Closed-loop Optimal Experiment Design: the Partial Correlation Approach

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Abstract— **We consider optimal experiment design for parametric prediction error system identification of linear time-invariant systems in closed loop. The optimisation is performed jointly over the controller and the external input. We use a partial correlation approach, i.e. we parameterize the set of "admissible controller" - "external input" pairs by a finite set of matrix-valued trigonometric moments. Our main contribution is twofold. First we derive a description of the set of admissible finite-dimensional moments by a linear matrix inequality. Optimal input design problems with semi-definite constraints and criteria which are linear in these moments can then be cast as semi-definite programs and solved by standard semi-definite programming packages. Secondly, we develop algorithms to recover the controller and the power spectrum of the external input from the optimal moment vector. This furnishes the user a complete and very general procedure to solve the input design problems of the considered class. Our results can be applied to multi-input multi-output systems, but for pedagogical reasons we present here the single-input single-output case. We also assume that the true system is in the model set.**

I. Introduction

Optimal input design for system identification has seen an intense development in this decade. This advance was initiated by the appearance of modern convex optimisation methods in the nineties, most notably semidefinite programming. Accordingly, most of the recent work in optimal input design focuses on casting different input design problems as semidefinite programs. Once an optimisation problem is available in the standard format of a semi-definite program, it can be solved by commercially or freely available solvers. One of the pioneering contributions introducing semidefinite programming into system identification was [1]. For further motivation and an extensive reference list we refer to [2].

However, converting optimisation problems into semi-definite programs is often far from trivial. Sometimes this is due to the NP-hardness of the problem. If a semi-definite description cannot be obtained, one usually tries to relax the problem in order to construct a semi-definite approximation. Often such a relaxation is easily at hand, but nothing about its quality is known. Furthermore, usually the relaxation yields only a bound on the optimal value of the cost criterion, but no clue on how to construct a suboptimal design.

Here we consider optimal input design for parametric closed-loop identification of linear time-invariant (LTI) systems. We consider a broad class of problems, namely when the joint optimal design of both controller and external input for the identification experiment is sought. A semi-definite description of optimal input design problems in this class has for years been elusive. The main obstacle is that the set of input-controller pairs constituting valid experimental setups could not be described in a way that is suitable for a semi-definite program formulation. One of our main results is to provide such a description. Rather than solving a particular problem, we thus present a scheme to solve a broad class of problems. Accordingly, the focus of our contribution will not be on the constraints and the cost function defining a particular optimal input design problem instance, but rather on a proper description of the feasible set of input-controller pairs. Plugging his particular constraints and cost criteria into our scheme, the user obtains a standardized semi-definite program which he can handle by commercially or freely available solvers. We allow the system to have multiple inputs and outputs (MIMO), but impose the condition that the true system is within the model structure. For pedagogical reasons and space limitations, we shall treat the single-input single-output (SISO) case only in this conference paper.

The description of an optimal input design problem as a semi-definite program is principally determined by the choice of the *design variables*, i.e. those quantities whose values the solver has to optimize. Different requirements have to be imposed on the choice of these variables. Firstly, the space of design variables has to be of *finite dimension*, even if there are possibly infinitely many degrees of freedom available for the design of the experiment. Secondly, the design variables have to enter both the constraints and the cost criterion of the design problem *linearly*. Thirdly, the set of *admissible* vectors of design variables has to be semidefinite representable, i.e., the condition on the vector of design variables to correspond to a realizable experiment design has to be equivalent to the satisfaction of a linear matrix inequality (LMI), possibly involving additional auxiliary variables.

The degrees of freedom relevant for closed-loop input design problems are the external excitation signal fed into the system and the controller in the loop. The external input usually enters into the cost criteria and the constraints in the form of its power spectrum, the controller in the form of its transfer function. Both can easily be converted into a joint power spectrum of some signals present in the loop.

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These spectra are frequency-dependent functions and as such infinite-dimensional objects. Their infinitely many degrees of freedom have to be condensed into a finite-dimensional vector of design variables. Two basic approaches to the choice of the design variables can be distinguished.

One is the *finite dimensional spectrum parametrization* (see e.g. [2], [3], [4]). Here the spectrum is developed into an infinite series and the design variables are given by the truncated vector of coefficients. If the basis functions are rational, by the Kalman-Yakubovich-Popov (KYP)-lemma ([5]) the positivity of the spectrum is expressed by an LMI, leading to a semi-definite relaxation of the original input design problem. This relaxation is an *inner* approximation, i.e., every feasible point of the relaxation is also feasible for the original problem. This has the advantage that the optimal solution of the relaxation yields a realizable input design. However, this approach considers only a finite-dimensional subspace cut out of the infinite-dimensional variety of possibly useful spectra. This results in a performance loss, as the optimisation procedure returns only suboptimal solutions. For simple systems this loss can be compensated by considering a long truncated coefficient sequence, but in practice the gap is significant and optimisation over large enough numbers of coefficients computationally prohibitive.

The other approach is the *partial correlation approach* (see e.g. [6], [2]), which is in some sense dual to the finite dimensional spectrum parametrization. Namely, an infinite sequence of *linear functionals* on the space of spectra is considered, and the design variables are the values of a finite number of these functionals on the spectrum in question. These values are called (generalized) *moments* ([7], [8]) of the spectrum. The linear functionals are chosen in such a way that both the constraints and the cost function depend only on a finite number of moments. Geometrically, the optimisation is performed over a finite-dimensional *projection* of the infinite-dimensional cone of possible spectra, as opposed to a finite-dimensional *section* in the finite dimensional spectrum parametrization approach. Each point in the finite-dimensional truncated moment space thus still corresponds to an infinite set of spectra rather than a single spectrum, and the points of the finite-dimensional moment cone exhaust all possible spectra. Thus the partial correlation approach does not suffer from the performance loss characteristic of the finite dimensional spectrum parametrization approach.

The role played by the KYP-lemma in the finite-dimensional spectrum parametrization is played by the Carathéodory-Fejer theorem in the partial correlation approach. This theorem implies that a given finite sequence of moments is generated by a positive power spectrum if and only if it satisfies a certain LMI ([9]). In this case the feasible set of moment vectors is semi-definite representable. However, the Carathéodory-Fejer theorem for matrix-valued moments holds *only* if no restrictions are imposed on the spectrum other than to produce the truncated sequence of moments under consideration, and positivity. In other words, a finite sequence of moments can be extended to an infinite sequence of moments of a positive spectrum if and only if it satisfies the LMI condition, but no additional constraint on the moments of this extension can be guaranteed to be satisfied. The partial correlation approach has been used to cast a variety of input design problems in closed-loop with fixed controller as semi-definite programs. However, here we treat the more ambitious problem where the controller is not fixed, but is also part of the optimal design.

In the case considered in this contribution, namely when the optimisation is performed over both controller and external input, the Carathéodory-Fejer theorem is no longer valid. The reason is that, in the closed-loop case with variable controller, constraints have to be imposed on the matrix-valued joint power spectrum under consideration, essentially pertaining to the required stability of the resulting closed-loop system. These constraints translate into additional constraints on the infinite moment sequence of the spectrum, which the Carathéodory-Fejer theorem can no longer guarantee to be satisfied. Therefore the LMIs furnished by this theorem are still necessary, but no more sufficient for a moment vector to correspond to a valid closed-loop experiment design. Thus the partial correlation approach leads to an *outer* semi-definite relaxation of the original input design problem. Solving this relaxation yields a lower bound on the optimal value of the cost criterion, but the furnished optimal moment vector does a priori not correspond to a valid experiment design. Accordingly, the partial correlation approach was mentioned as potentially useful in [10], [11], but conditioned on the availability of a proper description of the feasible set. The finite dimensional spectrum parametrization approach had thus to be proposed as a remedy: in [11] a solution is proposed for the closed-loop experiment design problem where stability of the loop is enforced through the use of a Youla parameter, which is approximated by a finite dimensional parametrization.

Our main contribution is to provide theoretical results to overcome this difficulty. In particular, we show that if a finite sequence of moments satisfies the additional constraints mentioned in the previous paragraph, then the LMI condition given by the Carathéodory-Fejer theorem not only guarantees the existence of a general extension of this moment sequence, but the existence of an extension which satisfies also the constraints. The main tool of our proof is the partial positive definite matrix completion theorem from [12]. Complementing the proof of exactness of the outer semi-definite relaxation provided by the Carathéodory-Fejer theorem, we provide a procedure to recover a valid input design from the optimal moment vector furnished by the relaxation. This allows to employ the partial correlation approach to its full extent, and to solve the considered class of optimal input design problems without the performance loss inherent to the finite dimensional spectrum parametrization approach.

The remainder of the paper is structured as follows. In the next section we define the class of input design problems to be solved. In Section 3 we provide a semi-definite description of the feasible set of truncated moment sequences. In Section 4 we develop methods to recover a set of valid experimental conditions from a given truncated moment sequence. Finally, in Section 5 we present an example. In the Appendix we provide the theoretical framework for the main result of Section 3, namely some results on chordal graphs and the associated partial positive definite matrix completion theorem.

II. Problem formulation

In this section we define the class of optimal input design problems we are going to study. We intend to perform parametric prediction error identification of a SISO LTI system in closed loop. The system dynamics is given by

$$y = G_0(q)u + H_0(q)e. \tag{1}$$

Here G_0 is the plant transfer function, H_0 the noise transfer function, q the forward-shift operator, e a zero mean white noise with variance λ_0, u is the input, and y is the output of the system. We assume H_0 to be stable and inversely stable. The transfer functions $G_0(z), H_0(z)$ are embedded in a model structure $G(z;\theta), H(z;\theta)$ and correspond to some true parameter value θ_0, $G_0(z) = G(z;\theta_0)$, $H_0(z) = H(z;\theta_0)$.

The parameter vector θ_0 is to be identified by an experiment consisting in closing the loop according to the relation

$$u = -K(q)y + r, \tag{2}$$

where r is an external excitation signal, and collecting a set of input-output data u, y. The design variables at our disposal are thus the signal r and the feedback controller $K(q)$. The estimator $\hat{\theta}$ of the true parameter value θ_0 is then evaluated as the minimizer of some prediction error criterion.

Our goal is to design an experiment by choosing an external input r and a controller K such that some cost function of r, K is minimized and some constraints on the pair (r, K) are satisfied. As mentioned in the introduction, our focus will not be on the cost criterion and the constraints of the optimal input design problem, but rather on a description of the set of feasible pairs (r, K) which is suitable for processing by a semi-definite program solver.

Following [2], we first pass from the quantities r, K to the spectra Φ_u, Φ_{ue}, which, as long as we work in the frequency domain and use formulas that are asymptotic in the number of data, yield an equivalent description of the experimental conditions. The power spectrum Φ_r of r and the controller K determine Φ_u, Φ_{ue} by the formulas

$$\begin{aligned} \Phi_u(\omega) &= \lambda_0|(1+KG_0)^{-1}KH_0|^2 + |1+KG_0|^{-2}\Phi_r(\omega), \\ \Phi_{ue}(\omega) &= -\lambda_0(1+KG_0)^{-1}KH_0, \end{aligned} \tag{3}$$

where the transfer functions on the right-hand side are evaluated at $z = e^{j\omega}$. On the other hand, Φ_r and K can be recovered from Φ_u, Φ_{ue} by the formulas

$$\Phi_r = |1+KG_0|^2(\Phi_u - \lambda_0^{-1}|\Phi_{ue}|^2), \tag{4}$$
$$K = -\Phi_{ue}(\lambda_0 H_0 + G_0\Phi_{ue})^{-1}. \tag{5}$$

Parametrizing the experimental setup by the joint power spectrum

$$\Phi_{\chi_0} = \begin{pmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{ue}^* & \lambda_0 \end{pmatrix} \tag{6}$$

of the signals u, e instead of the quantities r, K has the advantage that the feasible set becomes convex, which is a prerequisite for a semi-definite representation [2].

Within the framework of the partial correlation approach, the ultimate design variables are a finite set of moments of the joint power spectrum Φ_{χ_0}, which we now introduce. Consider a polynomial $d(z) = \sum_{l=0}^m d_l z^l$ of degree m, $m \geq 0$, such that the coefficients d_l are real, obey $d_0 \neq 0$, $d_m \neq 0$, and the polynomial $d(z)$ has all roots outside of the closed unit disk. Define 2×2 matrices

$$m_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1}{|d(e^{j\omega})|^2}\Phi_{\chi_0}(\omega)e^{jk\omega}\,d\omega \tag{7}$$

for integral k. The matrices m_k defined by (7) are called the *generalized moments* of the spectrum Φ_{χ_0}. Note that the moments m_k are real and obey the relation $m_k = m_{-k}^T$.

The cost criterion and the constraints of the design problem have to be expressible in a tractable manner in terms of these moments, as expressed by the following assumption.

Assumption 1: There exist integers $N \geq 0, n \geq m \geq 0$ and N auxiliary variables $x_1, x_2, \dots, x_N$ such that the constraints of the input design problem can be written as a linear matrix inequality

$$\mathcal{A}(m_0, m_1, \dots, m_n, x_1, \dots, x_N) \succeq 0 \tag{8}$$

in the moments m_k, $k = 0, \dots, n$ and the auxiliary variables x_l, $l = 1, \dots, N$, and the cost function of the input design problem is given by a linear function

$$f_0(m_0, \dots, m_n, x_1, \dots, x_N) = \sum_{k=0}^n \langle C_k, m_k\rangle + \sum_{l=1}^N c_l x_l, \tag{9}$$

where C_k are fixed matrices, c_l are fixed reals, and $\langle A, B\rangle = trace(AB^T)$.

The conditions of Assumption 1 are representative of a wide variety of problem formulations in open and closed-loop optimal experiment design; see e.g. [1],[2],[11]. In particular, all classical designs (D-optimal, A-optimal, L-optimal etc.) fall within the framework of Assumption 1.

To see this, we observe that under some mild assumptions the asymptotic (in the number N of data) average sample information matrix of the experiment is given by [13]

$$\overline{M} = \frac{1}{2\pi\lambda_0}\int_{-\pi}^{+\pi} F(e^{j\omega})\Phi_{\chi_0}(\omega)F^*(e^{j\omega})\,d\omega, \tag{10}$$

where

$$F(e^{j\omega}) = \frac{1}{H_0(e^{j\omega})}\left[\frac{\partial G(e^{j\omega},\theta)}{\partial\theta} \quad \frac{\partial H(e^{j\omega},\theta)}{\partial\theta}\right]_{\theta=\theta_0} \tag{11}$$

If the model structure is rational and $d(z)$ is chosen as the least common denominator of the elements of $F^*(z)$, then (10) is affine in a finite number of generalized moments $m_0, m_1, \dots, m_n$ defined in (7): see e.g. [2] for details. Now a typical experiment design problem, known as the *weighted trace design problem*, is

$$\min_{\Phi} tr\{C(\theta_0)W\} \tag{12}$$

where Φ is either $\Phi_u(\omega)$ in open loop or $\Phi_{\chi_0}(\omega)$ in closed loop, $C(\theta_0)$ is the asymptotic per sample covariance matrix evaluated at the true θ_0, and W is a positive semi-definite (PSD) weighting matrix, typically the Hessian of the identification criterion. Such criterion has been studied e.g. in [1], [2]. Now, using the Schur complement, the optimal design problem (12) can be transformed into $\min_\Phi trX$, subject to

$$\begin{bmatrix} X & W^{1/2} \\ W^{1/2} & \bar{M} \end{bmatrix} \succeq 0. \tag{13}$$

Thus, the criterion (12) can be re-expressed in the form of Assumption 1, where f_0 takes the simple form $f_0 = \sum_{k=1}^{l} x_{kk}$ where l is the size of X.

III. The set of feasible moments

In this section we develop an algorithm to solve optimal input design problems satisfying Assumption 1. In order to cast such problems as semi-definite programs, we derive a semi-definite description of the set of truncated generalized moment sequences $(m_0, \dots, m_n)$ which correspond to realizable experiment designs. This is one of the main results of the paper; lack of such a description inhibited the implementation of the partial correlation parametrization as a tractable semi-definite program. The main tool for our proof will be the positive matrix completion theorem introduced and formulated in the Appendix.

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. The moment matrices m_k, defined by formula (7), depend on the spectra Φ_u, Φ_{ue}, which in turn determine the experimental conditions. However, not all pairs (Φ_u, Φ_{ue}) correspond to valid experiment designs. Besides positivity of the joint power spectrum Φ_{χ_0} we have to impose that the transfer functions from the signals r, e to the signals u, y are stable. This requires that the function $f_{ue} : \mathbb{T} \to \mathbb{C}$, defined by the cross spectrum Φ_{ue} by means of $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$, can be extended to a holomorphic function outside of the unit disc, including the point at infinity (compare also [11]).

Let us divide the 2×2 matrix moments m_k into its 4 elements $m_{k,11}, m_{k,12}, m_{k,21}, m_{k,22}$, all belonging to $\mathbb{R}$.

Theorem 1: Assume the notations of Assumption 1. Let $(m_0, \dots, m_n)$ be an $(n+1)$-tuple of 2×2 matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k = 1, \dots, n$. Then the following two sets of conditions are equivalent.

1. There exist 2π-periodic scalar-valued distributions Φ_u, Φ_{ue} satisfying the following conditions.

a) the distribution $\Phi_{\chi_0}(\omega)$ defined by (6) is complex hermitian, satisfies $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$ for all ω, and is positive semi-definite;

b) the function $f_{ue} : \mathbb{T} \to \mathbb{C}$, defined by $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$, is extendable to a holomorphic function outside the unit disc, including the point at infinity;

c) the matrices $m_{-n}, \dots, m_n$ are related to Φ_{χ_0} by formula (7).

2. The matrices $m_{-n}, \dots, m_n$ satisfy the following conditions.

a) $m_{k,22} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega$ for $k = -n, \dots, n$;

b) $\sum_{l=0}^{m} d_l m_{k-l,21} = 0$ for $k = 1, \dots, n$;

c) the block-Töplitz matrix

$$T_n = \begin{pmatrix} m_0 & m_1^T & \ddots & m_{n-1}^T & m_n^T \\ m_1 & m_0 & \ddots & m_{n-2}^T & m_{n-1}^T \\ \ddots & \ddots & \ddots & & \\ m_n & m_{n-1} & \ddots & m_1 & m_0 \end{pmatrix}$$

is positive semi-definite.

Proof: Let us first show the implication 1. $\Rightarrow$ 2. Condition 2a follows from 1. by definition of Φ_{χ_0} and formula (7). Condition 2c is the Carathéodory-Fejer criterion necessary (and sufficient) for $(m_0, \dots, m_n)$ to be the truncated moment sequence of some positive spectrum, and thus follows from 1a.

Let us prove that 2b is implied by 1. We have

$$m_{k,21} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\Phi_{ue}(-\omega)}{d(e^{j\omega})d(e^{-j\omega})} e^{jk\omega}\, d\omega.$$

It follows that

$$\sum_{l=0}^{m} d_l m_{k-l,21} = \frac{1}{2\pi j} \int_{\mathbb{T}} \frac{f_{ue}(z^{-1})}{d(z)} z^{k-1}\, dz.$$

By condition 1b the function $f_{ue}(e^{-j\omega})$ is extendable to a holomorphic function inside the unit disc. Since all zeros of $d(z)$ are outside the closed unit disc, the ratio $f_{ue}(z)/d(z)$ is also holomorphic inside the unit disc. It follows that $\sum_{l=0}^{m} d_l m_{k-l,21} = 0$ for all $k > 0$.

In a similar manner it follows that the elements $m_{k,22} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega$ have to satisfy

$$\sum_{l=0}^{m} d_l m_{k-l,22} = 0, \qquad k > 0. \tag{14}$$

Let us now turn to the implication 2. $\Rightarrow$ 1.
To this end, we first show that we can extend the finite moment sequence $m_0, \dots, m_n$ to an infinite moment sequence, such that for every $n' > n$, the truncated moment sequence $m_0, \dots, m_{n'}$ still satisfies condition 2.

We will proceed by induction. Assume that conditions 2a – 2c hold for some $(n'+1)$-tuple $(m_0, \dots, m_{n'})$ of matrices, where $n' \geq m$. Here m is the degree of the polynomial d, and in condition 2c we have the matrix $T_{n'}$ instead of T_n, defined in a similar way. We shall show that there exists a matrix $m_{n'+1}$ such that the $(n'+2)$-tuple $(m_0, \dots, m_{n'+1})$ also satisfies these conditions.

In order to satisfy conditions 2a and 2b we set

$$\begin{aligned} m_{-(n'+1),22} = &\ m_{n'+1,22} \\ = &\ \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0}{|d(e^{j\omega})|^2} e^{j(n'+1)\omega}\, d\omega, \end{aligned} \tag{15}$$

$$m_{-(n'+1),12} = m_{n'+1,21} = -d_0^{-1} \sum_{l=1}^{m} d_l m_{n'+1-l,21}. \tag{16}$$

It remains to show the existence of elements $m_{n'+1,11} = m_{-(n'+1),11}$ and $m_{n'+1,12} = m_{-(n'+1),21}$ such that the resulting block-Töplitz matrix $T_{n'+1}$ is PSD.

This is a classical positive matrix completion problem, which can be dealt with by the results presented in the Appendix. To this end, we consider the matrix $T_{n'+1}$ as a partially specified matrix. The entries of $T_{n'+1}$ which appear in $T_{n'}$ as well as the entries defined by (15), (16) are specified, while the entries $m_{n'+1,11}, m_{n'+1,12}$ are unspecified. We denote by $G=(V,E)$ the graph corresponding to the known entries of the matrix $T_{n'+1}$, with labels $1,2,\dots,2(n'+2)$. We shall proceed in two steps: first we show that this graph is a chordal graph; next we show that the partially specified matrix $T_{n'+1}$ is partial PSD. It will then follow from Proposition 3 of the Appendix that $T_{n'+1}$ is PSD completable.

Step 1. The diagonal elements of $T_{n'+1}$ are all known. Consider now, for $k=1,\dots,2(n'+2)$ the set of vertices corresponding to the k-th diagonal element of $T_{n'+1}$ and the known elements below it, i.e. the set $\{k\}\cup\{l>k\,|\,(k,l)\in E\}\subset V$. It is easy to see that, for each $k=1,\dots,2(n'+2)$, this set forms a clique. Hence, the numbering of the vertex set V corresponding to the numbering of the rows of $T_{n'+1}$ is a perfect elimination ordering of the vertices of G (see Appendix). Therefore, by Proposition 1 in the Appendix, G is a chordal graph.

Step 2. We now show that the partially specified matrix $T_{n'+1}$ is partial PSD. To this end, we have to find the maximal cliques of G and to check whether the corresponding fully specified principal submatrices of $T_{n'+1}$ are PSD. By Proposition 2 in the Appendix, the maximal cliques can only be of the form $\{k\}\cup\{l>k\,|\,(k,l)\in E\}$ for some vertex k. We thus only have to test for maximality the cliques of this specific type for all vertices k. The only maximal cliques are given by the values $k=1$ and $k=3$. All other vertices lead to cliques which are contained in these two. The value $k=3$ corresponds to the largest fully specified lower right subblock of $T_{n'+1}$, which equals $T_{n'}$ and is PSD by the induction hypothesis. The value $k=1$ corresponds to the principal submatrix of $T_{n'+1}$ built of the first $2(n'+1)$ and the last row and column. It remains to show that this submatrix, denote it by T_C, is also PSD.

In order to prove this, we first define a $[2(n'+1)+1]\times[2(n'+1)+1]$ matrix

$$S=\begin{pmatrix} & I_{2(n'+1)} & 0_{2(n'+1)\times 1} \\ 0_{1\times[2(n'+1-m)+1]} & d_m\ 0\ d_{m-1}\ 0\dots\ d_1 & d_0 \end{pmatrix}.$$

Here $0_{k\times l}$ denotes a $k\times l$ zero matrix, while 0 denotes a scalar zero. Clearly the matrix S is nonsingular, therefore positive semi-definiteness of T_C is equivalent to positive semi-definiteness of the product $\tilde{T}_C=ST_CS^T$. It is easily seen that the upper left $2(n'+1)\times 2(n'+1)$ corner of $\tilde{T}_C$ equals that of T_C, which is given by $T_{n'}$. By condition 2b, which by virtue of (16) is valid also for $k=n'+1$, and by (14), which is valid for $k=1,\dots,n'$ by condition 2a and for $k=n'+1$ by (15), the lower left $1\times 2(n'+1)$ corner of $\tilde{T}_C$ is zero. Finally, the lower right element of $\tilde{T}_C$ equals

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\lambda_0}{|d(e^{j\omega})|^2}\sum_{k,l=0}^{m}d_kd_le^{j(k-l)\omega}\,d\omega=\lambda_0.$$

Therefore $\tilde{T}_C=\mathrm{diag}(T_{n'},\lambda_0)\succeq 0$, hence T_C is PSD, and the partially specified matrix $T_{n'+1}$ is partial PSD. It follows from Proposition 3 that $T_{n'+1}$ is PSD completable, i.e. there exist elements $m_{n'+1,11}, m_{n'+1,12}$ which make $T_{n'+1}$ PSD. Thus the sequence $(m_0,\dots,m_{n'+1})$ constructed in this way satisfies also condition 2c.

So far we have extended the given finite sequence $(m_0,\dots,m_n)$ to an infinite sequence of moments m_k satisfying condition 2. The construction of the distributions Φ_u,Φ_{ue} now follows classical lines.

Consider the space $\mathbf{P}$ of 2×2-matrix-valued complex hermitian polynomials $P(z)$ defined on the unit circle. For a polynomial $P(z)$ of degree r, define coefficient matrices P_k by $P(z)|d(z)|^2=\sum_{k=-(r+m)}^{r+m}P_kz^k$; they are of size 2×2 with $P_{-k}=P_k^*$. By the Carathéodory-Fejer criterion, the polynomial $P(z)|d(z)|^2$, and equivalently $P(z)$, is PSD for every $z\in\mathbb{T}$ if and only if there exists a PSD complex hermitian matrix $T^P=(T^P_{kl})_{k,l=0,\dots,r+m}$ of size $2(r+m+1)$, partitioned into blocks T^P_{kl} of size 2×2, such that $P_i=\sum_{k-l=i}T^P_{kl}$, for all $i=-(r+m),\dots,r+m$.

We define a linear functional $\mathcal{L}$ on the space $\mathbf{P}$ by $P(z)\mapsto\sum_{k=-(r+m)}^{r+m}\langle P_k,m_k\rangle$. If $P(z)$ is PSD on $\mathbb{T}$, then the value of the functional $\mathcal{L}$ equals $\langle T_{r+m},T^P\rangle\geq 0$, where T_{r+m} is the PSD block-Töplitz matrix from condition 2c, built on the moments $m_0,\dots,m_{r+m}$. By the M. Riesz extension theorem [14], we can extend $\mathcal{L}$ to a linear functional $\mathcal{L}'$ on the space of all continuous matrix-valued complex hermitian functions $F(z)$ defined on the unit circle, in such a way that $\mathcal{L}'$ is nonnegative on all functions obeying $F(z)\succeq 0$ for all $z\in\mathbb{T}$. Since the polynomial functions are dense in the continuous functions on $\mathbb{T}$, the extension $\mathcal{L}'$ is unique.

By the Riesz representation theorem, the functional $\mathcal{L}'$ corresponds to a matrix-valued distribution Φ_{χ_0} on the unit circle. The properties of Φ_{χ_0} required in condition 1. readily follow from the properties 2a – 2c of the infinite sequence of moments m_k. In particular, since the coefficients of the Laurent expansion of f_{ue} are bounded by the positivity of Φ_{χ_0}, and the coefficients at the positive powers are zero by condition 2b, this series converges outside of the unit disc to a holomorphic function. This completes the proof. ∎

Theorem 1 furnishes a semi-definite description of the set of feasible truncated generalized moment sequences $(m_0,\dots,m_n)$. Condition 2a determines the elements $m_{k,22}$ explicitly. Condition 2b yields linear relations on the elements $m_{k,21}$, while condition 2c amounts to an LMI. The optimal input design problem defined in Assumption 1 is thus turned into the following semi-definite program.

$$\min\left(\sum_{k=0}^{n}\langle C_k,m_k\rangle+\sum_{l=1}^{N}c_lx_l\right)\qquad(17)$$

with respect to the set of constraints

$$\mathcal{A}(m_0,m_1,\dots,m_n,x_1,x_2,\dots,x_N)\succeq 0,$$

$$m_{k,22}=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\lambda_0}{|d(e^{j\omega})|^2}e^{jk\omega}\,d\omega,\quad k=-n,\dots,n,$$

$$\sum_{l=0}^{m} d_l m_{k-l,21} = 0, \quad k = 1,\dots,n, \tag{18}$$

$$\begin{pmatrix} m_0 & m_1^T & \ddots & m_{n-1}^T & m_n^T \\ m_1 & m_0 & \ddots & m_{n-2}^T & m_{n-1}^T \\ \ddots & \ddots & \ddots & & \\ m_n & m_{n-1} & \ddots & m_1 & m_0 \end{pmatrix} \succeq 0,$$

where $m_{-k} = m_k^T$.

By solving this semi-definite program, the user obtains the optimal truncated moment sequence $(m_0,\dots,m_n)$ and the optimal value of the cost function. In order to perform the identification experiment, the external input signal r and the controller K must now be computed from the optimal moment matrices m_k. This task is tackled in the next section.

IV. Recovery of the experimental setup

In this section we assume that a sequence of moment matrices $(m_0,\dots,m_n)$ satisfying conditions 2a – 2c of Theorem 1 is given. We are interested in constructing a controller K and a power spectrum Φ_r such that the quantities Φ_u, Φ_{ue} defined by (3) reproduce the given moments $m_0,\dots,m_n$ by the formula (7).

Let us first give an informal motivation to the algorithm presented in this section. Our strategy will consist of splitting the moments into sums $m_k = m_k^r + m_k^e$ in a way that separates the contributions of the signals r and e to the joint power spectrum Φ_{χ_0}. By (3), the contribution of r in Φ_{χ_0} is given by

$$\Phi_{\chi_0}^r = \begin{pmatrix} |1+KG_0|^{-2}\Phi_r(\omega) & 0 \\ 0 & 0 \end{pmatrix}.$$

Inserting this matrix instead of Φ_{χ_0} into formula (7), we see that only the upper left element $m_{k,11}^r$ of the resulting moments m_k^r is nonzero. The sequences m_k^r and m_k^e have to correspond to positive power spectra. By the Carathéodory-Fejer criterion, the block-Töplitz matrices T_n^r, T_n^e constructed of these sequences have thus to be PSD. We perform the splitting such that T_n^e becomes singular, which, as we will see below, allows us to construct a rational realization of the joint spectrum Φ_{ue}. We now show that such splitting is always possible.

Theorem 2: Let $(m_0,\dots,m_n)$ be a sequence of 2×2 real matrices satisfying conditions 2a – 2c of Theorem 1. Then there exists a splitting $m_k = m_k^r + m_k^e$, $k = 0,\dots,n$, such that m_k^r and m_k^e are real for all k and

i) the matrices m_k^r are of the form $\begin{pmatrix} m_{k,11}^r & 0 \\ 0 & 0 \end{pmatrix}$;

ii) the block-Töplitz matrices T_n^r, T_n^e constructed of the matrices m_k^r and m_k^e, respectively, are positive semi-definite;

iii) the matrix T_n^e is singular.

Proof: Consider the $(n+1)\times(n+1)$ principal submatrix of the block-Töplitz matrix T_n formed by the elements $(m_{k-l,22})_{k,l=0,\dots,n}$ of T_n. By a permutation of the rows and columns of T_n, bring this submatrix into the lower-right corner and denote the resulting matrix $\bar{T}_n$ with $\bar{T}_n^{22}$ denoting the lower-right block. By condition 2c the matrix T_n is PSD, and thus $\bar{T}_n$ is also PSD. Hence, the Schur complement of $\bar{T}_n^{22}$, denote it by M, is a PSD $(n+1)\times(n+1)$ matrix. Let now λ be the smallest eigenvalue of M; then $M-\lambda I_{n+1}$ is singular and PSD. Define $m_{0,11}^r = \lambda$, $m_{k,11}^r = 0$ for $k = 1,\dots,n$, and $m_{k,12}^r = m_{k,21}^r = m_{k,22}^r = 0$ for all k. Finally, define $m_k^e = m_k - m_k^r$. Then $T_n^r = diag(\lambda,0,\lambda,0,\dots,\lambda,0)$ and is therefore PSD, and so is $\bar{T}_{n,11}^r = diag(\lambda,\lambda,\dots,\lambda)$. The singularity and positivity of T_n^e follow from the singularity and positivity of $M-\lambda I_{n+1}$ and from properties of the Schur complement. ■

The construction of a singular $T_n^e = T_n - T_n^r$ above is by no means unique. It was obtained by defining moments m_k^r that are zero for all $k \neq 0$. This corresponds to a filtered white noise reference signal: $\Phi_r(\omega) = |d(e^{j\omega})|^2|1+KG_0|^2\lambda$: see (3)-(7). There are many other ways to produce a block-Töplitz matrix T_n^r that yields a singular $T_n^e = T_n - T_n^r$ with the desired properties, as we shall illustrate below.

Let us now turn to the moment matrices m_k^e. Since the block-Töplitz matrix T_n^e is singular, we have $T_n^e\mathbf{v} = 0$ with $\mathbf{v} = (p_n, q_n, p_{n-1}, q_{n-1},\dots,p_0,q_0)^T \in \mathbb{R}^{2(n+1)}$. Observe that the sequence $m_0^e,\dots,m_n^e$ still satisfies the conditions 2a – 2c of Theorem 1. We have seen in the previous section that in this case we can extend the sequence $m_0^e,\dots,m_n^e$ to an infinite sequence of moments m_k^e, such that for every $N \geq n$, the larger truncated sequence $m_0^e,\dots,m_N^e$ still satisfies conditions 2a – 2c of Theorem 1. However, the singular matrix T_n^e now appears $N-n+1$ times as a principal submatrix of the larger PSD block-Töplitz matrix T_N^e constructed of the sequence $m_0^e,\dots,m_N^e$. This implies that we can write out two sets of $2N-n+1$ linear homogeneous equations:

$$\sum_{l=0}^{n}(m_{k+l,11}^e p_l + m_{k+l,12}^e q_l) \quad = \quad 0, \tag{19}$$

$$\sum_{l=0}^{n}(m_{k+l,21}^e p_l + m_{k+l,22}^e q_l) \quad = \quad 0 \tag{20}$$

for all $k = -N,\dots,N-n$. Since N can be arbitrarily large, (19), (20) must ultimately hold for every integer k.

Define the transfer functions $p(z) = \sum_{l=0}^{n} p_l z^{-l}$, $q(z) = \sum_{l=0}^{n} q_l z^{-l}$. Then the distribution $\Phi_{\chi_0} - \Phi_{\chi_0}^r$ corresponding to the moment sequence m_k^e must satisfy the relation

$$\frac{\Phi_{\chi_0}(\omega) - \Phi_{\chi_0}^r(\omega)}{d(z)d(z^{-1})}\begin{pmatrix} p(z^{-1}) \\ q(z^{-1}) \end{pmatrix} = 0.$$

This implies

$$\Phi_{ue}(\omega) \quad = \quad -\lambda_0 p^{-1}(z)q(z), \tag{21}$$

$$\Phi_u(\omega) - \frac{1}{|1+KG_0|^2}\Phi_r(\omega) \quad = \quad \lambda_0^{-1}\Phi_{ue}\Phi_{ue}^*. \tag{22}$$

for all ω, with $z = e^{j\omega}$. Using (5) then allows to express the controller K in terms of the polynomials $p(z)$ and $q(z)$:

$$K(z) = \frac{q(z)}{p(z)H_0(z) - q(z)G_0(z)}. \tag{23}$$

Consider the special choice of $T_n^r = diag(\lambda, 0, \lambda, 0, \ldots, \lambda, 0)$ described above, let $\mathbf{v}$ be the corresponding solution (defined up to a scalar factor) of $(T_n - T_n^r)\mathbf{v} = 0$, and $p(z), q(z)$ be the polynomials defined by $\mathbf{v}$. Then the controller $K(z)$ defined by (23) and the spectrum $\Phi_r(\omega) = |d(e^{j\omega})|^2|1 + KG_0|^2\lambda$ are one possible set of valid experimental conditions that reproduce the given truncated moment sequence. The degree of the controller will then be determined by (23) and will depend on the specific model structure adopted for G_0, H_0.

In general, the set of possible solutions for $\mathbf{v}$ will depend on the chosen model structure and on the complexity of the controller: the lower the complexity of the controller, the smaller the dimension of the space of potential solutions for the vector $\mathbf{v}$. We illustrate this for an ARX model structure.

Consider an ARX model structure $G_0 = \frac{B}{A}$ and $H_0 = \frac{1}{A}$, with $A = 1 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}$, and $B = b_1 z^{-1} + \cdots + b_{n_b} z^{-n_b}$, and let K be factored into $K = \frac{N}{D}$ where N and D are polynomials. For a vector $\mathbf{x} \triangleq (x_0, x_1, \ldots, x_n)^T \in \mathbb{R}^{n+1}$ we now introduce the following $(n+m) \times m$ matrix:

$$S_m(\mathbf{x}) = \begin{pmatrix} x_n & 0 & \ldots & \ldots & 0 \\ \vdots & x_n & 0 & & \vdots \\ x_0 & & \ddots & \ddots & \vdots \\ 0 & x_0 & & \ddots & 0 \\ \vdots & & \ddots & & x_n \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & x_0 \end{pmatrix}$$

We can then write, using (23), that $A(z)q(z) = N(z)$ and $p(z) - B(z)q(z) = D(z)$. Introducing the vectors $\mathbf{p} \triangleq (p_n, \ldots, p_0)^T$, $\mathbf{q} \triangleq (q_n, \ldots, q_0)^T$, $\mathbf{a} \triangleq (1, a_1, \ldots, a_{n_a})^T$, $\mathbf{b} \triangleq (0, b_1, \ldots, b_{n_b})^T$, these equations can then be written as a set of $2(n+1) + n_a + n_b$ linear equations as follows:

$$\begin{pmatrix} \mathbf{0}_{(n+n_a+1)\times(n+1)} & S_{n+1}(\mathbf{a}) \\ S_{n+1}(\mathbf{e_1}) & -S_{n+1}(\mathbf{b}) \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{n} \\ \mathbf{d} \end{pmatrix} \quad (24)$$

where $\mathbf{n} \triangleq (n_{n_n+n_a}, \ldots, n_0)^T$ and $\mathbf{d} \triangleq (d_{n_n+n_b}, \ldots, d_0)^T$. The set of equations (19)-(20) can similarly be rewritten as

$$\begin{pmatrix} \bar{T}_{n,11} - \bar{T}_{n,11}^r & \bar{T}_{n,12} \\ \bar{T}_{n,21} & \bar{T}_{n,22} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \quad (25)$$

with $\bar{T}_n$ and $\bar{T}_n^r$ as defined above. The bottom part of (25) represents $n+1$ constraints on the $2(n+1)$ unknown elements of $\mathbf{p}$ and $\mathbf{q}$. If in the $2(n+1)+n_a+n_b$ equations (24), we fix k of the controller coefficients (to zero, say, in order to reduce the controller complexity), then this adds k linear constraints on the $2(n+1)$ elements of $\mathbf{p}$ and $\mathbf{q}$, yielding a solution space of dimension $n + 1 - k$. Fixing n of them at predetermined values will yield a unique, up to multiplication by a common factor, solution for $\mathbf{p}$ and $\mathbf{q}$. However, there is no guarantee that there exists a Töplitz matrix $\bar{T}_{n,11}^r$ such that the first $n + 1$ equations (25) are satisfied for this solution. Thus, the procedure is to first compute a Töplitz matrix $\bar{T}_{n,11}^r$ that makes matrix in (25) singular. The resulting solutio! n set $\{\mathbf{p},\ \mathbf{q}\}$ then determines a corresponding set of admissible controllers via (24).

V. EXAMPLE

Let us apply our results to the optimal experiment design problem posed in [15]. Consider the true system $y = \frac{B(z)}{A(z)}u + \frac{1}{A(z)}e$ with $G_0 = \frac{B(z)}{A(z)} = \frac{0.1047z^{-1}+0.0872z^{-2}}{1-1.5578z^{-1}+0.5769z^{-2}}$. Here y is the output, subject to the energy constraint $\bar{E}y^2(t) = 1$, and e is white Gaussian noise with variance $\lambda_0 = 0.01$. The system is to be identified within an ARX model structure of order two. The length of the data set is $N = 1000$. The aim is to minimize the worst-case ν-gap of the uncertainty region around the identified model corresponding to a confidence level of $\alpha = 0.95$ (for details see [15]).

We compare two experimental setups, namely the optimal open loop configuration and the optimal closed loop configuration. The optimal input power spectrum Φ_u for the open-loop experiment was computed with the methods presented in [15]. The optimal closed loop experimental configuration was computed using the results of this paper. The worst-case ν-gap, as a function of the average sample information matrix (10) [15], depends on the moments m_0, m_1, m_2, m_3 defined by (7) with $d(z) = 1 + a_1 z + a_2 z^2$. The output power constraint is affine in the elements of the matrices m_k, $k = 0, 1, 2$. Hence it is sufficient to optimize over the moment matrices $m_0, \ldots, m_n$ up to the order $n = 3$. We perform the decomposition $T_3 = T_3^e + T_3^r$ of the optimal block-Töplitz moment matrix T_3 such that the trace of T_3^r is maximized. Then T_3^e will automatically have a nontrivial kernel. From T_3^r we compute a multisine reference input signal r and from the kernel of T_3^e a 6-th order controller K according to (24). The computed external input signal r and the controller K are a realization of the optimal closed loop experimental setup described by the moment matrix T_3.

In a Monte-Carlo simulation, 750 runs were performed with each configuration, and the worst-case ν-gap of the corresponding uncertainty regions was recorded. Its mean over 750 runs for the optimal open loop and closed loop experiments equals 0.0547 and 0.0553, respectively. Thus, for the chosen value of λ_0, the optimal open loop and the optimal closed loop experiments perform equally well. In fact, even though the theoretical value for the worst-case ν-gap is smaller for the closed loop optimal experiment than for the open loop one, the values are equal to within the precision of the optimization algorithm. If we consider the graph of the worst-case ν-gap as function of the moments, the optimal moment matrix T_3 is situated in a narrow, long and very flat valley, which includes also the optimal open loop experimental configuration.

The above picture can be observed for a wide range of signal-to-noise ratios. It changes dramatically, however, if the constraints make an open loop experimental setup impossible, i.e., if the signal-to-noise ratio for the optimal open loop experiment tends to zero. For a noise variance $\lambda_0 = 0.016$, the input in an open loop configuration can only have a very small power, because the power of the noise term in the output is close to 1. The mean over 750 runs of the worst-case ν-gap of the identified uncertainty regions for the optimal open loop and closed loop experiments was 0.0999 and 0.0871, respectively. Clearly in this case the optimal closed loop experiment performs much better than the optimal open loop experiment. In addition, in cases where the plant G_0 is unstable, only a closed loop design is feasible.

VI. Conclusion

Based on the partial positive definite matrix completion theorem, we have produced an optimal solution to the closed-loop optimal experiment design problem, where the optimization is performed jointly over the controller and the spectrum of the external excitation. The solution is expressed as a finite number of moments, from which we have shown how to recover an optimal controller and reference spectrum. Our simulations showed that for identification with the aim of minimization of the worst-case ν-gap of the identified uncertainty region, open loop design is nearly optimal if the signal-to-noise ratio is not too low. Future research has to show whether this is a peculiarity of the particular plant used in the simulation or whether this is a general phenomenon.

Appendix

In this section we provide the theoretical framework for one of our main results, Theorem 1. Consider the problem of completing a real symmetric matrix, only part of whose entries are specified, to a full PSD matrix. This is known as the *positive matrix completion problem.*

We first introduce some graph-theoretic notions, which are necessary to describe the specification pattern of the matrices. Let $G = (V, E)$ be an undirected graph, with V the set of vertices, of cardinality n, and $E \subset V \times V$ the set of edges. The graph G is said to be *chordal* if every cycle of length not less than 4 has a chord, i.e. an edge linking two vertices of the cycle, but not being part of the cycle. A subset $C \subset V$ of vertices such that $(C, E \cap C \times C)$ is a complete subgraph of G, i.e. such that every pair of vertices in C is linked by an edge, is said to be a *clique*. A clique is said to be *maximal* if it is not contained in any strictly larger clique. A numbering of the elements of the vertex set V with labels $1, \dots, n$ is said to be a *perfect elimination ordering* if for all $k = 1, \dots, n$ the set $\{k\} \cup \{l > k \,|\, (k, l) \in E\} \subset V$ forms a clique. We have the following classical resu! lts.

Proposition 1: [16] A graph G has a perfect elimination ordering if and only if it is chordal.

Proposition 2: [17, Section 59.3.1] Let G be a chordal graph with its vertices arranged in a perfect elimination ordering. Then for every maximal clique C of G, there exists a vertex k such that $C = \{k\} \cup \{l > k \,|\, (k, l) \in E\}$.

We shall now introduce matrix completion problems with specification patterns defined by graphs. Associated to a graph $G = (V, E)$ on n vertices is a set of partially specified real symmetric matrices of size $n \times n$. A partially specified matrix M is in this set if i) the diagonal entries M_{kk} of M are specified, ii) the entries M_{kl} and M_{lk} are specified if and only if $(k, l) \in E$, and in this case $M_{kl} = M_{lk}$. A partially specified matrix M with specification pattern defined by the graph G is said to be *partial positive semi-definite* if for every clique $C \subset V$ the fully specified principal submatrix $(M_{kl})_{k,l \in C}$ of M is PSD. Obviously it is sufficient to demand positive semi-definiteness only of submatrices corresponding to *maximal* cliques. A partially specified matrix M is said to be *positive semi-definite completable* if there exists a specification of the unspecified entries of M such that the resulting fully specified matrix is PSD. Clearly partial positive semi-definiteness is a necessary condition for PSD completability. The following result from [12] characterizes the cases when this is also sufficient.

Proposition 3: Let the partial PSD matrix M have a specification pattern defined by a chordal graph G. Then M is PSD completable. On the other hand, let G be a graph which is not chordal. Then there exists a partial PSD matrix M with specification pattern defined by G which is not PSD completable.

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