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On The Structure of Digital Controllers in Sampled-Data Systems with FWL Consideration

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A sampled-data control system consists of the continuous-time plant $P(s)$, and a sampled-data controller composed of a sampler (A/D converter) S, the digital controller $C_d(z)$ and a hold (D/A converter) H. An antialiasing filter $F(s)$ must be introduced before the sampler S [1]. One important issue is the stability of this sampled-data system. Chen and Francis [2] showed that under generic assumptions, namely, $P(s)F(s)$ having no unstable pole/zero cancellation and the sampling rate being non-pathological, the sampled-data system is L_2 input/output stable *if and only if* an equivalent discrete-time feedback system is exponentially stable, where $H_d(z)$ is the equivalent discrete-time plant obtained by discretizing $P(s)F(s)$.

Suppose that the discrete-time system $H_d(z)$ is strictly proper and has a realization (A_z, B_z, C_z). Let (Φ, L, K, D) be a realization of the discrete-time controller $C_d(z)$, where $\Phi \in R^{n_c \times n_c}$, $L \in R^{n_c \times m_c}$, $K \in R^{l_c \times n_c}$ and $D \in R^{l_c \times m_c}$, and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be the realization of the equivalent discrete-time closed-loop system. It can be shown that

$$\bar{A} = \begin{pmatrix} A_z & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} B_z & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} D & K \\ L & \Phi \end{pmatrix} \begin{pmatrix} C_z & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \triangleq M_0 + M_1 X M_2 \triangleq \bar{A}(X), \quad (1)$$

where X is the *system matrix* of the digital controller $C_d(z)$, and all the $\mathbf{0}$s / $\mathbf{I}$s are zero / identity matrix of proper order. It turns out that the sampled-data system is stable *if and only if* all eigenvalues of $\bar{A}$, denoted by $\{\lambda_k^0 \triangleq \lambda_k(\bar{A})\}$, are in the interior of the unit circle.

The stability may be lost when the digital controller $C_d(z)$ is implemented with a Digital Control Processor (DCP) due to the Finite Word Length (FWL) effects. This problem is more serious when fast sampling is used [3]. The system matrix of the actually implemented digital controller is $X + \Delta X$ instead of X, where each element of ΔX is bounded by $\epsilon/2$,

$$\mu(\Delta X) \triangleq \max_{i,j} |\Delta X(i,j)| \leq \epsilon/2. \quad (2)$$

In [4], the effects of FWL implemented digital controller was studied on the degradation of the cost function of a LQG control system while these effects were analyzed in [5] on the stability and performance of sampled-data systems.

There are two main aspects involved in controller implementations: the first one is concerned with estimating the smallest word length B_s^{min} that ensures stability. This is concerned with solving the following stability robustness problem (see [5]):

$$\mu_0(X) \triangleq inf\{\mu(\Delta X) : \bar{A}(X) + M_1 \Delta X M_2) \ is \ unstable\}. (3)$$

The other aspect is concerned with the optimal controller structure problem. We note that the controller realizations are not unique. In fact, if (Φ_0, L_0, K_0, D) is a realization of the controller $C_d(z)$, so is (Φ, L, K, D), where $\Phi = T^{-1}\Phi_0 T$, $L = T^{-1}L_0$, $K = K_0 T$ for any nonsingular real matrix (similarity transformation) T. The controller system matrix X is also a function of T. All such X form a set S_X. Once an initial realization is given, different *structures* (*realizations*) X correspond to different similarity transformations T.

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Noting that $\mu_0(X)$ is a function of the controller structure X, the interesting problem is to find out those structures such that $\mu_0(X)$ is maximized:

$$\max_{X \in S_X} \mu_0(X). \quad (4)$$

Computing explicitly the value for $\mu_0(X)$ seems very hard and is still an open problem, while the following real stability radius problem has been well studied and was solved by Qiu *et al* [6]:

$$r_R(X) \triangleq \inf\{\bar{\sigma}(\Delta X) : \bar{A}(X) + M_1 \Delta X M_2 \text{ } is \text{ } unstable\}, (5)$$

where $\bar{\sigma}(\Delta X)$ denotes the maximal singular value of ΔX. Although the closed-loop remains stable for perturbation ΔX, such that $\bar{\sigma}(\Delta X) < r_R(X)$, it should be noted that since $\mu(\Delta X) < \bar{\sigma}(\Delta X)$ (with the difference potentially quite large depending on ΔX) $r_R(X)$ and $\mu(\Delta X)$ can not be compared directly to assess robustness of stability [5]. With a statistical approach considering the elements of ΔX to be normally distributed in the interval $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, a lower bound of $\mu_0(X)$ based on $r_R(X)$ was derived by Fialho and Georgiou in [5]. With regard to the second aspect, an approach based directly on the real stability radius is not suitable since $r_R(X)$ is not a *tractable* function of X.

Since $\mu_0(X)$ is not a tractable function of X (that is the transformation matrix T), the *optimal structure* problem defined by (4) can not be solved so far. To overcome this, based on a first order approximation, a *tractable* measure is derived in terms of the pole sensitivity of the closed-loop with respect to the controller structure perturbation ΔX. This measure, on the one hand, has no rigorous connection with $\mu_0(X)$ due to the approximation, but on the other hand it is a reasonable measure for studying the controller structure problem given that it is connected to a lower bound of $\mu_0(X)$ *when* $\mu_0(X)$ *is very small* (otherwise, it does not makes much sense to study the controller structure problem) and that there is no *tractable* rigorous measure available so far. *This measure is in fact a tradeoff between rigour and computational tractability.* The structures that maximize this measure provide a better stability behavior if not the best in the sense given by (4). The optimal structure problem is re-defined by maximizing this measure with respect to all possible controller structures. A class of solutions to this problem is presented. From practical considerations, the realizations that yield a good performance against the FWL effects meanwhile possess as many trivial parameters such as $0, 1$ and -1 as possible are desired. This issue is discussed and the sparse structure problem is investigated. An algorithm is derived for finding the so-called *sparse quasi-optimal structures.* For details, we refer to the full version of this paper.

REFERENCES

[1] T. Chen and B. A. Francis, 'Input-Output Stability of Sampled-Data Systems', *IEEE Trans. on Automatic Control*, Vol. 36, No.1. pp. 50-58, January 1991.

[2] T. Chen and B. A. Francis, 'Sampled Data Optimal Design and Robust Stabilization', *J. Dynamic Systems, Measurement and Control*, Dec. 1992.

[3] R.H. Middleton and G.C. Goodwin, *Digital Control and Estimation: A Unified Approach*, Prentice Hall, Englewood Cliffs, New Jersey, 1990.

[4] P. Moroney, A.S. Willsky and P.K. Houpt, 'The Digital Implementation of Control Compensators: The Coefficient Word Length Issue,' *IEEE Trans. on Automatic Control*, Vol. AC-25, No. 4, pp. 621-630, August 1980.

[5] Ian J. Fialho and Tryphon T. Georgiou, 'On stability and performance of sampled-data systems subject to word length constraint', *IEEE Trans. on Automatic Control*, Vol. AC-39, pp.2476-2481, Dec. 1994.

[6] L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young and J.C. Doyle, 'On the real structured stability radius', *in Proc. of 12th World IFAC Congress*, Vol. 8, pp. 71-78, July 1993, Sydney, Australia.