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Adaptation and Robustness in Predictive Control

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Abstract

The explicit connection is made between the (nonadaptive) control law design stage of Predictive Adaptive Control and recent techniques of Linear Quadratic Gaussian control with Loop Transfer Recovery for robustness enhancement. The inherent controller design robustness of these methods is examined in terms of the nonadaptive closed loop properties, that is of the class of open loop perturbations to which the closed loop system is robust and of the effected input spectrum to the plant in closed loop. With this latter information, we then pose the question of to which plant model does the Recursive Least Squares identification stage converge under this input. The revelation is that in many circumstances the identified model is precisely that which yields the best satisfaction of the previous robustness criterion for the control law. In this way we demonstrate the robust interplay of the identifier and of the controller in this class of adaptive control methods.

1 Introduction

Predictive adaptive control laws have achieved a significant level of acceptablility and practical success in industrial process control applications. Their *raison d'être* is typically advanced as an heuristic generalization of minimum variance adaptive control, where the control, u_k, is chosen to minimize the plant output, y_{k+1}, variance from a reference, w_{k+1}, i.e. $E(y_{k+1} - w_{k+1})^2$, at each time k [1]. This was generalized firstly to include minimization not just of the process output variance but of the sum of this variance plus a small quantity of input variance, $E[(y_{k+1} - w_{k+1})^2 + \lambda u_k^2]$, at each k [2]. This allowed extension to handle some nonminimum phase systems. More recently, this control objective was altered to include minimization of a criterion involving outputs and inputs further into the future of the system, see e.g. [3]. Specifically, u_k to u_{k+N_u-1} are chosen to minimize,

$$\begin{aligned} J(N_1, N_2, N_u) &= E\{\sum_{j=N_1}^{N_2} [y_{k+j} - w_{k+j}]^2 + \sum_{j=1}^{N_u} \lambda_j [u_{k+j-1}]^2\} \quad (1) \\ \text{subject to } u_{k+i} &= 0, \quad i = N_u, \ldots, N_2. \end{aligned}$$

Through the inclusion of more distant future information into the criterion it is felt that better ability to control broader and more difficult classes of plants is achieved. The identification component of these adaptive controllers is usually chosen to be a variant of Recursive Least Squares (RLS), with additional devices such as deadzones, signal normalization and parameter projection.

If the same criterion (1) is minimized at each time instant k with only the first control u_k being applied to the system, then the resultant control law becomes a receding horizon Linear Quadratic (LQ) law. Thus for a time-invariant plant, the control law solving this problem will yield asymptotically a fixed state variable feedback controller, see [3,4,5]. Further analysis and choice of design variables admits the possibility of equating this receding horizon LQ problem with an infinite horizon LQ problem and, thereby, guaranteeing closed loop stability for the nonadaptive law. This has been discussed by the authors in [5] using the monotonicity methods of Poubelle [6].

In order to solve for the control value, u_k, for this problem it is necessary to generate predictions of the future values of the plant output, y_{k+i}, and then one may write explicitly a set of linear equations for the control sequence $\{u_{k+i}; i = 0, \ldots, N_u - 1\}$ involving the Hankel matrix of impulse response parameters of the plant. No notion of plant state needs to be introduced since this solution is only in terms of output predictions and desired signal values. The mechanism of generating these predictions, however, does perform the same rôle as does a state estimator or observer, [7,8]. Further, Mohtadi [7] has stated that the selection of the predictor/observer polynomial, $C(z)$, in practice has a dramatic influence upon the success of the adaptive controller. Hence the 'predictive' part of the predictive controller is seen as incorporating the observer associated with the state feedback solution of an LQ problem. Thus the predictive control strategy is composed of the same elements as a linear state variable feedback strategy with incomplete measurements and a quadratic cost, which we denominate (somewhat too generally) as Linear Quadratic Gaussian (LQG) control.

With the recognition that the predictive control criterion frequently has a direct interpretation as an equivalent infinite horizon LQG problem, one is led to ask whether this inherent control strategy might be robust as a nonadaptive control law. Middleton *et al.* [9] have advanced that robust adaptive control should commence as adaptive robust control and here we have associated with an empirically-derived adaptive control law a nonadaptive version (LQG) which has been the subject of considerable recent robustness analysis. Specifically, LQG Loop Transfer Recovery (LTR) [10] theory has been developed to allow the recovery in LQG designs of the robustness known to be present in full state feedback LQ control. We shall interpret the predictive controller design in terms of the LQG/LTR theory and show how the rôle of particular elements of the design, such as λ in (1), is the same as that of certain other variables in LTR methods.

The robustness of LQG/LTR control systems is measured by the ability to maintain closed loop stability in the face of multiplicative perturbations to the open loop nominal plant. That is to say, should the actual open loop plant differ from the nominal plant used for the controller design, then LQG/LTR control preserves closed loop stability provided the multiplicative perturbation to the nominal plant satisfies certain frequency domain constraints. We shall interpret the multiplicative error between the plant and its nominal model as being the relative error of the frequency response of the open loop system.

In adaptive control systems, one is concerned not just with controller design and its robustness to externally imposed modelling errors, but also with on-line plant identification where two facets of the same issue arise: 'What is the effect of the control law upon the model identified in closed loop?' and/or 'To what extent is the identifier capable of providing a model compatible with the controller robustness?'. We shall apply the Return Difference Equalities (EDR) of optimal filtering and control, i.e. LQG, to derive the spectrum of the input signal applied to the plant in closed loop. With LTR control methods this spectrum has a particularly simple form. When this input spectrum data is now used with Ljung's theory of frequency domain evaluation of least squares identification criteria, we gain a view on the implied

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weighting of the model fit achieved through the coupled use of RLS and LQG control. The surprise here is that the model fit achieved via the use of this control law dovetails precisely acccording to a relative error criterion. That is, the identified model frequency response will have smallest error from the actual plant frequency response exactly in accordance with the requirements of the LQG/LTR robustness result. In this way, the adaptation helps to tune the model to fit the controller robustness, thereby improving closed loop robustness overall. This will be compared to H_∞ (or worst case) methods to show that this robust interplay between controller and identifier has the potential to achieve considerably better control.

It is our view that this work represents the first study in which the robustness theory of LQ control has been utilized in an adaptive control procedure incorporating the features of both the controller and the identifier components. It describes the coupled behaviour of these two elements and begins with an empirically verified but heuristically derived practical algorithm and makes full body contact with modern robustness and identification theories. This yields both an understanding of why the practical algorithms have been found to perform well and also of how they might be modified to include recent advances in sophisticated controller and identifier design.

The structure of the paper is as follows. Section 2 is devoted to an exploration of the connection between the nonadaptive predictive control law and LQG methodology. Section 3 treats LQG robustness and LTR design. Section 4 concerns the interpretation of predictive control as LQG/LTR and Section 5 deals with the interplay between the controlled closed loop and the identifier. Section 6 concludes.

2 Predictive Control and LQG

2.1 The Control Law

Let the open loop plant have a state variable description as follows,

$$\begin{aligned} x_{k+1} &= Fx_k + Gu_k \qquad (2)\\ y_k &= Hx_k, \qquad (3)\end{aligned}$$

and let the control law be obtained by minimization of the standard finite horizon LQ criterion,

$$J(N) = \mathrm{E}\left\{\sum_{j=1}^{N}\left[x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}\right]\right\} \qquad (4)$$

We make the following observations concerning this control law;

- a reference signal, w_k, may be introduced into the criterion (4) to yield a tracking problem [13]. This solution will have the same stability properties as the regulation problem but will include a reference signal precompensator.
- comparing the criteria (1) and (4) with $w_k = 0$, it is readily seen that they are equivalent provided $N = N_2$ and the weighting matrices associated with the state and with the control are, respectively, $Q = H^T H$ and $R = \lambda I$, ignoring the differences in horizons N_2 and N_u. Incorporating these horizons and N_1 requires the introduction of some zero Q values and some infinite R's for certain time indices [5].
- the solution of this standard finite horizon LQ problem may be stated in terms of a time-varying linear state variable feedback law,

$$u_{k+N-j} = -(G^T P_{j-1} G + R_{j-1})^{-1} G^T P_{j-1} F \hat{x}_{k+N-j}. \qquad (5)$$

Here $\hat{x}_k$ is a state estimate produced by an observer, P_j is the solution of the Riccati difference equation.

$$P_{j+1} = F^T P_j F - F^T P_j G (G^T P_j G + R_j)^{-1} G^T P_j F + Q_j \qquad (6)$$

- the control law (5), in the receding horizon control situation, only results in u_k being applied from this entire finite time solution. A new finite horizon problem is solved for u_{k+1} with the same horizon. Thus a stationary control law arises with a gain only dependent upon P_{N-1}. This is the principle behind receding horizon LQ control [12].
- the asymptotic stability of the closed loop with a receding horizon LQ control law is not guaranteed, since the finite horizon subproblems do not have any connection to infinite horizon properties such as stability. The asymptotic stability is assured for the closed loop of an infinite horizon LQ problem with $[F, Q^{1/2}]$ stabilizable.
- using monotonicity arguments, it is possible to modify slightly the predictive control criterion so that the resultant closed loop is asymptotically stable, see [5]. This modification proceeds by showing that the receding horizon solution, with a given Q and R, can be made to be the solution of an infinite horizon problem with a larger Q and the same R. An infinite time time-invariant LQ problem has a solution,

$$\begin{aligned} P &= F^T P F - F^T P G (G^T P G + R)^{-1} G^T P F + Q. \qquad (7)\\ u_k &= -(G^T P G + R)^{-1} G^T P F \hat{x}_k. \qquad (8)\end{aligned}$$

2.2 The State Estimator

The linear state variable control laws above are all implemented using state estimates, $\hat{x}_k$, since full state information is not available. We make the following remarks concerning the generation of these state estimates;

- because the plant is assumed strictly proper it is possible to produce state estimates using an observer with a direct feedthrough term without encountering algebraic loop problems. Such an observer has the form,

$$\hat{x}_{k+1} = (F - MHF)\hat{x}_k + (G - MHG)u_k + My_{k+1} \qquad (9)$$

where the eigenvalues of $F - MHF$ may be arbitrarily placed by choice of M provided $[F, HF]$ is an observable pair.

- if we presume that output measurements are corrupted by noise, as is the state, then an alternative derivation of an observer may be carried out based upon optimal state estimation, Kalman Filtering (KF) [14]. Then the filter gain, M, is designed via the solution of the filtering algebraic Riccati equation as follows, with process noise covariance Q_o and independent measurement noise covariance R_o,

$$\begin{aligned} M &= \Sigma H^T (H\Sigma H^T + R_o)^{-1} \qquad (10)\\ \Sigma &= F\Sigma F^T - F\Sigma H^T (H\Sigma H^T + R_o)^{-1} H\Sigma F^T + Q_o. \qquad (11)\end{aligned}$$

- it is possible to extend the KF to deal with coloured measurement noise and with correlated process and measurement noise [22].
- note the structural similarity (duality actually) between the LQ control solution and the KF solution. At least this is true inasmuch as Riccati difference equations of similar form are used. The actual duality exists between the LQ control of a plant with delay and the Kalman one-step-ahead predictor for the same plant.
- the observer/KF with direct feedthrough is not dual to the LQ controller for a plant with delay.
- by careful choice of the state coordinate basis, it is possible to formulate state estimates directly as future plant output predictions [8]. In this formulation, the observer characteristic polynomial is identified with the predictor polynomial. Thus the predictor component of predictive control may be incorporated within the structure of an observer.

2.3 Summary

The nonadaptive predictive control law (perhaps with some slight mod-

$$G(z) = -K \times \left\{I + (I - MH)(F + GK)[zI - (I - MH)(F + GK)]^{-1}\right\} \times MH(zI - F)^{-1}G. \quad (22)$$

For the KF, or output preserving problem,

- with no state feedback

$$G(z) = -H(zI - F)^{-1}M \quad (23)$$

- with an observer possessing no direct feedthrough term

$$G(z) = -H(zI - F)^{-1}GK(zI - F + MH - GK)^{-1}M, \quad (24)$$

- with an observer possessing a direct feedthrough term

$$G(z) = -H(zI - F)^{-1}GK \times \left\{I + (I - MH)(F + GK)[zI - (I - MH)(F + GK)]^{-1}\right\} \times M. \quad (25)$$

This alteration from the ideal $G(z)$ has the potential effect that all robustness of the closed loop may be lost, since we no longer have an automatic lower bound upon $|I+G(e^{j\omega})|$. It may well be the case that the new closed loop is actually more robust than the ideal. What is at issue is that the 'guaranteed' margin implied by the Return Difference Equalities is lost.

3.4 Loop Transfer Recovery

Loop Transfer Recovery (LTR) refers to a design methodology whereby the robustness guarantee of LQ and KF can be recovered for certain plants operating under LQG. The better known theory for LTR is in continuous time, see [10,19], where the KF and LQ are dual. The design methodology runs as follows;

1. perform an LQ design according to normal rules.
2. design an observer by using the Kalman Filter with covariance matrices $Q_o = GG^T$ and $R_o = \rho I$, with the positive real parameter $\rho \to 0$.
3. if the open loop plant, $P(s) = H(sI-F)^{-1}G$, is minimum phase, i.e. it possesses no zeros which are in the right half complex plane, then $G_{LQG}(s) \to G_{LQ}(s)$ as $\rho \to 0$ for all s. If the $P(s)$ is nonminimum phase then this convergence does not occur. Here $G_{LQG}(s)$ refers to the continuous time equivalent of (21) or (22) (There is no distinction between these two in continuous time.) while $G_{LQ}(s)$ is the equivalent of (20).

An alternative design procedure is the dual of the above;

1. perform a KF design according to normal rules.
2. design a state variable feedback via LQ with weighting matrices $Q = H^T H$ and $R = \lambda I \to 0$.
3. if $P(s)$ is minimum phase then $G_{LQG}(s) \to G_{KF}(s)$ as $\lambda \to 0$ for all s. Here $G_{LQG}(s)$ refers to the continuous time equivalent of (24) or (25) while $G_{KF}(s)$ is the equivalent of (23).

Several points are in order here;

- The principle of these LTR methods is that by performing a singular optimal filtering (control) design in the LQG controller, the extra dynamics appearing in $G_{LQG}(s)$ are forced to cancel with the (stable) open loop plant zeros, thereby yielding the required LQ or KF $G(s)$, which, by our previous arguments, possesses an inherent degree of robustness.
- Since the open loop plant and the controller/observer system are both strictly proper in continuous time, the convergence of return differences above involves the appearance of successively larger gain matrices M or K as $\rho \to 0$ or $\lambda \to 0$.
- While the method guarantees nothing for nonminimum phase plants, there abound claims that the methodology performs well with many such systems. The question is then what value of ρ or λ is a suitable stopping point. Recall the nominal design is LQG and this is simply a robustness enhancement measure. That is, the LQG controller design guarantees stability for the nominal plant and our aim here is to modify this design procedure in such a fashion as to guarantee a measure of robustness along with the nominal stability but without departing from the LQG design methodology.
- The robustness recovered by this procedure is that of LQ or KF which is not necessarily worse nor better than that of the LQG design.
- The robustness is achieved at the expense of performance, since the nominal behaviour is detuned from the natural LQG settings, Q, R, Q_o, R_o.
- It is generally agreed, amongst the cogniscienti, that the latter (KF followed by singular optimal control) method is preferable to the LQ-first approach. This is because the KF-first method preserves the plant output signal in the recovery and this is usually more closely related to control objectives than preservation of the control input properties.

In discrete time, the situation is similar but deceptively different, as has been investigated by Maciejowski [18] with the following remarks being pertinent.

Remark 1 *In discrete time there is a distinction between the Kalman Filter (KF) and the Kalman one-step-ahead predictor (KP). This distinction does not exist in continuous-time.*

- *The Kalman Filter involves no delay. The state estimate, $\hat{x}_{k|k}$, of the state at time k given input-output data up to and including time k is produced. The Kalman predictor yields $\hat{x}_{k|k-1}$.*
- *The KF therefore possesses a direct feedthrough term.*
- *If the open loop plant is strictly proper then the KF may be used to implement an LQG feedback law without the appearance of algebraic loops.*
- *The Kalman predictor is dual to the LQ control of a plant with delay. The KF is <u>not</u> dual to this LQ problem!*

Remark 2 *LTR is only possible in discrete time using the true Kalman Filter followed by singular optimal control design.*

- *The issue of correct relative degree in $G_{LQG}(z)$ and the appearance of large M or K in the singular design stage conflicts with stability requirements for discrete time systems. To be more precise, from root locus arguments one may see that large feedback gains in the overall controller will necessarily lead to closed loop poles exiting the (compact) stability domain.*
- *The nonduality of discrete time LQ and KF for a strictly proper plant causes the discrepancy between design approaches. It can be shown that if one restricts oneself to Kalman predictors (and not KF), then the complete equivalent of the continuous theory emerges with the robustness recovered being just that achievable by a controller possessing a delay [20]. Because of the extra delay in the loop vis-à-vis KF based controllers, these Kalman predictor based controllers are inherently less robust.*

ification for guaranteed stability) falls under the ambit of stationary infinite horizon LQ control implemented with the use of an observer. With only a slight abuse of notation we shall denominate this Linear Quadratic Gaussian (LQG) control in the sequel.

3 LQG Robustness and LTR

There has been considerable activity for an extended period on issues associated with the robustness of feedback control systems. Here we shall state a subset of results which pertain to the robustness of LQG control. The path we follow will be close to a single-input/single-output, discrete-time version of Lehtomaki *et al.* [15], Doyle [16], Stein and Athans [10], and Kwakernaak [17] and shall culminate with the work of Maciejowski [18]. Extensions to multivariable systems are direct from the referred works but would hinder clarity here.

3.1 Closed Loop Robustness

Here we shall consider criteria for the preservation of stability when a given feedback controller, $C(z)$, designed on the basis of a nominal plant, $P(z)$, is connected in a unity feedback closed loop with an actual plant, $\tilde{P}(z)$, which may differ from $P(z)$.

Denote the actual plant, $\tilde{P}(z)$, as being a multiple of the nominal $P(z)$,

$$\tilde{P}(z) = L(z)P(z), \tag{12}$$

where the multiplicative perturbation is $L(z)$ and pole-zero cancellations may occur in (12). Denote the nominal and actual controller/plant cascades by $P(z)C(z) = G(z)$ and $\tilde{P}(z)C(z) = \tilde{G}(z) = L(z)G(z)$. Let the corresponding open and closed loop characteristic polynomials be $\phi_{ol}(z)$, $\tilde{\phi}_{ol}(z)$, $\phi_{cl}(z)$ and $\tilde{\phi}_{cl}(z)$. Then we have the following result,

Theorem 1 *The closed loop characteristic polynomial $\tilde{\phi}_{cl}(z)$ has no zeros outside the unit circle if,*

- *$\phi_{ol}(z)$ and $\tilde{\phi}_{ol}(z)$ have the same number of zeros outside the unit circle,*
- *$\phi_{ol}(z)$ and $\tilde{\phi}_{ol}(z)$ have the same unit circle zeros,*
- *$\phi_{cl}(z)$ has no zeros outside or on the unit circle,*
- *the multiplicative plant perturbation, $L(z)$, and the nominal return difference, $1 + G(z)$, satisfy*

$$|L^{-1}(z) - 1| < \min\left[1, |1 + G(z)|\right] \text{ at each } z \in \Omega, \tag{13}$$

where Ω is a contour consisting of the unit circle indented around unit circle zeros of $\phi_{ol}(z)$.

This is a discrete-time version of a statement of Lehtomaki *et al.* [15], although it is a classical gain margin result in the single-input/single-output case.

This theorem links the closed loop robustness to multiplicative perturbation $L(z)$ (recall the nominal value of $L(z)$ is 1) with the value of the frequency response of the return difference, $[1 + G(z)]$, of the nominal controlled system. To gain a further appreciation of the statement, consider the following expression of the final inequality,

$$\begin{aligned} L^{-1}(z) - I &= [G(z)L(z)]^{-1}[G(z) - G(z)L(z)] \\ &= \tilde{G}^{-1}(z)[G(z) - \tilde{G}(z)] \\ &= \tilde{P}^{-1}(z)[P(z) - \tilde{P}(z)]. \end{aligned} \tag{14}$$

If the nominal controller is designed to yield a nonzero magnitude of the frequency response of the return difference, then the feedback system stability will be robust to a certain level of relative error between the nominal and actual plants.

3.2 LQ and KF Robustness

If we denote the feedback gain in the LQ solution (5) by K and regard the signal $z_k = -K\hat{x}_k$ as a fictitious output in a unity feedback representation of the LQ plant then one has for the cascaded plant/controller transfer function, $G(z)$ of the previous section,

$$G(z) = -K(zI - F)^{-1}G. \tag{15}$$

We may now state the discrete Return Difference Equality of optimal control. This is a relationship satisfied by the solution of an infinite horizon time-invariant LQ problem, i.e. by matrix P of the algebraic Riccati equation and the corresponding gain vector K of the state variable feedback. It follows simply from the algebraic Riccati equation that,

$$\begin{aligned} &R + G^T(z^{-1}I - F)^{-T}Q(zI - F)^{-1}G = \\ &\quad [I - K(z^{-1}I - F)^{-1}G]^T(G^TPG + R)[I - K(zI - F)^{-1}G] \end{aligned} \tag{16}$$

We consider this equality for z on the unit circle and note that the left hand side is a spectrum which consists of a strictly positive constant part, R, and a strictly proper nonnegative part. Thus from (16) we have directly that

$$|I - K(zI - F)^{-1}G| \geq \frac{R}{G^TPG + R} > 0 \qquad \text{for all } z \in \Omega. \tag{17}$$

Referring now to Theorem 1 and (15), we see that LQ full state feedback possesses a natural robustness to multiplicative perturbation of the plant system, because $I + G(z) = I - K(zI - F)^{-1}G$.

For the Kalman Filter, there exists the dual Return Difference Equality,

$$\begin{aligned} R_o \;+\; &H(zI - F)^{-1}Q_o(z^{-1}I - F)^{-T}H^T = \\ &[I - H(zI - F)^{-1}M](H\Sigma H^T + R_o)[I - H(z^{-1}I - F)^{-1}M]^T, \end{aligned} \tag{18}$$

where R_o is the measurement noise covariance matrix and Q_o is the process noise covariance matrix.
By direct analogy to the LQ case, we have that

$$|I - H(zI - F)^{-1}M| \geq \frac{R_o}{H\Sigma H^T + R_o} \qquad \text{for all } z \in \Omega. \tag{19}$$

Thus the Kalman Filter possesses an inherent degree of robustness to multiplicative mismodelling of the plant, so that stability of the filter (in terms of the behaviour of the deviation between predicted or filtered plant outputs and actual plant output) is preserved in the face of certain $L(z) \neq I$ modifications to the design plant.

3.3 Guaranteed LQ Margins with Observers

"There are none!" [16].

That is, it is possible to find examples of the complete loss of the above robustness to multiplicative perturbation when either the LQ controller is implemented using an observer, LQG, or the Kalman Filter is used when the plant is operating under state-estimate feedback. This phenomenon was probably first reported by Kwakernaak.

The issue is that the plant/controller cascade, $G(z)$ above, is replaced as follows for the LQ (so-called input preserving) problem,

- with direct state feedback, $\hat{x}_k = x_k$

$$G(z) = -K(zI - F)^{-1}G, \tag{20}$$

- with an observer possessing no direct feedthrough term

$$G(z) = -K(zI - F + MH - GK)^{-1}MH(zI - F)^{-1}G, \tag{21}$$

- with an observer possessing a direct feedthrough term

Remark 3 *The LTR return difference converges to the KF return difference as the LQ control weighting $\lambda \to 0$ if and only if the open loop plant $P(z)$ is minimum phase and minimum delay, i.e. no zeros outside the unit circle and $\det HG \neq 0$, i.e. a unit delay.*

To summarize, the discrete time LQG/LTR design follows;

1. Perform a Kalman Filter (with direct feedthrough) design for the open loop plant.

2. Conduct a singular optimal control design for the LQ feedback with weighting matrices $Q = H^T H$ and $R = \lambda I$.

3. If the open loop plant is minimum phase and minimum delay, then $G_{LQG}(z) \to G_{KF}(z)$ as $\lambda \to 0$ for all z. Otherwise one must cease the design at a nonzero value of λ — the empirical claim being that this works well for many systems. The limiting feature of these (singular) optimal control laws with nonminimum phase/delay systems is that the control signal magnitude becomes unmanageably large as $\lambda \to 0$.

4 Predictive Control as LQG/LTR

Our claim in this brief section is that Predictive Control, (PC), as a nonadaptive control law, implements *ipso facto* an LQG/LTR design. Specifically, we have shown the control objective of Generalized Predicitive Control (1) to be explicitly a receding horizon LQ criterion which, if asymptotic stability modifications are made, is identical to an infinite horizon LQ criterion with weighting matrices $Q \geq H^T H$ and $R = \lambda I$.

The positive value λ is selected as a somewhat arbitrary control input weighting designed to perturb the minimum variance criterion to prevent possibly unbounded control actions resulting with nonminimum phase systems. Our thesis here (as indicated by our notation) is that λ is a small correction to our desired controller criterion selected to achieve closed loop robustness to our design. That is the λs in Predictive Control and in LQG/LTR are the same.

To complete our presciption of Predictive Control as LQG/LTR, we note that the 'missing link' in the PC design is the specification of the predictor polynomials — $\mathcal{C}(z)$ in the terms of Mohtadi [7]. A major feature of PC is that the choice of $\mathcal{C}(z)$ is recognized as crucial for success and is often presumed to have been fixed before the control design stage and tuning is begun. Given the identity between these predictors and observers already enunciated in the Introduction and [3,7,8], we now are in a position to associate with each fraction of the PC design an equivalent from discrete time LQG/LTR. Mohtadi [7] advocates the use of $\mathcal{C}(z)$ which reflect the plant and measurement noise models and which implement delay-free predictions of the plant output. The connection to Kalman Filtering theory is clear and indeed suggests how this component of PC should be designed. By careful choice of KF weighting matrices it is possible to maintain the coordinate-free state variable controller design [21].

5 Adaptation and Robustness

The major application of Predictive Control laws is in the area of adaptive process control. That is the nonadaptive control law design is coupled with an on-line Recursive Least Squares (RLS) parameter estimator. The questions here then concern the interplay between the adaptation and the controller robustness achieved via the LQG/LTR connection. In this context the estimated model plays the rôle of the nominal plant, $P(z)$, while the actual plant is the perturbed $\tilde{P}(z)$. We first consider how the control signal, u_k, affects the selection of the nominal plant transfer function, $P(z)$, and then move on to consider the features of the control in the Predictive Controller or LQG/LTR.

According to the recent theory of Ljung [11], the minimization of a Least Squares prediction error criterion between asymptotically large, stationary data sets of plant input and output can, via Parseval's identity, be interpreted in terms of a frequency response deviation minimization. Specifically, we presume a plant structure

$$y_k = \tilde{P}(z)u_k + v_k, \tag{26}$$

where v_k is the measurement noise, and we presume a model structure

$$\hat{y}_k = P(z,\theta)u_k + H(z,\theta)\xi_k, \tag{27}$$

where ξ_k is the prediction error of the model (27) and $P(z,\theta)$ and $H(z,\theta)$ are parametrized transfer functions. The Least Squares solution is sought as follows

$$\min_{\theta\in\Theta} \mathrm{E}\left[H^{-1}(z,\theta)(y_k - P(z,\theta)u_k)\right]^2. \tag{28}$$

This may equally well be regarded as a frequency domain minimization

$$\min_{\theta\in\Theta} \int_{-\pi}^{\pi} \left[\left|\tilde{P}(e^{j\omega}) - P(e^{j\omega},\theta)\right|^2 \Phi_{uu}(\omega) + \Phi_{vv}(\omega)\right] \frac{d\omega}{|H(e^{j\omega},\theta)|^2}. \tag{29}$$

From (29) we see the rôle played in the selection of plant model, $P(z,\theta)$, by the input spectrum as well as the parts played by the noise spectrum and the class of noise models. For our (illustrative) purposes here we shall assume in what follows that we are operating with small noise, $\Phi_{vv}(\omega) \ll \Phi_{yy}(\omega)$, and that we have chosen no noise model, $H(z,\theta) = 1$. Further, we shall assume that the on-line use of RLS with prediction error updates yields the true Least Squares prediction error minimizing value of θ. Then the simplified identification criterion associated with the adaptation of the controller is

$$\min_{\theta\in\Theta} \int_{-\pi}^{\pi} \left|\tilde{P}(e^{j\omega}) - P(e^{j\omega},\theta)\right|^2 \Phi_{uu}(\omega)\, d\omega. \tag{30}$$

We next pose the question of what is the nature of $\Phi_{uu}(z)$ for PC or LQG/LTR. Recall that our control law is given by

$$u_k = -K\hat{x}_k + w_k, \tag{31}$$

where w_k is the reference signal and the state variable feedback gain, K, is given by the solution of a singular optimal predictive control problem, i.e. $R = \lambda \to 0$ and $Q = H^T H$ fixed. The closed loop transfer function between the external reference signal, w_k, and the control signal, u_k, is precisely the inverse of the (input preserving or LQ) return difference. For the nominal plant, the observer dynamics cancel from this transfer function to yield the closed loop relation

$$u_k = [I - K(zI - F)^{-1}G]^{-1} w_k. \tag{32}$$

To ascertain the spectrum of u_k we need to analyse this return difference with $Q = H^T H$ and $R = \lambda \to 0$, and for this we turn back to the return difference equality (16) with this substitution for Q and R,

$$\begin{aligned}\lambda + G^T(z^{-1}I - F)^{-T}H^T H(zI - F)^{-1}G = \\ [I - K(z^{-1}I - F)^{-1}G]^T(G^T PG + \lambda)[I - K(zI - F)^{-1}G]. \end{aligned} \tag{33}$$

Now notice that the left hand side of (33) consists of $\lambda + P^T(z^{-1})P(z)$ with λ small, and the right hand side is a constant multiple of the return difference with its conjugate transpose. Combining this with (32) we see that the LQG/LTR control law produces a closed loop control spectrum,

$$\begin{aligned}&\Phi_{uu}(\omega)\\ &= [I - K(e^{j\omega}I - F)^{-1}G]^{-1}\Phi_{ww}(\omega)[I - K(e^{-j\omega}I - F)^{-1}G]^{-1}\\ &\approx [H(e^{j\omega}I - F)^{-1}G]^{-1}\Phi_{ww}(\omega)[H(e^{-j\omega}I - F)^{-1}G]^{-1}\\ &= P^{-1}(e^{j\omega})\Phi_{ww}(\omega)P^{-1}(e^{-j\omega}),\end{aligned} \tag{34}$$

where $\approx$ means 'is approximately proportional to'. This expression for the nominal closed loop control spectrum stems directly from the singular optimal control element of its genesis.

The control spectrum from (34) may now be substituted into the identification criterion (30) to yield the equivalent adaptive predictive control identification criterion;

$$\min_{\theta\in\Theta}\int_{-\pi}^{\pi}\left|\frac{\tilde{P}(e^{j\omega})-P(e^{j\omega},\theta)}{\tilde{P}(e^{j\omega})}\right|^2\Phi_{ww}(\omega)\,d\omega. \qquad (35)$$

That is to say, using (14), the adaptation criterion is identical to the modelling criterion

$$\min_{\theta\in\Theta}\int_{-\pi}^{\pi}\left|L(e^{j\omega})-1\right|^2\Phi_{ww}(\omega)\,d\omega. \qquad (36)$$

To draw these many threads together at this stage, we have shown the following;

- The LQG/LTR control law is implicit in the statement of Predictive Control.
- This control law is naturally robust to unmodelled multiplicative plant perturbations, $L(z)$.
- The closed loop stability is maintained provided (13) is satisfied, that is, $|L^{-1}(z)-I|<|I+G(z)|$ for all $z\in\Omega$.
- The LQG/LTR control law engenders a closed loop control spectrum which causes the adaptation component to minimize a weighted integral of $\left|L(e^{j\omega})-1\right|^2\Phi_{ww}(\omega)$.
- The adaptation and control stages may thus be seen to be mutually supporting in terms of their effects vis-à-vis the closed loop robustness requirements.
- The natural robustness of LQ/KF is really only established using the return difference equalities and their overbounds. Similarly, the model fit only refers to the attempts to overbound the multiplicative error.
- One may use the above results to indicate how the reference signal should be chosen to achieve maximal robustness and also how one might choose identification prefilters better to enhance robustness, if extra information is available. By careful choice of w_k and prefilters, one may encourage the modelling to fit best precisely at those frequencies where $[I+G(z)]$ is small. In this way, designs exceeding straight H_∞ (worst case) robustness design can be achieved.
- With the establishment of Predictive Control within this framework, it becomes feasible to contemplate how more sophisticated (and more sophistical) control design methods might be incorporated into adaptive control.

6 Conclusion

We have followed through a tale of detective work isolating the connections between an existing popular adaptive control scheme and the recent theory of robustness of LQG systems to show that this control law embodies the elements of LQG/LTR design together with an interrelating adaptation phase whose objective is interpretable in terms of best fitting the control robustness criterion. In this fashion we have shown how adaptation and robustness may be mutually supportive and, further, have demonstrated both why this adaptive control method has been found to work well in many circumstances and how it might be modified better to take into account further plant knowledge and more sophisticated design.

The pleasing feature of this work is that full contact is established between practical adaptive control and sophisticated control design theories. This should reinforce both camps.

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