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MONOTONICITY AND ITS CONSEQUENCES FOR STABILITY OF RECEDING HORIZON LQ CONTROL

by

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ABSTRACT

A number of recently derived monotonicity results for the solutions of the Riccati Difference Equation (RDE) of Linear Quadratic (LQ) optimal control are briefly recalled. These results are then used to redesign receding horizon LQ optimal control schemes in such a way as to ensure asymptotic stability of the closed loop. The key device is to transform the RDE that accompanies every finite horizon LQ Control problem into a Fake Algebraic Riccati Equation and to ensure that it becomes the Algebraic Riccati Equation (ARE) of a legitimate related infinite horizon LQ problem.

1. Introduction

Recently the authors have derived sufficient and necessary conditions for the monotonicity of the solution of discrete-time (resp. continuous-time) Riccati Difference (resp.Differential) Equations (RDE) : see Poubelle et al., 1988. Although these results are of independent theoretical interest, their most remarkable practical application so far is that they provide a technique for guaranteeing closed loop stability of receding horizon linear quadratic (LQ) optimal control schemes, as will become clear later in this paper. This technique is based on converting the RDE of the given problem into an Algebraic Riccati Equation (ARE) of a related infinite horizon LQ problem, and has been called a Fake Algebraic Riccati Technique.

Receding horizon LQ optimal control has been studied for some brief period in the mid-1970's, and has seen a sudden rebirth in the mid-1980's in the guise of what is now called Predictive Control. This is nothing else but LQ optimal control formulated in an input-output framework and, therefore, with a very special choice of weighting matrices in the cost function. Predictive Control is mainly advocated for the adaptive control of processes, and has obviously proved to be effective and popular, its main merit being a simple and intuitive interpretation of its design variables. However, a major drawback with Predictive Control is that, even in its non-adaptive version, the stability of the closed loop cannot be guaranteed.

We shall show here that our monotonicity results and our Fake Algebraic Riccati technique are the appropriate tools for the stability analysis of Receding Horizon LQ control. We shall illustrate what causes Predictive Control to be potentially unstable, and how closed loop stability can always be guaranteed by enforcing monotonicity. These stability results are part of a thorough analysis of stability, performance and robustness of predictive receding horizon control methods, due to appear in a forthcoming book : see Bitmead, Gevers and Wertz (1989).

2. The equations of LQ optimal control.

We consider a linear state-space model

$$x_{t+1} = Fx_t + Gu_t \qquad (2.1)$$

$$y_t = Hx_t$$

with $x\in \mathbb{R}^n$, $u\in \mathbb{R}^m$, $y\in \mathbb{R}^p$ and a quadratic cost function :

$$J(N,x_t) = x_{t+N}^T P_0 x_{t+N} + \Sigma_{j=0}^{N-1}\{x_{t+j}^T Qx_{t+j} + u_{t+j}^T Ru_{t+j}\} \qquad (2.2)$$

Throughout this paper we shall make the following

<u>Assumption A</u>

1) $[F,G]$ is a stabilizable pair
2) $Q\geq 0$, $R>0$, $P_0\geq 0$

With the model (2.1) and the cost function (2.2), various optimal control strategies can be defined. They will most often require the solution of the following RDE from the initial condition P_0 :

$$P_{j+1} = F^TP_jF - F^TP_jG(G^TP_jG + R)^{-1}G^TP_jF + Q \qquad (2.3)$$

Here we shall study two such strategies.

<u>Infinite Horizon LQ Control</u>

This corresponds to iterating (2.3) until P_j converges to a steady state P_∞, or alternatively solving the following Algebraic Riccati Equation (ARE) for its maximal solution :

$$P_\infty = F^TP_\infty F - F^TP_\infty G(G^TP_\infty G + R)^{-1}G^TP_\infty F + Q \qquad (2.4)$$

The time-invariant optimal control law is then

$$u_t = -(G^TP_\infty G + R)^{-1}G^TP_\infty Fx_t \qquad (2.5)$$

$$:= -K_\infty x_t$$

for all t, with

$$K_\infty \triangleq (G^TP_\infty G + R)^{-1}G^TP_\infty F \qquad (2.6)$$

The stability of the closed loop state transition matrix $(F-GK_\infty)$ is then governed by the following result.

Proposition 2.1 (de Souza et al. 1986)
Under assumption A, the ARE (2.4) has a unique stabilizing solution P_∞, i.e.

$$|\lambda_i(F-G(G^TP_\infty G+R)^{-1}G^TP_\infty F)|<1 \quad \forall i,$$

if and only if the pair $[F, Q^{1/2}]$ has no unobservable mode on the unit circle. If so, and if in addition $P_0>0$ or $P_0 \geq P_\infty$, then

$$\lim_{j\to\infty} P_j = P_\infty$$

Receding Horizon LQ control

This corresponds to iterating (2.3) N times and implementing the time invariant control law

$$u_t = -(G^TP_{N-1}G+R)^{-1}G^TP_{N-1}Fx_t \qquad (2.7)$$
$$= -K_{N-1}x_t$$

for all t, with

$$K_{N-1} \stackrel{\Delta}{=} (G^TP_{N-1}G+R)^{-1}G^TP_{N-1}F \qquad (2.8)$$

Whereas infinite horizon LQ optimal control gives closed loop stability by construction (under the very weak conditions spelled out in Proposition 2.1), nothing in the construction of the receding horizon control law is aimed at guaranteeing closed loop stability, i.e. $(F - GK_{N-1})$ having all its eigenvalues strictly inside the unit circle. An interesting question, therefore, is : under what condition is $F - GK_{N-1}$ exponentially stable for some fixed horizon N, or for all $N\geq 0$, or for all N larger than some N_0 ?

One way to study the stability of $F - GK_{N-1}$, is to examine whether K_{N-1} can be viewed as the steady-state gain of a related infinite horizon LQ problem for which Proposition 2.1 provides a stability result. This is achieved by rewriting the RDE (2.3) as a Fake ARE (FARE) :

$$P_j = F^TP_jF - F^TP_jG(G^TP_jG+R)^{-1}G^TP_jF + \bar{Q}_j \qquad (2.9)$$

This is not so much a rewriting of the RDE as a definition for $\bar{Q}_j$

$$\bar{Q}_j \stackrel{\Delta}{=} Q - (P_{j+1} - P_j) \qquad (2.10)$$

Clearly we have, from Proposition 2.1 and (2.9),

Theorem 2.1
Under Assumption A, $(F - GK_{N-1})$ is exponentially stable if $\bar{Q}_{N-1}\geq 0$ and $(F,\bar{Q}_{N-1}^{1/2})$ has no unobservable mode on the unit circle.

As is apparent from (2.10), satisfaction of the two conditions on $\bar{Q}_{N-1}$ will essentially depend on whether $P_N \leq P_{N-1}$. We therefore turn now to our monotonicity results.

3. Monotonicity results

We consider the RDE (2.3) and we briefly recall some important monotonicity results of its solution sequence $\{P_j\}$, initially derived in Bitmead at al. (1985) and Poubelle et al. (1988).

Proposition 3.1
Consider the RDE (2.3). If $P_{j+1} \leq P_j$ for some j, then $P_{j+k+1} \geq P_{j+k}$ for all $k\geq 0$.

Proof : see Bitmead et al.(1985) and Poubelle et al.(1988).

Proposition 3.2
Consider the RDE (2.3). The sequence P_j is monotonically nonincreasing (resp. monotonically nondecreasing) if and only if $\bar{Q}_0 \geq Q$ (resp. $\bar{Q}_0 \leq Q$).

Proof : see Poubelle et al. (1988).

The proof of these results in discrete-time turned out to be very difficult, as it relies on finding a usable expression for $P_{j+1} - P_j$. In continuous-time, however, the proof is extremely simple and, given that these monotonicity results are of theoretical interest in their own right, we take a brief respite from our discrete time developments to present the continuous-time monotonicity results with their proof.

The continuous-time equivalent of (2.3) is the following Riccati Differential Equation (RDE) :

$$\dot{P}_t = F^TP_t + P_tF - P_tGR^{-1}G^TP_t + Q \qquad (3.1)$$

The closed loop state transition matrix is

$$A_t = F - GR^{-1}G^TP_t \qquad (3.2)$$

Differentiating (3.1) w.r.t. t yields

$$\ddot{P}_t = A_t^T\dot{P}_t + \dot{P}_tA_t \qquad (3.3)$$

Thus $\dot{P}_t$ satisfies a Lyapunov equation, whose solution can be written as

$$\dot{P}_t = \Phi_{t,t_0}\dot{P}_{t_0}\Phi_{t,t_0}^T \quad \text{for } t\geq t_0 \qquad (3.4)$$

where Φ_{t,t_0} is the fundamental matrix of A_t.
The FARE is defined as

$$F^TP_t + P_tF - P_tGR^{-1}G^TP_t + \bar{Q}_t = 0 \qquad (3.5)$$

with

$$\bar{Q}_t = Q - \dot{P}_t \qquad (3.6)$$

The following result from Poubelle et al.(1988) immediately follows from (3.4) and (3.6).

Proposition 3.3
1. Consider the RDE (3.1). If for some t_0, $\dot{P}_{t_0}\geq 0$ (resp. $\dot{P}_{t_0}\leq 0$) then $\dot{P}_t\geq 0$ (resp. $\dot{P}_t \leq 0$) for all $t\geq t_0$.
2. The solution $\{P_t\}$ of (3.1) is monotonically nonincreasing (resp. monotonically nondecreasing) if and only if $\bar{Q}_0 \geq Q$ (resp. $\bar{Q}_0 \leq Q$).

4. Stability analysis of Receding Horizon LQ

With the monotonicity results under our wings we can now return to the stability analysis of receding horizon LQ strategies. Collecting together (2.10), Theorem (2.1) and Proposition 3.1, we have the following result.

Theorem 4.1
Under Assumption A, $(F - GK_{N-1})$ has all its eigenvalues strictly inside the unit circle if any one of the following conditions hold :
i) $[F,Q^{1/2}]$ has no unobservable mode on the unit circle and for some j, $0\leq j\leq N-1$, $P_{j+1}\leq P_j$
or
ii) $P_N<P_{N-1}$

Comments

1) The conditions given here are sufficient conditions for closed loop stability; they are not necessary. De Souza (1988) has shown that monotonically increasing P_j can also lead to stabilizing closed loop transition matrices provided the second difference $P_{j+1} - 2P_j + P_{j-1}$ is negative definite.

2) We notice that the mechanism for proving closed loop stability of a receding horizon controller is to utilise the closed loop stability properties of a related infinite horizon controller. We have indeed replaced the designer's choice of Q by a larger $\bar{Q}_{N-1} = Q + P_{N-1} - P_N$ and applied the closed loop stability result of Proposition 2.1 to the ARE (2.9) (with j=N-1).

In the sequel we shall assume that $(F, Q^{1/2})$ has no unobservable mode on the unit circle.
This is always possible since Q is a design choice : it suffices to choose Q>0. We shall therefore now discuss ways to achieve $P_{j+1} \le P_j$ for some j.

5. How to enforce monotonic decreasing Pj ?

We first note from Proposition 3.2 that it is possible to force P_j to be monotonically nondecreasing from j = 0. It suffices to choose P_0 such that

$$\bar{Q}_0 = P_0 - F^T P_0 F + F^T P_0 G (G^T P_0 G + R)^{-1} G^T P_0 F \ge Q \quad (5.1)$$

One way to achieve this is by trial and error : choose P_0 and compute $\bar{Q}_0$; check whether (5.1) is satisfied.
Clearly P_0 must be larger than P_∞ (by Proposition 3.1) and therefore one might be tempted to think that cloosing $P_0 = \alpha I$ with α large enough would yield the desired monotonicity; however, this does not always work, as shown in Bitmead et al.(1985). A more systematic procedure is to iterate n steps (n is the order of the system) of the RDE for $P_j^{-1} \triangleq W_j$ with initial condition $W_0 = 0$.

More precisely, one iterates a RDE for $W^*_{j+1} = W_j + GR^{-1}G^T$. Straightforward calculations show that W^*_j obeys the following RDE

$$W^*_{j+1} = F^{-1}W^*_j F^{-T} - F^{-1}W^*_j F^{-T} Q^{1/2}$$
$$[I + Q^{1/2}F^{-1}W^*_j F^{-T}Q^{1/2}]^{-1} Q^{1/2}F^{-1}W^*_j F^{-T} + GR^{-1}G^T \quad (5.2)$$

The closed loop matrix can then be written

$$\bar{F}_j = F - G(G^T P_j G + R)^{-1} G^T P_j F \quad (5.3a)$$
$$= F - GR^{-1} W^{*-1}_{j+1} F \quad (5.3b)$$

We then have the following result, initially derived by Kwon and Pearson (1978).

Proposition 5.1

Consider the system (2.1) with F invertible, Q≥0, R>O and [F,G] controllable, and let W_j^* be the solution of the RDE (5.2) with $W^*_0 = 0$. Then for all j≥n, W^*_{j+1} is nonsingular and $\bar{F}_j$ defined by (5.3b) has all its eigenvalues strictly inside the unit circle.

We see that the choice of initial condition $W^*_0 = 0$ may be interpreted as the effective method for choosing P_0 as infinity in order to achieve guaranteed monotonic decrease. Since W^*_j is monotonically nondecreasing, for all j≥n

$$P_j = (W^*_{j+1} - GR^{-1}G^T)^{-1}$$

is monotonically nonincreasing.
Therefore choosing

$$P_0 = (W^*_{n+1} - GR^{-1}G^T)^{-1}$$

is a proper way of guaranteeing monotonic non-increasing P_j for all j≥0.

6. Assessement of predictive control methods

Predictive control methods, which have been given various names by various authors, are linear quadratic optimal control methods formulated in terms of inputs and outputs only. For finite prediction horizons, their solution can be obtained by inverting a matrix made up of impulse response elements without solving a RDE. We refer the reader to Clarke, Mohtadi and Tuffs (1987) for a prototype of these methods.

Here we shall only present the simplest form of the predictive control criterion to illustrate the closed loop stability problem. The cost function (2.2) is replaced by

$$J(N,N_u) = \sum_{i=0}^{N} y^T_{t+j} y_{t+j} + \sum_{i=0}^{N_u} \lambda u^T_{t+j} u_{t+j} \quad (6.1)$$

subject to $u_{t+j} = 0$, $j = N_{u+1},\dots,N$, where N is the prediction horizon, N_u is the control horizon, and $\lambda>0$ is a scalar weighting on the control.

A comparison between (6.1) and (2.2) immediately shows that for $N_u = N$ (6.1) is a special case of (2.2) with the following choices for Q, R and P_0 :

$$Q = H^T H, \; P_0 = H^T H, \; R = \lambda I \quad (6.2)$$

We first notice that the closed loop stability of the predictive control algorithm will be governed by the eigenvalues of $\bar{F}_j$ in (5.3a) where P_j is the solution of the RDE (2.3) with the substitutions (6.2). Since $P_0 = H^T H$ is equivalent with $P_{-1} = 0$, it immediately follows that the P_j sequence is monotonically nondecreasing. Hence, the stability result of Theorem 4.1 cannot be used since the monotonicity of $\{P_j\}$ goes the wrong way.

The effect of imposing $u_{t+j} = 0$, $j = N_u + 1, \dots, N$ is to use a time varying cost function in which $R = \lambda I$ for $j = 0, \dots, N_u$ and $R = \infty$ for $j = N_u + 1, \dots, N$. This effectively corresponds with solving the first $N_2 - N_u$ iterations of the RDE (2.3) with $R = \infty$, and the last N_u with $R = \lambda I$.
With $R = \infty$, the RDE becomes a Lyapunov equation

$$P_{j+1} = F^T P_j F + Q \quad (6.3)$$

One might hope that this strategy would, at the end of the first $N_2 - N_u$ Lyapunov iterations, yield a solution

$$P_{N_2-N_u} = \sum_{i=0}^{N_2-N_u} (F^T)^i H^T H F^i \quad (6.4)$$

which would be larger than the P_∞ solution of (2.4) with $R = \lambda I$ and $Q = H^T H$, thereby potentially producing a monotonically decreasing sequence $P_{N_2-N_u+j}$, j≥0, from that point onwards. Even though the hoped for monotonic decrease cannot be guaranteed, we have the following remarkable result :

Theorem 6.1
Denote P^R_∞ the maximal solution of the ARE (2.4) and P^∞_j the solution of the Lyapunov equation (6.3) with the same F and Q. Then, provided

1) $[F, (G^T P^R_\infty G)^{-1} G P^R_\infty F]$ is observable,

2) $[F, Q^{1/2}]$ is stabilizable or $P^\infty_0 \geq 0$,

3) F has no eigenvalues which are reciprocal pairs,
there exists a finite K such that $P^\infty_j \geq P^R_\infty$ for all $j \geq K$.

Proof
See Bitmead et al. (1989).

This Theorem is of theoretical interest in its own right. Although it gives us hopes of achieving closed loop stability through a time-varying strategy, it does not really solve the problem. Indeed, it does not tell us how many steps of the Lyapunov equation are necessary for its solution to exceed P^R_∞ (i.e. how large $N_2 - N_u$ should be). More importantly, it does not guarantee that initializing the RDE (2.3) with such a solution exceeding P^R_∞ will entail monotonic decrease from there on.

This conclusion has led us to advocate replacing the predictive control criterion (6.1) by one involving a penalty $x^T_{t+N} P_0 x_{t+N}$ on the final state with a proper selection of P_0. Closed loop stability can then always be guaranteed. The introduction of a state implies that an observer may have to be introduced and that a RDE (or an ARE) will have to be solved, resulting in a slightly higher computational load than with the predictive control implementation of Clarke et al.(1987), where the regulator coefficients are obtained by inverting a matrix of impulse response elements. These computational issues are further discussed in Bitmead, Gevers, Wertz (1989).

7. Concluding remarks

We have presented sufficient and necessary conditions for monotonicity of the solution sequence of a RDE. Although these monotonicity results were first derived in a Kalman filtering context (see Bitmead et al., 1985), they have now proved to be of use in analyzing the stability of receding horizon LQ control schemes. In fact we have shown that receding horizon LQ can always be made closed loop stable by a proper selection of the penalty P_0 on the final state.

REFERENCES

Bitmead R.R., Gevers M.R., Petersen I.R. and Kaye R.J., "Monotonicity and stabilizability properties of solutions of the Riccati difference equation : propositions, lemmas, theorems, fallacious conjectures and counterexamples. Systems and Control Letters, 5:309-315, 1985.

Bitmead R.R., Gevers M. and Wertz V., "The Thinking man's GPC (Generalized Predictive Controller)", to appear 1989.

Clarke D.W., Mohtadi C. and Tuffs P.S., "Generalized predictive control : Parts i and ii." Automatica, 23:137-160, 1987.

De Souza C.E., Gevers M. and Goodwin G.C., "Riccati equations in optimal filtering of nonstabilizable systems having singular state transition matrices", IEEE Trans. Automatic Control, vol. AC-31, pp. 831-838, Sept. 1986.

Kwon W.H. and Pearson A.E., "On feedback stabilization of time-varying discrete linear systems." IEEE Trans. Automatic Control, vol.AC-23, pp.479-481, 1978.

Poubelle M.A., Bitmead R.R. and Gevers M., "Fake Algebraic Riccati Techniques and Stability", IEEE Trans.Auto Control, vol.33, n°4, pp. 379-381, April 1988.