(\rho,t). \quad (74)$$

in the reference signal $r_1(t)$ at the instants 0s and 5s is visible on $y_2(t)$.

In terms of experimental cost, the IFT controller is obtained after 6 iterations (and a total of 30 experiments) compared to 8 iterations (and a total of 8 experiments) for the CbT controller.

5 Conclusions

This paper has considered the use of the correlation-based approach for tuning decoupling linear multivariable controllers. The parameters are tuned by either solving a correlation equation or minimizing the square of a cross-correlation function. The diagonal controllers attempt to fulfill the desired output specifications, while the off-diagonal controllers aim at decoupling. In contrast to the approaches where decouplers and diagonal controllers are designed sequentially, the design of decouplers and controllers is done *simultaneously* here. The tuning of all decouplers and controllers is achieved by performing a single experiment per iteration regardless of the number of inputs and outputs, since all reference signals can be excited simultaneously. This feature represents an advantage over some other data-driven methods such as IFT, where the required number of experiments per iteration increases with the number of inputs and outputs. In addition, perfect decoupling can be achieved using CbT while, in the case of data-driven control design methods that minimize a norm of the error signal, there is a trade-off between satisfying the decoupling specifications and noise rejection.

The variance of the estimated controller parameters has been compared for the two cases of simultaneous and sequential excitations. This analysis shows that, due to the fact that decoupling is imposed as a design criterion, simultaneous excitation of all references has a negative effect on the variance of the estimated controller parameters. More accurate estimates require require that n_y experiments per iteration be performed. In fact, one must choose between low experimental cost (simultaneous excitation) and better accuracy of the estimated parameters (sequential excitation). Simulation results illustrate the features and applicability of this controller tuning method for LTI MIMO systems.

References

[1] T. Agnoloni and E. Mosca. Controller falsification based on multiple models. *International Journal of Adaptive Control and Signal Processing*, 17:163–177, 2003.

[2] K. J. Åström and B. Wittenmark. *Adaptive Control*. Addison-Wesley, 1989.

[3] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.

[4] M. C. Campi, A. Lecchini, and S. M. Savaresi. Virtual reference feedback tuning: a direct method for the design of feedback controllers. *Automatica*, 38(8):1337–1346, 2002.

[5] L. Gerencsér, Z. Vágó, and H. Hjalmarsson. Randomization methods in optimization and adaptive control. In B. Pasik-Duncan, editor, *Lecture Notes in Control and Information Sciences*, volume 280/2002, pages 137–154. Springer-Verlag, Berlin, Germany, 2002.

[6] L. Gerencsér, Z. Vágó, and H. Hjalmarsson. Randomized iterative feedback tuning. In *15th IFAC World Congress*, Barcelona, Spain, July 2002.

[7] H. Hjalmarsson. Efficient tuning of linear multivariable controllers using iterative feedback tuning. *International Journal of Adaptive Control and Signal Processing*, 13:553–572, 1999.

[8] H. Hjalmarsson, M. Gevers, S. Gunnarsson, and O. Lequin. Iterative feedback tuning: Theory and application. *IEEE Control Systems Magazine*, pages 26–41, 1998.

[9] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.

[10] H. Jansson, H. Hjalmarsson, and A. Hansson. On methods for gradient estimation in IFT for MIMO systems. In *15th IFAC World Congress*, Barcelona, Spain, July 2002.

[11] A. Karimi, L. Mišković, and D. Bonvin. Iterative correlation-based controller tuning with application to a magnetic suspension system. *Control Engineering Practice*, 11(9):1069–1078, 2003.

[12] A. Karimi, L. Mišković, and D. Bonvin. Iterative correlation-based controller tuning. *International Journal of Adaptive Control and Signal Processing*, 18:645–664, 2004.

[13] C. T. Kelley. *Iterative Methods for Linear and Nonlinear Equations*. SIAM, Philadelphia, USA, 1995.

[14] C. T. Kelley. *Iterative Methods for Optimization*. SIAM, Philadelphia, USA, 1999.

[15] L. Mišković, A. Karimi, and D. Bonvin. Correlation-based tuning of a restricted-complexity controller for an active suspension system. *European Journal of Control*, 9(1):77–83, 2003.

[16] L. Mišković, A. Karimi, D. Bonvin, and M. Gevers. Direct closed-loop identification of 2×2 systems: Variance analysis. In *CD-ROM Proc. of 14th IFAC Symp. on System identification*, pages 893–898, Newcastle, Australia, March 2006.

[17] M. B. Nevelson and R. Z. Khasminskii. Stochastic approximation and recursive estimation. In B. Silver, editor, *Translations of Mathematical Mongraphs*, volume 47. American Mathematical Society, Providence, Rhode Island, 1973.

[18] B. Ninness and F. Gustafsson. A unifying construction of orthonormal bases for system identification. *IEEE Transactions on Automatic Control*, 42(4):515–521, 1997.

[19] H. Robbins and S. Monro. A stochastic approximation method. *Ann. Math. Stat.*, 22:400–407, 1951.

[20] M. G. Safonov and T-C. Tsao. The unfalsified control concept and learning. *IEEE Trans. on Automatic Control*, 42(6):843–847, 1997.

[21] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control - Analysis and Design*. John Wiley & Sons, Chichester, England, 1996.

[22] T. Söderström and P. Stoica. Instrumental variable methods for system identification. In A. V. Balakrishnan and M. Thoma, editors, *Lecture Notes in Control and Information Science*. Springer-Verlag, Berlin, 1983.

[23] T. Söderström and P. Stoica. *System Identification*. Prentice-Hall, U.K., 1989.

[24] J. C. Spall and J. A. Cristion. Model-free control of nonlinear stochastic systems with discrete-time measurements. *IEEE Trans. on Automatic Control*, 43(9):1198–1210, 1998.

[25] Q. G. Wang. Decoupling control. In *Lecture Notes in Control and Information Sciences*, volume 285/2003. Springer-Verlag, Berlin, Germany, 2003.

[26] M. Yeddanapudi and A. F. Potvin. *Nonlinear Control Design Blockset: User's Guide*. The Mathworks Inc., Natick, MA, 1997.

[27] K. Zhou and J. C. Doyle. *Essentials of robust control*. Prentice-Hall, N.Y., 1998.