C41

IFAC Symp. on Adaptive Systems in Control and Signal Processing, Glasgow, 1989, pp. 129-134

C 41

OPTIMAL CONTROL REDESIGN OF GENERALIZED PREDICTIVE CONTROL

Robert R. Bitmead, Department of Systems Engineering, Australian National University, G.P.O. Box 4, Canberra, Australia.
Michel Gevers, Laboratoire d'Automatique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium.
Vincent Wertz, Laboratoire d'Automatique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium.

Abstract. Since Generalized Predictive Control (GPC) is a special case of Receding Horizon Linear Quadratic (LQ) Optimal Control, we study the stability and performance properties of LQ optimal controllers with a view to assessing those of GPC. This leads us to suggest some modifications of GPC in order to guarantee closed loop stability and good predictable performance.

Keywords. Optimal control; predictive control; stability.

1. Introduction

Recently, the application of Predictive Control methods in the adaptive control of industrial processes has risen to prominence with many successful applications being reported and a wealth of algorithm modifications advanced. For simplicity we shall refer to the family of these methods as Generalized Predictive Control (GPC), a term coined by Clarke *et al.* [CMT87]. Several theses exist to explain the acceptability of these methods for process control problems, but the most persuasive of these, for us, is that the GPC approach implements a sophisticated control law (Linear Quadratic Optimal) while preserving a manageable number of design variables whose effects are usually easily tied to the dynamical properties of the closed-loop system. Nevertheless, there is a significant number of cases where GPC does not yield desired dynamical responses and a better understanding of its *modus operandi* is required. Our aim here is to provide both an understanding of the mechanisms involved in GPC and methods for assuring adequate dynamical response.

We study the behaviour of GPC as a non-adaptive control law. In this way we hope to delineate its potential modes of behaviour without the complication of adaptation. In [BGW89], the adaptive version is considered; it rests squarely on many of the properties derived in the nonadaptive case. We give our results without proof; they can all be found in [BGW89].

2. Generalized Predictive Control as Optimal Control

2.1 The GPC Method of Clarke *et al.*

Predictive methods in adaptive control have arisen in various guises coming independently from several quarters. The version which appears to have had the most acceptability is that derived by Clarke, Mohtadi and Tuffs [CMT87] as a generalization of the one-step-ahead optimal Generalized Minimum Variance (GMV) strategy of Clarke and Gawthrop [CG79], which itself was a generalization of the Minimum Variance optimal control strategy of Åström and Wittenmark [AW73]. We take as our reference on GPC a version of the algorithm of [CMT87].

As our primary interest here will be with stability and performance issues which can equally well be advanced with scalar systems as with multiinput-multioutput, we shall for simplicity deal with single input-single output systems. We suppose that a model of the linear plant is given in the following ARMAX form:

$$A(q^{-1})y_t = B(q^{-1})u_{t-1} + C(q^{-1})\xi_t \qquad (1)$$

where u_t, y_t and ξ_t are the plant input signal, output signal and disturbance process and A, B and C are polynomials in the unit delay operator q^{-1}, with A and C monic. We note that [CMT87] assigns integrators to components of the model. These integrators and the C polynomial play a rôle in the derivation of plant output predictors.

A partially constrained quadratic optimal control criterion is posed in terms of the inputs and outputs (and not explicitly the state) of this linear model. The cost function to be minimized is:

$$\begin{aligned} J(N_1, N_2, N_u) &= E\{\sum_{j=N_1}^{N_2} [y_{t+j}]^2 + \sum_{j=1}^{N_u} \lambda_j [u_{t+j-1}]^2\} \qquad (2) \\ \text{subject to } u_{t+i} &= 0, \quad i = N_u, \ldots, N_2 \end{aligned}$$

where N_1 is the minimum costing horizon, N_2 is the maximum costing horizon, and N_u is the control costing horizon. The positive constants λ_j weight the relative importance of control and output energies. The use of the expectation in (2) is made to indicate that the control values chosen are calculated conditioned on data available up to and including time t. We have, for reasons of simplicity and brevity, chosen a cost function that reflects a regulation objective rather than a tracking objective, since the stability and performance properties, which are the main objects of this paper, are independent of whether the desired output is zero or some nonzero reference signal. The more complete tracking problem is treated in [BGW89].

We denote the predictions of y_{t+j} from data available at time t, assuming future controls (including u_t) are zero, by $\hat{y}_{t+j|t}$. The vector $\mathbf{f}$ is then composed of these 'free response' predictions:

$$\mathbf{f} = [\hat{y}_{t+1|t}, \hat{y}_{t+2|t}, \ldots, \hat{y}_{t+N_2|t}]^T. \qquad (3)$$

Next define the vector of future control inputs $\tilde{\mathbf{u}}$:

$$\tilde{\mathbf{u}} = [u_t, u_{t+1}, \ldots, u_{t+N_u-1}]^T, \qquad (4)$$

and define the vector of predicted controlled plant outputs:

$$\hat{\mathbf{y}} = [\hat{y}_{t+1}, \hat{y}_{t+2}, \ldots, \hat{y}_{t+N_2}]^T. \qquad (5)$$

That is, the values $\hat{y}_{t+j}$ are the predicted values of the plant output taking into account both the free response predicted output $\tilde{y}_{t+j|t}$ and the control effects after time t. The predicted input-output relationship of the plant can then be written as:

$$\hat{\mathbf{y}} = \mathbf{G}\tilde{\mathbf{u}} + \mathbf{f}, \tag{6}$$

where the $N_2 \times N_u$ matrix $\mathbf{G}$ is composed of the impulse response parameters, g_j, of the plant model,

$$\mathbf{G} = \begin{pmatrix} g_0 & 0 & \dots & 0 \\ g_1 & g_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{N_u-1} & g_{N_u-2} & \dots & g_0 \\ \vdots & \vdots & & \vdots \\ g_{N_2-1} & g_{N_2-2} & \dots & g_{N_2-N_u} \end{pmatrix}. \tag{7}$$

(We have taken N_1 equal to one for simplicity in (7). The effect of altering N_1 is to delete rows from the top of G.) The quadratic minimization of (2) yields:

$$\tilde{\mathbf{u}} = -(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{I})^{-1}\mathbf{G}^T\mathbf{f}. \tag{8}$$

Equation (8) provides the future control input signal for times t to $t + N_u - 1$ as an open-loop strategy based upon information available at time t. The mechanism utilised for closing the loop and forcing a feedback control in GPC is to implement only the first element of $\tilde{\mathbf{u}}$, i.e. u_t. This procedure is known as *Receding Horizon Control*. As a result, the control gain calculated in (8) remains fixed and only the vector $\mathbf{f}$ is updated at every time.

The computation of the elements, g_j, of $\mathbf{G}$ and of the filters producing the elements $\tilde{y}_{t+j|t}$ appearing in $\mathbf{f}$ require the solution of $2N_2$ Diophantine equations, which may be solved recursively in j, the prediction horizon. We refer the reader to [CMT87] for the details of these calculations.

2.2 Linear Quadratic Optimal Control

We now briefly present Linear Quadratic (LQ) controller design, our aim being to construct linkages between the GPC method and the LQ method, so that results from LQ theory can be brought to bear on possible redesigns of GPC. LQ control is formulated in terms of state-space models rather than simpler input-output models. Consequently, we consider a linear state-space model,

$$\begin{aligned} x_{t+1} &= Fx_t + Gu_t & (9) \\ y_t &= Hx_t. & (10) \end{aligned}$$

Again, we shall consider the LQ regulator problem only, because the stability arguments to be advanced in the next section are dependent only upon the regulator problem characteristics. The tracking problem is easily dealt with by state augmentation and we refer the reader to [BGW89].

The quadratic cost criterion which we seek to optimize in the LQ regulator problem is given by

$$J(N, x_t) = x_{t+N}^T P_0 x_{t+N} + \sum_{j=0}^{N-1} \left\{ x_{t+j}^T Q_{N-j-1} x_{t+j} + u_{t+j}^T R_{N-j-1} u_{t+j} \right\}. \tag{11}$$

This is a finite horizon optimal control problem. We note that we have not invoked stochastic uncertainties here, although this can be easily incorporated.

The solution of the LQ optimal regulator problem may be given directly in closed-loop form as follows. One iterates the Riccati Difference Equation (RDE),

$$P_{j+1} = F^T P_j F - F^T P_j G (G^T P_j G + R_j)^{-1} G^T P_j F + Q_j \tag{12}$$

from the initial condition P_0 and implements the control sequence given by,

$$\begin{aligned} u_{t+N-j} &= -(G^T P_{j-1} G + R_{j-1})^{-1} G^T P_{j-1} F x_{t+N-j} \quad (13) \\ & j = 1, \dots, N \end{aligned}$$

The optimal cost, $J(N, x_t)^*$, is given as follows by P_N,

$$J(N, x_t)^* = x_t^T P_N x_t. \tag{14}$$

A special case of (11) is when Q and R are constant matrices and N is allowed to approach infinity. If such a stationary infinite horizon LQ problem is formulated then the solution above alters in that, subject to P_0 being nonnegative definite and $[F, G]$ being stabilizable, P_j converges to a constant matrix P_∞, as j goes to infinity, which is the maximal solution of the Algebraic Riccati Equation (ARE),

$$P_\infty = F^T P_\infty F - F^T P_\infty G (G^T P_\infty G + R)^{-1} G^T P_\infty F + Q. \tag{15}$$

A time invariant control law is therefore defined,

$$u_{t+j} = -(G^T P_\infty G + R)^{-1} G^T P_\infty F x_{t+j}. \tag{16}$$

To implement these control laws requires that the state x_t be available at time t for construction of the control signal. As the state is not always perfectly measurable, an alternative control law is used,

$$\begin{aligned} u_{t+N-j} &= -(G^T P_{j-1} G + R_{j-1})^{-1} G^T P_{j-1} F \hat{x}_{t+N-j} & (17) \\ u_{t+N-j} &= -(G^T P_\infty G + R)^{-1} G^T P_\infty F \hat{x}_{t+N-j}, & (18) \end{aligned}$$

where $\hat{x}_t$ is a state estimate produced by an observer.

With a strictly proper plant it is possible to use an observer with direct feedthrough without encountering algebraic loop problems. It has the form,

$$\hat{x}_{t+1} = (F - MHF)\,\hat{x}_t + (G - MHG)\,u_t + My_{t+1} \tag{19}$$

where the eigenvalues of $F - MHF$ may be arbitrarily placed by choice of M provided $[F, HF]$ is observable.

Just as the GPC design is nominally formulated in a stochastic framework, it is possible to formulate both the LQ optimal control problem and the observer construction in such a framework. The direct feedthrough observer (19) is then the analogue of the true Kalman filter while the more usual observer with a delay is analogous to the Kalman one-step ahead predictor.

The solution of the LQ control problem, in the case where x_t is not fully observed, therefore reduces to the design of a control gain matrix sequence of (17), via the solution of the Riccati difference equation (12), coupled with the design of a state estimator gain matrix M.

2.3 GPC interpreted as LQ Control

It is immediately apparent from the comparison between the criteria (2) and (11) that the GPC problem formulation should fit within the framework of the LQ problem. Indeed, the finite horizon control signal $\tilde{\mathbf{u}}$ of GPC with constant λ is the same as that of LQ formed by taking $N_2 = N$ and,

$$\begin{aligned} Q_j &= \begin{cases} H^T H & \text{if } j = 0, \dots, N_2 - N_1 \\ 0 & \text{if } j = N_2 - N_1 + 1, \dots, N_2 - 1 \end{cases} & (20) \\ R_j &= \begin{cases} \infty I & \text{if } j = 0, \dots, N_2 - N_u - 1 \\ \lambda I & \text{if } j = N_2 - N_u, \dots, N_2 - 1 \end{cases} & (21) \\ P_0 &= H^T H. & (22) \end{aligned}$$

The optimization problems (2) and (11) with these substitutions are identical, and therefore $\tilde{\mathbf{u}}$ of (8) and $\{ u_{t+j}, j = 0, \ldots, N_u\}$ are identical, even though the former is computed in open-loop and the latter in closed-loop.

Now recall that in the GPC control law only the control u_t is applied at time t and the same control gain is used at the next instant. In terms of the LQ problem it is then clear that the GPC control signal is given by,

$$u_t^{GPC} = -(G^T P_{N-1} G + \lambda I)^{-1} G^T P_{N-1} F \hat{x}_t, \qquad (23)$$

where P_{N-1} is derived as the solution of the RDE (12) with variables as specified above in (20)-(22).

To complete the establishment of GPC as a subset of LQ control, we remark that the rôle of the predictions $\mathbf{f}$ in GPC is entirely analogous to the rôle of the state estimate $\hat{x}_t$ in the LQ formulation, and the eigenvalues of $F - MHF$ in the observer equation (19) are the same as the zeros of the polynomial C used to construct the predictors. In particular, if $C(q^{-1})$ is chosen to be unity then the equivalent observer will be deadbeat.

3. Stability and Performance Properties of Receding Horizon LQ Control

Having established that the GPC control law can be obtained as the solution of a particular LQ problem, we now study some of the stability and performance issues connected with LQ control systems, which will be followed by an analysis of how GPC satisfies or fails to satisfy the requirements for adequate behaviour.

3.1 Stability of the Infinite Horizon LQ Controller

The fundamental LQ asymptotic stability result derives from the stationary infinite horizon regulator problem and the properties of the solution of the Algebraic Riccati Equation (15) (ARE). We have [dSGG86],

Theorem 1 *Consider the ARE*

$$P = F^T P F - F^T P G (G^T P G + R)^{-1} G^T P F + Q \qquad (24)$$

where $[F, G]$ is stabilizable, $[F, Q^{1/2}]$ has no unobservable modes on the unit circle, $Q \geq 0$ and $R \geq 0$.
Then there exists a unique, maximal, nonnegative definite symmetric solution $\bar{P}$, which is stabilizing, i.e. $F - G(G^T \bar{P} G + R)^{-1} G^T \bar{P} F$ has all its eigenvalues strictly within the unit circle.

The hopes of receding horizon strategists rest firmly on the following convergence result,

Theorem 2 *Consider the ARE (24) above and its stabilizing solution $\bar{P}$, and consider the RDE*

$$P_{j+1} = F^T P_j F - F^T P_j G (G^T P_j G + R)^{-1} G^T P_j F + Q. \qquad (25)$$

Then, provided $[F, G]$ is stabilizable, $[F, Q^{1/2}]$ is detectable and $P_0 \geq 0$, $P_j \to \bar{P}$ as $j \to \infty$.

The conventional wisdom of receding horizon has been to invoke the above results to argue that, provided J is taken sufficiently large, $F - G(G^T P_j G + R)^{-1} G^T P_j F$ will have its eigenvalues all within the unit circle, for any $j \geq J$. The issue however has always been: how big a value of J needs to be taken and how can this be affected by choice of P_0.

Following Mademoiselle Poubelle [PBG88], we shall rewrite the RDE (25) as a Fake ARE (FARE), and use the ARE stability Theorem 1 to establish stability of receding horizon LQ controllers via monotonicity of the solutions of the RDE. We begin by rewriting the RDE as an ARE,

$$P_j = F^T P_j F - F^T P_j G (G^T P_j G + R)^{-1} G^T P_j F + \tilde{Q}_j. \qquad (26)$$

Notice that this is not so much a rewriting of the RDE as a definition for $\tilde{Q}_j$:

$$\tilde{Q}_j = Q - (P_{j+1} - P_j). \qquad (27)$$

Clearly, we have from Theorem 1 and (26),

Theorem 3 *Consider the FARE (26) defining the matrix $\tilde{Q}_j$. If $\tilde{Q}_j \geq 0$, $[F, G]$ is stabilizable, $[F, \tilde{Q}_j^{1/2}]$ is detectable, then P_j is stabilizing, i.e.*

$$\bar{F}_j = F - G(G^T P_j G + R)^{-1} G^T P_j F \qquad (28)$$

has all its eigenvalues within the unit circle.

This theorem forms the centrepiece of our developments to translate stability properties from the ARE arena to the RDE.

3.2 Stability of Finite Horizon LQ via Monotonicity

We first establish some monotonicity properties of the solution sequence, $\{P_j\}$, of the RDE (25). They are an amalgam of results from [BGPK85], [PBG88].

Theorem 4 *If $P_{j+1} \leq P_j$ for some j, then $P_{j+k+1} \leq P_{j+k}$ for all $k \geq 0$.*
If $P_{j+1} \geq P_j$ for some j, then $P_{j+k+1} \geq P_{j+k}$ for all $k \geq 0$.

With these monotonicity results under our wings, we now turn back to their implications for the stability of closed loop systems derived from the solutions of the RDE.

Theorem 5 *Consider the RDE (25). If $[F, G]$ is stabilizable, $[F, Q^{1/2}]$ is detectable, $P_{j+1} \leq P_j$ for some j then $\bar{F}_k$ given by (28) with P_k is stable for all $k \geq j$.*

This result will be the key to our stability arguments. Later we shall seek initial conditions which cause the P_j sequence to be monotonically nonincreasing *ab initio*, thereby yielding stability of $\bar{F}_j$ for all j.

3.3 Stabilizing Feedback Strategies

The questions naturally arising from the monotonicity results of the previous section focus on how one could (or should) choose initial conditions, P_0, for the RDE in order to achieve a monotonic nonincreasing sequence of P_j and, thereby, closed loop stability whatever the choice of finite horizon for the receding horizon LQ strategy.

It is apparent from Theorems 2 and 4, that any attempt to achieve stability properties of the solution P_j of the RDE via monotonic nonincreasing behaviour needs to commence with P_0 greater than $\bar{P}$. Indeed, one might be tempted to suggest that arbitrarily large P_0 could always yield the desired stability. This need not be true, as has been studied in [BGPK85]. However, there does exist an intelligent choice for large P_0; it involves the choice of effectively infinite P_0 as implemented by specifying

$$P_0^{-1} = W_0 = 0, \qquad (29)$$

and then iterating an alternative form of the RDE for $P_j^{-1} = W_j$, or more precisely for $W_{j+1}^* = W_j + GR^{-1}G^T$. Straightforward calculations show that W_j^* obeys the following RDE

$$\begin{aligned} W_{j+1}^* = {} & F^{-1} W_j^* F^{-T} - F^{-1} W_j^* F^{-T} Q^{1/2} \\ & \times [I + Q^{1/2} F^{-1} W_j^* F^{-T} Q^{1/2}]^{-1} Q^{1/2} F^{-1} W_j^* F^{-T} \\ & + GR^{-1}G^T. \end{aligned} \qquad (30)$$

The closed loop matrix $\bar{F}_j$ of (28) may be written as,

$$\begin{aligned} \bar{F}_j &= F - G(G^T P_j G + R)^{-1} G^T P_j F \\ &= F - GR^{-1} W_{j+1}^{*}{}^{-1} F. \end{aligned} \quad (31)$$

We shall now see that taking zero initial conditions for the W_j^* equations entails monotonic nonincreasing P_j for $j \geq n$, and hence stability. This result was initially derived by Kwon and Pearson [KP78].

Theorem 6 *Consider the state-variable system above of dimension n and assume that $R > 0$, $Q \geq 0$, F is invertible and $[F, G]$ is controllable. Further consider W_j^* the solution of the RDE (30) with initial condition $W_0^* = 0$. Then,*

$$\begin{aligned} K_j &= -R^{-1} G^T W_{j+1}^{*}{}^{-1} F & (32) \\ &= -(R + G^T W_j^{-1} G)^{-1} G^T W_j^{-1} F & (33) \end{aligned}$$

yields $F + GK_j$ with all its eigenvalues strictly inside the unit circle for all $j \geq n$.

We see that this feedback strategy identifies a mechanism for initializing a specific finite-horizon LQ optimal control problem in such a fashion as to ensure asymptotic stability if implemented as a receding horizon law. The choice of initial condition $W_0^* = 0$ may be interpreted as the effective method for choosing P_0 as infinity in order to achieve guaranteed monotonic decrease.

3.4 Time-varying strategies with receding horizon

Section 2.3 has indicated that GPC can implement an LQ control law which does not have fixed Q and R matrices over the whole horizon. In particular, the ploy of using several steps with $R = \infty$ is frequently used in GPC by selecting N_u appropriately. We shall study the possible effects on stability of combining a number of initial steps of the RDE with infinite R value followed by steps of the finite R RDE. When $R = \infty$, the RDE (25), becomes a Lyapunov equation,

$$P_{j+1} = F^T P_j F + Q, \quad (34)$$

possessing the obvious solution,

$$P_j = (F^T)^j P_0 F^j + \sum_{i=0}^{j-1} (F^T)^i Q F^i. \quad (35)$$

For stable F the solution P_j above converges as $j \to \infty$. For unstable F, P_j does not converge.

We have some elementary results on the solutions of such infinite R RDE's.

Theorem 7 *Denote the solution of the RDE with infinite R and initial condition P_0^∞ by P_j^∞ and denote the solution with finite R by P_j^R. Then if $P_0^\infty \geq P_0^R$ we have,*

$$P_j^\infty \geq P_j^R.$$

A simple corollary of the above theorem is that, if P_∞^∞ exists (which it will iff F is stable), then $P_\infty^\infty \geq P_\infty^R$. Thus one might envisage that the choice of several steps of $R = \infty$ might yield a P_j value exceeding P_∞^R and thereby the possibility of producing a monotonically decreasing sequence of P_{j+k} from that point onwards. Indeed, we have the following remarkable result,

Theorem 8 *With the same notation as above, provided*

- *$[F, (G^T P_\infty^R G + R)^{-1} G^T P_\infty^R F]$ is observable, and*
- *$[F, Q^{1/2}]$ is stabilizable, or $P_0^\infty \geq 0$,*
- *F has no eigenvalues which are reciprocal pairs,*

there exists a K such that $P_k^\infty \geq P_\infty^R$ for all $k > K$.

Although the above theorem certainly gives us hopes of achieving closed loop stability through a time varying strategy, it does not quite solve the problem. Indeed, it does not tell us how many steps of the $R = \infty$ equation are necessary for its solution to exceed P_∞^R and, more importantly, it does not guarantee that initializing the finite R RDE with such a solution exceeding P_∞^R will produce monotonic decrease and hence stability. Hence stability must still be tested *a posteriori* in any case.

We next move on to identify a further advantageous aspect of LQ problem solutions possessing specifically related weighting matrices. Our aim will be to quantify the achieved performance of these control systems.

3.5 Comparative Performance of LQ schemes

Our aim in this section is to present a result on the achieved performance of LQ control schemes (as opposed to the designed performance) and then to link this with the notions of monotonicity and receding horizon designs.

We begin by defining three successive quadratic performance measures associated with the state equation (9) and with a single infinite horizon quadratic criterion having constant Q and R

$$J(Q, R) = \lim_{N \to \infty} \sum_{j=0}^{N-1} \left\{ x_j^T Q x_j + u_j^T R u_j \right\}. \quad (36)$$

Definition 1 *The optimal control performance J_{opt} is the minimal value of $J(Q, R)$.*

From our earlier theory, this $J(Q, R)$ is achieved by applying the LQ optimal feedback control derived by solving the ARE with matrices F, G, Q, R.

Definition 2 *The designed control performance J_{des} is the optimal value of that LQ criterion associated with a design problem using a possibly different weighting matrix pair $\bar{Q}$, $\bar{R}$, i.e. J_{opt} for $J(\bar{Q}, \bar{R})$.*

Again, our earlier theory says that this J_{des} is computable by solving an ARE with matrices F, G, $\bar{Q}$ and $\bar{R}$.

Definition 3 *The achieved control performance J_{ach} is the value of $J(Q, R)$ computed when the control law designed with matrices $\bar{Q}$ and $\bar{R}$ is applied to the state-variable system.*

Thus J_{ach} is the achieved performance, as measured by the $J(Q, R)$ criterion, of an optimal controller designed with differing weighting matrices $\bar{Q}$ and $\bar{R}$.

We then have the following result, which is the dual of a result of Nishimura [Nis67] in the case of optimal filtering.

Theorem 9 *With the above definitions of performance measures, we have*

$$\bar{Q} \geq Q, \ \bar{R} \geq R \quad \text{implies} \quad J_{des} \geq J_{ach} \geq J_{opt}.$$

A complementary result involving $\bar{Q} \leq Q$ is not possible because of a lack of ordering between J_{des} and J_{ach}.

The consequence of Theorem 9 is that, in terms of the FARE (26), receding horizon strategies may be viewed as infinite horizon LQ controllers with Q matrix given by $\bar{Q}_{N-1}$ where N is the horizon. To achieve closed loop performance from the finite horizon control law approximating well that of the infinite horizon law derived with the same Q and R, Theorem 9 dictates that $\bar{Q}$ should exceed

Q. This further reinforces the approach to receding horizon LQ problems which relies on ensuring the monotonic convergence of P_j to the ARE solution from above, since these are associated via (27) with $\bar{Q} \geq Q$. Our thesis is that such controllers will demonstrate closed loop performance consistent with that to be expected from an infinite horizon LQ controller with similar Q and R. If in addition the horizon N is increased, then $P_{N,ach}$ approaches closer to $P_{\infty,opt}$ (or at least its bound does so).

In this section we have arugued that closed loop stability and close approximation of infinite horizon dynamics can both be assured provided an LQ law is designed to yield monotonically nonincreasing P_j matrices. In the next section we return to the explicit GPC problem and combine the LQ formulation of GPC from Section 2 with the design objectives for receding horizon from this section.

4. Reassessment of GPC

The popularity of GPC, particularly in the adaptive context, arises because of its provision of sophisticated control design but with a minimum of design variables, each of which is readily related to desired closed loop performance. Hence the availability of many 'rules of thumb' for the selection of parameters such as λ, N_u etc. The issue arises naturally as to whether the closed loop system achieves the design objective and whether it demonstrates the dependence on parameters implied by these simplified rules. We begin by analysing the stability properties of GPC, and we limit our analysis to the standard case where $\lambda_j = \lambda$ in (2).

4.1 Stability Properties of GPC

Recall that Theorem 5 states that a receding horizon strategy constructed with monotonically nonincreasing P_j will yield an asymptotically stable closed loop. We have the following contrary result for GPC.

Theorem 10 *Consider the solution, P_j, of the RDE (12) associated with the GPC control law having $N_u = N_2$, $N_1 = 1$, ie. $Q_j = H^TH$, $R_j = \lambda I$ and $P_0 = H^TH$. Then, for all $j \geq 0$,*

$$P_{j+1} \geq P_j,$$

i.e. P_j is always monotonically nondecreasing.

The import of this theorem is that GPC with full control horizon N_u cannot produce monotonically decreasing P_j, whatever the value of N_2. This should then make us somewhat suspicious about GPC's ability always to produce stabilizing control laws. We now provide an example which illustrates controller misbehaviour in GPC.

Consider the second order system with transfer function polynomials given by,

$$\begin{aligned} A(q^{-1}) &= 1 - 4q^{-1} + 4q^{-1} = (1-2q^{-1})^2 \\ B(q^{-1}) &= q^{-1} - 1.999q^{-2}. \end{aligned}$$

It has a state-variable description as follows

$$F = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, G = (1 \quad 0), H = (1 \quad .001).$$

This system is unstable, with open loop poles both at -2, and possesses a near pole-zero cancellation, which makes it nearly undetectable.

We select as control cost weighting factor $\lambda = .1$ and compute the solution of the RDE (12) from $P_0 = H^TH$ and with the substitutions $R_j = \lambda I$, $Q_j = H^TH$ for all j. We also compute the closed loop pole positions, λ_{cl}, resulting from the LQ receding horizon control law. These are listed below for several representative values.

$$\begin{aligned} P_0 &= \begin{pmatrix} 1 & .001 \\ .001 & .000001 \end{pmatrix}, & \lambda_{cl} &= 1.0005, \ 1.9990 \\ P_{20} &= \begin{pmatrix} 4.2799 & .069 \\ .069 & .10477 \end{pmatrix}, & \lambda_{cl} &= .3819, \ 1.9839 \\ P_{24} &= \begin{pmatrix} 10.433 & 10.026 \\ 10.026 & 16.216 \end{pmatrix}, & \lambda_{cl} &= .3819, \ .9161 \end{aligned}$$

It is clear from this example that the convergence speed of the solution of the RDE is very slow from $P_0 = H^TH$ when the pair $[F, H]$ is almost nondetectable. The first stabilizing controller is achieved only after 24 iterations, i.e. $N_2 = 24$.

This example illustrates the monotonically nondecreasing nature of the GPC RDE solution, and it shows the potential difficulty arising from the choice of P_0 as H^TH.

There are several ways to solve this dilemma in utilising a receding horizon LQ controller by, for example, selecting a different Q and/or P_0 matrix etc. But then the computational niceties of GPC evanesce. For comparison, the Kwon-Pearson controller for the above system yields closed loop stability for any $N_2 \geq 2$. Applying (30)-(32) to our example, we get:

$$K_2 = (-.1085 \quad -.7710)$$

$$F + GK_2 = \begin{pmatrix} .392 & -.771 \\ -.25 & .5 \end{pmatrix},$$

yielding closed loop eigenvalues $\lambda_{cl} = .0034, .8881$.

The above results indicate some negative aspects of the GPC algorithm in that the P_j sequence will always be monotonically nondecreasing, which does not augur well either for guaranteeing stability or for assuring performance. The use of time-varying weighting matrices by a choice of $N_u < N_2$ can alleviate some problems by accelerating the initial increase of P_j. The strategy in using $N_u < N_2$ is to allow a fixed number of steps to be taken with infinite R and then to switch to finite R iterations of the RDE, operating under the presumption that this latter phase of the RDE will be initialized from above the steady-state solution and therefore be likely to yield monotonically decreasing P_j. As already explained in Section 3.4, the problem with this strategy, as with GPC in general, is that it is essentially impossible to ascertain how many steps of each kind are necessary to achieve the desired result.

It is therefore no surprise that all except one nontrivial stability result of Clarke *et al.* [CMT87] are for GPC controllers obtained when N_2 goes to infinity. The nontrivial finite horizon stability theorem of Clarke *et al.* is as follows.

Theorem 11 *[MC86] The closed loop under GPC is stable if*

- *$N_1 \geq n$, $N_u = N_2 - N_1 + 1 \geq n$, with n the order of the system, and*
- *$\lambda = \epsilon$ where ϵ is a sufficiently small positive constant.*

We remark that, while the theorem does not rely upon limiting arguments with time indices, it still requires the selection of a 'vanishingly small' constant ϵ.

With the same example as before, we illustrate how small ϵ may have to be. The system has order $n = 2$; we take $N_1 = 2$, $N_2 = 4$, $N_u = 2$ and controller LQ weightings (Q, R) as $(H^TH, \infty I)$, $(H^TH, \infty I)$, $(0, \epsilon I)$, $(0, \epsilon I)$. A computer analysis using Pro-Matlab shows that, in order for closed loop stability to be achieved, it is necessary to take $\epsilon \leq 5.2 \times 10^{-13}$!

In summary it is apparent that the techniques provided for the GPC method to deliver stability and/or closed loop performance are either asymptotic in nature or involve the careful selection of various constants in a way which would be hard to formalize in a general context. The methods of LQ state-variable feedback do allow more flexibility and we now briefly present some of the issues involved in a full LQ design with a view to delineating the respective costs and benefits vis-à-vis GPC for prospective applications.

4.2 Full LQ Design

The attractive feature of GPC is that it yields access to the sophisticated properties of LQ control without necessitating a complete expertise in this subject. This is achieved by precluding some of the design choices or recouching them in terms of process parameters. The drawback is that it can be difficult to guarantee both closed loop stability and performance.

Our thesis is that LQ state-variable feedback design may be formulated in a similar fashion to GPC requiring the same level of expertise. However, LQ designs possess the capability to guarantee both the closed loop stability and performance properties as was developed in Section 3. In addition, the LQ formalism admits considerably more flexibility through the manipulation of observer dynamics, quadratic cost weightings without artificial constraint, and frequency weighting if desired. Of course, the price to pay for this added flexibility is in the cost of computation, but it is our belief that there are many instances where the computational constraint is not an active one.

For the application of LQ control, a decision must be made between finite horizon control, based on the selection of Q, R and P_0 and the iteration of the RDE for a number of steps, or infinite horizon control, requiring the solution of the ARE. For the computation of the solution of the RDE, clearly matrix multiplication routines are required and, if the Kwon-Pearson type of control law is implemented, matrix inversion routines will be needed. The dimensions of the matrices involved are determined by the state dimension chosen for the plant model and the prediction horizon incorporated into a tracking criterion. For infinite horizon LQ control, the solution of the ARE is necessary and present methods for performing this are based upon the QR methods for the computation of the eigenvalues and eigenvectors of the 'Hamiltonian' matrix associated with the LQ problem. This matrix has dimension twice that of the plant.

Our conclusion is that for moderately sized problems the numerical overheads of LQ vis-à-vis GPC need not be constraining and, in an adaptive context, are of the same order of magnitude as the parameter estimation computational cost.

The criteria for GPC predictor dynamics and for LQ state estimator dynamics are identical. In computational terms, the design of an observer is straightforward, and the computation of the state estimates involves several matrix additions and multiplications. The GPC case requires the solution of several Diophantine equations and the computation of filtered input and output signals. In rough orders of magnitude terms, this is slightly, but not significantly, more work than for the LQ observer.

5. Conclusion

The verdict of our analysis of GPC versus LQ design in a chiefly nonadaptive framework is that LQ can subsume the GPC approach while providing guarantees of closed loop stability, closed loop performance and predictable behaviour modification in response to alteration of controller design parameters. Further, this extension of GPC need not introduce any additional constants requiring specification in the design if Q is taken as $H^T H$ and R as λI, but can be easily adjusted to allow such finer manipulation of design as necessary. There is a computational cost with LQ as compared with GPC, especially for N_u small, but for moderately large systems, these costs need not constrain the implementation of the control law.

In an adaptive implementation of GPC and LQ controllers, the control law is modified very frequently and thus computational limits must be respected. Even in this case, it is our view that the burden of LQ controller calculation is roughly equivalent to the burden of computing a Least Squares parameter identifier, and so is not likely to affect achievable sampling rates by orders of magnitude. Indeed, appealing to an apocryphal duality, one might expect that parameter estimation and controller design should generate the same computational load.

In [BGW89], our stability and performance analysis of GPC and LQ control, which we have briefly outlined in this paper, has been further extended to an examination of the robustness issues of GPC and LQ controllers with a view to ascertaining the anticipated behaviour of these control systems in an adaptive implementation. This has led us to suggest a number of simple and theoretically based principles and procedures for the design of a Candidate Robust Adaptive Predictive controller.

References

[AW73] K.J. Astrom and B. Wittenmark. On self-tuning regulators. *Automatica*, 9:185–199, 1973.

[BGPK85] R.R. Bitmead, M. Gevers, I.R. Petersen, and R.J. Kaye. Monotonicity and stabilizability properties of solutions of the Riccati difference equation: propositions, lemmas, theorems, fallacious conjectures and counterexamples. *Systems and Control Letters*, 5:309–315, 1985.

[BGW89] R.R. Bitmead, M. Gevers, and V. Wertz. *The Thinking Man's GPC.* Snodfart Press, Palo Alto, CA USA, 1989.

[CG79] D.W. Clarke and P.J. Gawthrop. Self-tuning control. *Proc. IEE*, 123:633–640, 1979.

[CMT87] D.W. Clarke, C. Mohtadi, and P.S. Tuffs. Generalized predictive control: Parts i and ii. *Automatica*, 23:137–160, 1987.

[dSGG86] C.E. de Souza, M. Gevers, and G.C. Goodwin. Riccati equations in optimal filtering of nonstabilizable systems having singular state transition matrices. *IEEE Trans. Automatic Control*, AC-31:831–838, 1986.

[KP78] W.H. Kwon and A.E. Pearson. On feedback stabilization of time-varying discrete linear systems. *IEEE Trans. Automatic Control*, AC-23:479–481, 1978.

[MC86] C. Mohtadi and D.W. Clarke. Generalized predictive control, lq or pole-placement: a unified approach. *25th IEEE Conf. on Decision and Control, Athens*, 1536–1541, 1986.

[Nis67] T. Nishimura. On the a *priori* information in sequential estimation problems. *IEEE Trans. Automatic Control*, AC-12:123–125, 1967.

[PBG88] M.A. Poubelle, R.R. Bitmead, and M. Gevers. Fake algebraic Riccati equations and stability. *IEEE Trans. Automatic Control*, AC-33:379–381, 1988.