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UNIFICATION OF DISCRETE AND CONTINUOUS TIME STOCHASTIC ADAPTIVE CONTROL ALGORITHMS.

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Abstract . In this paper, we present a formulation of a continuous time stochastic adaptive control algorithm and explore the relationship with classical continuous time deterministic schemes and with discrete time stochastic adaptive control algorithms. We give a complete global stability theory and highlight the technical difficulties which arise in the continuous time stochastic case.

Keywords. Adaptive systems, stochastic systems, identification, model reference control.

INTRODUCTION

In recent years, considerable progress has been made towards the unification of many adaptive control algorithms which have been proposed over the last fifteen years. Particular emphasis has been given to the problem of unifying discrete and continuous time deterministic adaptive control schemes see, for example, Goodwin and Mayne, (1987)) and in unifying discrete deterministic and stochastic schemes (Goodwin and Sin (1984)). However, up to now, continuous time stochastic control algorithms have only been derived with restrictive assumptions, which make them appear very different from their discrete counterparts (see for example Chen (1985), Moore (1986) and Chen and Moore (1987)).

We believe that there is a need for a formulation of a continuous time adaptive stochastic control algorithm which is close to similar formulations in the deterministic or discrete-time cases. Indeed, as pointed out in Moore (1986), such an algorithm would give greater insight into the discrete stochastic case when fast sampling is employed and into the continuous time deterministic case under unbounded, or wide band, noise.

Hence, in this paper, we present a formulation of a continuous-time stochastic adaptive control algorithm which is closely related to the discrete time formulation given for example in Goodwin and Sin (1984). We establish global stability of the algorithm and highlight the technical difficulties which arise in the continuous time case. As in deterministic systems, a central issue is that of the relative degree of the system input-output transfer function. Up to now, the only known "solutions" treat the case of relative degree zero. The direct extension of these results to larger relative degrees seems difficult.

This paper gives a solution to the problem. This solution depends on additional properties of the parameter estimator and these depend upon using a new algorithm structure which differs from that used in discrete time.

The paper is organized as follows. Section 2 presents and discusses the continuous-time stochastic model that will be used. Section 3 discusses the predicted output based on pseudoregressions. The estimation algorithm and some of its properties are presented in section 4. An indirect adaptive control algorithm is presented in section 5 together with a proof of global stability.

THE MODEL

When dealing with continuous time stochastic processes, even the choice of an appropriate model becomes difficult. A commonly used model is the following state space innovations representation

$$dx_t = Ax_t dt + Bu_t dt + Kd\omega_t \qquad (2.1a)$$

$$dy_t = Cx_t dt + d\omega_t \qquad (2.1b)$$

where $w_t \triangleq \int_0^t d\omega_t$ is a Wiener process with incremental covariance $\sigma^2 dt$, i.e. $E[\omega_t^2 - \omega_s^2] = \sigma^2(t-s)$. We shall denote by $\mathcal{F}_t$ the increasing σ-algebra generated by $\{\omega_\tau / 0 \le \tau \le t\}$. This model has been used extensively in the non-adaptive literature (see e.g. Astrom, 1970) but also more recently in the adaptive case (Moore 1986 and Gevers et. al 1987). Notice that y_t is in fact the integral of the output of this model.

One of the aims of this paper is to establish links between continuous-time and discrete-time stochastic adaptive control algorithms. In the latter case, input-output (ARMAX) descriptions have been preferred, and for this reason we shall also use an input-output representation in this paper. This is a compact formal notation for the corresponding state space results as given in Gevers et. al (1987). One way to define an input output representation for (2.1) is to use the integral operator as in Chen (1985) or Chen and Moore (1987). However, the use of an integral operator makes the treatment of continuous time stochastic processes apparently different from the treatment of continuous time deterministic processes. To make the connections more apparent, we will introduce another formal representation which uses the differential operator.

Let $\rho \triangleq \frac{d}{dt}$, $z_t \triangleq \frac{dy_t}{dt}$, $\omega_t \triangleq \frac{dw_t}{dt}$. Then (2.1) can be rewritten as

$$A(\rho)z_t = B(\rho)u_t + C(\rho)\omega_t \qquad (2.2a)$$

where

$$A(\rho) = \rho^n + a_{n-1}\rho^{n-1} + \ldots + a_0 \tag{2.2b}$$

$$B(\rho) = b_m\rho^m + \ldots + b_0 \tag{2.2c}$$

$$C(\rho) = \rho^n + c_{n-1}\rho^{n-1} + \ldots + c_0 \tag{2.2d}$$

n is the order of the process and $n-m\geq 1$ is the relative degree of the input-output transfer function. The model (2.2) contains an abuse of notation, since we "differentiate" a Wiener process. However, this is purely a compact notational for the system described by (2.1). In future, unless otherwise specified, all polynomials will be polynomials in ρ.

Let $E(\rho)$ be any stable monic polynomial

$$E(\rho) \triangleq \rho^n + e_{n-1}\rho^{n-1} + \ldots + e_0 \tag{2.3}$$

Then (2.3a) can be rewritten as

$$Ez_t = (E-A)z_t + Bu_t + C\omega_t, \text{ or} \tag{2.4}$$

$$z_t = \bar{\psi}_t^T\theta_0 + \omega_t \tag{2.5}$$

$$\bar{\psi}_t^T \triangleq [\rho^{n-1}\frac{z_t}{E},\ldots,\frac{z_t}{E},\rho^{n-1}\frac{u_t}{E},\ldots,\frac{u_t}{E},\rho^{n-1}\frac{\omega_t}{E},\ldots,\frac{\omega_t}{E}] \tag{2.6}$$

$$\theta_0^T \triangleq [e_{n-1}-a_{n-1},\ldots,e_0-a_0,\ b_{n-1},\ldots,b_0, c_{n-1}-e_{n-1},\ldots,c_0-e_0] \tag{2.7}$$

Let $\hat{\theta}_t$ be any $\mathcal{F}_t$-measurable estimate of θ_0. Then we define the "prediction" $\hat{z}_t$ of z_t by using pseudo regressions as:

$$\hat{z}_t = \psi_t^T\hat{\theta}_t \tag{2.8}$$

where

$$\psi_t^T \triangleq [\rho^{n-1}\frac{z_t}{E},\ldots,\frac{z_t}{E},\rho^{n-1}\frac{u_t}{E},\ldots,\frac{u_t}{E},\rho^{n-1}(\frac{z_t-\hat{z}_t}{E}),\ldots,\frac{z_t-\hat{z}_t}{E}] \tag{2.9}$$

Finally we define

$$\tilde{\theta}_t \triangleq \hat{\theta}_t - \theta_0\ ;\ \eta_t \triangleq z_t-\hat{z}_t-\omega_t \tag{2.10}$$

η_t can be viewed as the "noise reduced" prediction error. Since there is an instantaneous feedthrough of white noise into the output dy_t (see 2.1b)), obviously the best one can hope for is to prove that η_t goes to zero in some sense, but not $z_t-\hat{z}_t$. It will be useful for future use to establish a relationship between η_t and $\psi_t^T\tilde{\theta}_t$. From (2.5) and (2.8) to (2.10), it follows that

$$\frac{C}{E}\eta_t = -\psi_t^T\tilde{\theta}_t \tag{2.11}$$

<u>Comment 2.1</u>

As pointed out above, all the expressions from (2.2) are shorthand notations that should be interpreted as integral equations – see, for example, Gevers et. al (1987) for the appropriate state space forms.

ESTIMATION ALGORITHM

Motivated by the discrete time algorithm given in Goodwin and Sin (1984), we introduce the following estimation algorithm

$$\rho\hat{\theta}_t = \frac{\psi_t}{r_t}(z_t - \psi_t^T\hat{\theta}_t) \tag{3.1}$$

where r_t is a suitably chosen normalization constant.

In discrete time, r_t is usually defined as the cummulative sum of $\psi_\tau^T\psi_\tau$. This ensures, amongst other things, that $\psi_\tau^T\psi_\tau/r_t$ is bounded. However, if we replace the sum by an integral in continuous time then we loose this boundedness property. We therefore modify the usual definition of r_t as follows:

$$r_t \triangleq \sup_{0\leq\tau\leq t}\psi_\tau^T\psi_\tau + \int_0^t\psi_\tau^T\psi_\tau\,d\tau + c_0\ ;\ c_0>0 \tag{3.2}$$

Note that this re-establishes the boundedness property as required in our subsequent analysis.

Noting that $d\tilde{\theta}_t = d\hat{\theta}_t$ and using the definition of η_t given in (2.10), we can write that $\tilde{\theta}_t$ is the solution of the following stochastic differential equation

$$d\tilde{\theta}_t = \frac{\psi_t}{r_t}\eta_t\,dt + \frac{\psi_t}{r_t}d\omega_t \tag{3.3}$$

We shall next establish properties of the estimator which hold irrespectively of the control law that will be used. (Compare with the deterministic case in Goodwin and Mayne 1987). We shall need the following assumption.

<u>Assumption 1</u>:

$C(s)$ is strictly minimum phase and the filter $E(s)$ is chosen such that

1) $\mathrm{Re}\,\sigma_i(E(s)) \leq -\alpha < 0$, $i=1,\ldots,n$ where $\sigma_i(E)$ are the roots of E.

2) $\frac{C}{E}$ is input strictly passive, i.e. $\exists\,\beta > 0$ such that $\forall\, T > 0, \int_0^T y_\tau u_\tau d_\tau \geq \int_0^T u_\tau^2 d\tau$ where y_t is the output of the filter $\frac{C}{E}$ driven by u_t.

<u>Comment 3.1</u>

Note that the filter E is crucial in satisfying the above passivity condition. In discrete time, E is often implicitely assumed to be q^n. However, the continuous time theory indicates that this choice will be inadequate when fast sampling is employed.

<u>Comment 3.2</u>

Another difficulty in continuous time is that one must show that finite escapes do not occur. This is straightforward in discrete time provided zero divides are avoided. (Caines and Meyn (1985)). However, in continuous time it is necessary to show that the solutions of (3.3) possess moments of arbitrary high order for all finite T. This is done in the following Lemma.

Lemma 3.1: The solutions of (3.3) satisfy

$$E\left\{\sup_{0\le t\le T} |\tilde{\theta}_t|^{2p}\right\} < (K + 3\,E\left\{|\tilde{\theta}_0|^{2p}\right\})e^{KT}$$

for any $p > 0$, where K depends only on p and T.

Proof. The result depends upon showing that the coefficients of (3.3) are linearly bounded. The result then follows from Lemma 3.8, page 140 of Gihman and Skorohod (1979). See Gevers et. al (1987).

This result implies, in particular, that $\tilde{\theta}_t$ is almost surely bounded for all finite t, i.e. the solutions of (3.3), almost surely, have no finite escape time.

We can then establish the following result which gives properties of the parameter estimator:

Theorem 3.1: Under assumption 1, the following properties hold for the model (2.2) and the estimator (3.1),(3.2):

(i) $$\limsup_{t\to\infty} \|\tilde{\theta}_t\| \le K_1 < \infty \quad \text{a.s.} \tag{3.4}$$

(ii) $$\lim_{t\to\infty} \int_0^t \frac{\eta_\tau^2}{r_\tau}dt \le K_2 < \infty \quad \text{a.s.} \tag{3.5}$$

(iii) For all finite Δ, $$\lim_{t\to\infty}\sup_{0\le T\le\Delta} \|\hat{\theta}_{t+T} - \hat{\theta}_t\| = 0 \quad \text{a.s.} \tag{3.6}$$

Proof. See Appendix A.

Comment 3.3:

The above proof uses the result in equation (A.1)

It is instructive to compare expression (A.1) obtained from the Ito rule (Wong, 1971) with the coresponding discrete time result.

The discrete time form of (3.1) is (Goodwin, Ramadge, Caines (1981)):

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \Delta\frac{\psi_{t-1}}{r_{t-1}}(z_t - \psi_{t-1}^T\hat{\theta}_{t-1})$$

Squaring both sides we obtain

$$\tilde{\theta}_t^2 = \tilde{\theta}_{t-1}^2 + \Delta^2\frac{\|\psi_{t-1}\|^2}{r_{t-1}^2}[\eta_t+\bar{\omega}_t]^2 + 2\Delta\frac{(\psi_{t-1}^T\tilde{\theta}_{t-1})}{r_t}[\eta_t+\bar{\omega}_t]$$

where η_t is the deterministic part of the error and $\bar{\omega}_t$ is the discrete time noise. It is argued in Salgado, Middleton, Goodwin (1987) that if an appropriate pre-sample filter is used, then $\bar{\omega}_t$ will have variance of the order of $\frac{\sigma^2}{\Delta}$. Thus

$$\tilde{\theta}_t^2 - \tilde{\theta}_{t-1}^2 = \Delta^2\left\{\frac{\|\psi_{t-1}\|^2}{r_{t-1}^2}\eta_t^2 + \frac{2\|\psi_{t-1}\|^2}{r_{t-1}^2}\eta_t\bar{\omega}_t\right\}$$
$$+ 2\Delta\frac{\psi_{t-1}^T\tilde{\theta}_{t-1})}{r_t}[\eta_t+\bar{\omega}_t] + \frac{\Delta^2\|\psi_{t-1}\|^2}{r_{t-1}^2}\bar{\omega}_t^2 \tag{3.7}$$

Comparing this result with that given in (A.1), we see that, in the limit as $\Delta\to 0$, the first term disappears since it is of order Δ^2, the second term becomes the first term in (3.7) and $\frac{\Delta^2\|\psi_{t-1}\|^2}{r_{t-1}^2}\bar{\omega}_t^2$ becomes $\frac{\|\psi\|^2}{r^2}\sigma^2 dt$ on application of the Ito rule.

We can now establish a continuous time equivalent of the discrete-time stochastic key technical lemma given in Goodwin and Sin (1984).

Lemma 3.2 (Stochastic Key Technical Lemma)

Suppose the estimator is such that (3.5) is satisfied, and the controller is such that the following growth condition is satisfied:

$$\frac{r_t}{t} \le C + \frac{K}{t}\int_0^t \eta_\tau^2 d\tau \quad \text{a.s.} \tag{3.8}$$

where r_t, η_t are defined before and C, K are finite positive constants. Then

(i) $$\limsup_{t\to\infty} \frac{r_t}{t} \le K_3 < \infty \quad \text{a.s.} \tag{3.9}$$

(ii) $$\lim_{t\to\infty} \frac{1}{t}\int_0^t \eta_\tau^2 d\tau = 0 \quad \text{a.s.} \tag{3.10}$$

Proof. (i) From (3.8) and the nonnegativity of η_τ^2 and r_τ it follows that

$$\frac{r_t}{t} \le C + K\int_0^t \frac{\eta_\tau^2}{r_\tau}\frac{r_\tau}{\tau}d\tau$$

The result then follows from the Bellman-Gronwall lemma (see e.g. Desoer and Vidyasager 1975) using (3.5).

(ii) Suppose first that $\lim_{t\to\infty} r_t = \infty$. Then, by the continuous time Kronecker (Gevers et. al 1987):

$$\lim_{t\to\infty}\frac{1}{r_t}\int_0^t \eta_\tau^2 d\tau = 0 \quad \text{a.s.} \tag{3.11}$$

Hence, using (3.9) and (3.11)

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_\tau^2 d\tau = \lim_{t\to\infty}\frac{r_t}{t}\frac{1}{r_t}\int_0^t \eta_\tau^2 d\tau = 0 \quad \text{a.s.} \tag{3.12}$$

Alternatively, if $\lim_{t\to\infty} r_t \le K_4 < \infty$ for some K_4, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_\tau^2 d\tau \le \lim_{t\to\infty}\frac{1}{t}\frac{K_4}{r_t}\int_0^t \eta_\tau^2 d\tau \le \lim_{t\to\infty}\frac{K_4}{t}\int_0^t \frac{\eta_\tau^2}{r_\tau}d\tau = 0 \quad \text{a.s.}$$

Using (3.5).

Q.E.D.

Comment 3.3

Comparing the properties of our estimator with those proved for the continuous-time deterministic algorithms in Goodwin and Mayne (1987), we observe that we have not proved the uniform boundedness of $\frac{\eta_t}{r_t^{1/2}}$ or of $\dot{\theta}$; neither have we proved $\dot{\theta} \in L_2$. The uniform boundedness of $\frac{\eta_t}{r_t^{\frac{1}{2}}}$ is not needed in the subsequent analysis ($\frac{\eta_t}{r_t^{\frac{1}{2}}} \in L_2$ suffices).

As for $\dot{\theta}$, it does not exist in our stochastic framework, but the conditions on $\dot{\theta}$ in Goodwin and Mayne (1987) are replaced by the weaker condition (3.6).

AN INDIRECT ALGORITHM

We now consider a general class of control laws in state feedback form:

$$u_t = -\psi_t^T \beta + y_t^* \tag{4.1}$$

where β denotes a control law parameter vector and where y_t^* denotes a bounded reference signal. This gives a strictly proper controller. In the adaptive case, we replace β by $\hat{\beta}$ which will be evaluated as a suitable function of $\hat{\theta}$. This gives

$$u_t = -\psi_t^T \hat{\beta}_t + y_t^* \tag{4.2}$$

We make the following assumptions about the control law:

Assumptions

C.1 : $\|\hat{\beta}\|$ is uniformly bounded by $\|\hat{\theta}\|$

C.2 : $\hat{\beta}$ has a bounded derivative w.r.t. $\hat{\theta}$

C.3 : Let

$$\hat{\beta} = [\hat{p}_{n-1},\ldots,\hat{p}_0, \hat{\ell}_{n-1},\ldots,\hat{\ell}_0, \hat{h}_{n-1},\ldots,\hat{h}_0]^T$$

and form

$$\hat{L} = \hat{\ell}_{n-1}\,\rho^{n-1} + \ldots + \hat{\ell}_0;\ \hat{Q} = E + \hat{L}$$

$$\hat{P} = \hat{p}_{n-1}\,\rho^{n-1} + \ldots + \hat{p}_0$$

$$\hat{H} = \hat{h}_{n-1}\,\rho^{n-1} + \ldots + \hat{h}_0$$

Then

$\forall t$, $\hat{Q}\,\hat{A} + \hat{P}\hat{B} = \hat{A}^*(t)$ where $\hat{A}^*(t)$ is continuous, has a uniform stability margin, i.e. $\mathrm{Re}\,\lambda_i(\hat{A}^*(t)) \le -\beta < 0$ i=1, ..., 2n and $\lim_{t\to\infty} \sup_{0\le T\le\Delta} \|\hat{A}^*(t+T) - \hat{A}^*(t)\| = 0$ for some $\Delta > 0$.

A special case of the above class of control laws is the adaptive linear quadratic optimal controller structure with an infinite horizon. $\hat{Q}$ and $\hat{P}$ are then computed as follows:

Let

$$\hat{\theta} = [\hat{g}_{n-1},\ldots,\hat{g}_0, \hat{b}_{n-1},\ldots,\hat{b}_0, \hat{f}_{n-1},\ldots,\hat{f}_0] \tag{4.3}$$

and define

$$\hat{a}_i = e_i - \hat{g}_i;\ \hat{c}_i = e_i + \hat{f}_i;\ i=0, \ldots, n-1.$$

$$\hat{A}(\rho) = \rho^n + \hat{a}_{n-1}\rho^{n-1} + \ldots + \hat{a}_0$$

$$\hat{B}(\rho) = \hat{b}_{n-1}\,\rho^{n-1} + \ldots + \hat{b}_0$$

$$\hat{C}(\rho) = \rho^n + \hat{c}_{n-1}\,\rho^{n-1} + \ldots + \hat{c}_0$$

Then $\hat{Q}$, $\hat{P}$ satisfy an equation of the form $\hat{Q}\hat{A} + \hat{P}\hat{B} = \hat{A}^*(t)$ where $\hat{A}^*(t)$ depends upon $\hat{A}$, $\hat{B}$ and $\hat{C}$ and the weights used in the performance index.

In this case, assumptions C.1 to C.3 are satisfied provided (a) $\hat{\theta}_t$ is bounded, and (b) no "near pole-zero cancellations" occur between $(\hat{A}, \hat{B})$ and $(\hat{A}, \hat{C})$.

Comment 4.1

Several strategies have been described elsewhere to avoid the problem of "near pole-zero cancellations" in the above algorithm. For example, one idea presented in Middleton et. al. (1987) for the deterministic case is to consider a set of convex regions in the parameter space in which no pole-zero cancellations occur. A seperate estimator is then run in each of these regions with an orthogonal projection facility to ensure that the estimates remain inside the corresponding convex region. If we assume that the true system lies inside at least one of these regions then it is readily shown that properties (i) to (iii) of Lemma 3.1 will hold for at least one of the estimators. If one then monitors $\int_0^t \frac{\eta_t^2}{r_\tau} d\tau$ and chooses the estimator having the least value of this function (with some hysterisis to avoid swapping between estimators infinitely often), then the essential properties of the parameter estimator are retained together with the additional feature that near pole-zero cancellations are now impossible. The full proof of these facts follows by combining the ideas in Middleton et. al (1988) with the results in Theorem 3.1.

$\nabla\nabla\nabla$

We now show that the regression ψ_t can be written as the state of a linear time-varying system driven by the prediction error e, and the reference signal y^*.

From the model (2.1), the controller (4.2), the definition of ψ_t ((2.9), and the definition of the errors η_t and e_t, we can write:

$$d\psi_t = A_t\,\psi_t dt + B_1\,(\eta_t dt + d\omega_t) + B_2\,y_t^* dt \tag{4.4}$$

where

$$A_t = \left[\begin{array}{cccc|cccc|cccc}
-\hat{a}_{n-1} & \cdots & \cdots & -\hat{a}_0 & 0 & \cdots & \hat{b}_{n-1} \cdots & \hat{b}_0 & \hat{f}_{n-1} & \cdots & \cdots & \hat{f}_0 \\
\vdots & & & 0 & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & I_{n-1} & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
-\hat{p}_{n-1} & \cdots & \cdots & -\hat{p}_0 & -\hat{q}_{n-1} & \cdots & \cdots & -\hat{q}_0 & -\hat{h}_{n-1} & \cdots & \cdots & -\hat{h}_0 \\
\vdots & & & \vdots & \vdots & & & 0 & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & I_{n-1} & & \vdots & \vdots & & & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -c_{n-1} & \cdots & \cdots & -c_0 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & I_{n-1} & & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{array}\right] \tag{4.5a}$$

$$B_1^T = [1 \;\; 0 \;\; \dots \;\; 0 \;\; 1 \;\; 0 \;\; \dots \;\; 0] \tag{4.5b}$$

$$B_2^T = [0 \;\; \dots \;\; 0 \;\; 1 \;\; 0 \;\; \dots \;\; \dots \;\; 0] \tag{4.5c}$$

where B_1^T has 1's in the 1^{st} and $(2n+1)^{st}$ positions and B_2^T has as 1 in the $(n+1)^{st}$ position.

We can then establish that the homogeneous part of (4.4) is exponentially stable.

Lemma 4.1: Consider the differential equation

$$\frac{d}{dt}\psi_t = A_t \psi_t \tag{4.6}$$

with A_t given by (4.5a). Assume that the $\hat{a}_i$, $\hat{b}_i$, $\hat{c}_i$ are estimated using the parameter estimator of Section 3 and that C.1 to C.3 holds. Then (4.6) is exponentially stable. a.s.

Proof. This result follows from Lemma 3 of Krisselmeier (1985) and the structure of A_t given in (4.5) – see Gevers et. al (1987) for details.

Q.E.D.

We then have the main result of this paper.

Theorem 4.1

Consider a system (2.1), the parameter estimator of Section 3 satisfying Assumption 1 and any control law of the form (4.2) satisfying C.1 to C.3. Then, for arbitrary finite initial conditions and an arbitrary, piecewise continuous, uniformly bounded reference input y_t^*, then:

(i) $\limsup \frac{1}{t}\int_0^t \|\psi_\tau\|^2 d\tau < \infty$ a.s.

(ii) $\limsup \frac{1}{t}\int_0^t u_\tau^2 d\tau < \infty$ a.s.

(iii) $\limsup \frac{1}{t}\int_0^t |Cx_\tau|^2 d\tau < \infty$ a.s.

Proof. Outline (see Gevers et. al (1987)).
We first establish a growth condition between y and ψ and using Lemma 4.1, i.e.

$$\frac{r_t}{t} \le K_6 + \frac{K_7}{t}\int_0^t \eta_\tau^2 d\tau \qquad \text{a.s.}$$

We then use the Stochastic Key Technical Lemma (Lemma 3.2) to complete the proof.

Comment 4.2

Note that Cx_t is the 'deterministic part' of the output. Thus, part (iii) of Theorem 4.2 establishes that this part of the output is sample mean square bounded. This is all that can be said about the output since it contains a Wiener process.

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CONCLUSIONS

The paper has analyzed a class of continuous time stochastic adaptive control algorithms and has shown that, under suitable conditions, they will be globally stable. The results for the discrete time–case and differences, where they exist, have been highlighted.

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Appendix A: Proof of Theorem 3.1.

(a) Starting from (3.3) and using Ito's rule (Wong 1971), we get

$$d(\tilde{\theta}_t^T \tilde{\theta}_t) = 2\tilde{\theta}_t^T \frac{\psi_t}{r_t}(\eta_t dt + d\omega_t) + \sigma^2 \frac{\|\psi_t\|^2}{r_t^2} dt \tag{A.1}$$

$$= -\frac{2}{r_t}(-\tilde{\theta}_t^T \psi_t \eta_t - \epsilon\eta_t^2)dt - 2\epsilon\frac{\eta_t^2}{r_t}dt$$

$$+ 2\tilde{\theta}_t^T \frac{\psi_t}{r_t} d\omega_t + \sigma^2 \frac{\|\psi_t\|^2}{r_t^2} dt \tag{A.2}$$

for some $\epsilon > 0$.

Defining $\xi_t \triangleq -\tilde{\theta}_t^T \psi_t \eta_t - \epsilon \eta_t^2$ and integrating (A.2) yields

$$\tilde{\theta}_t^T\tilde{\theta}_t = \tilde{\theta}_s^T\tilde{\theta}_s - 2\int_s^t \frac{\xi_\lambda}{r_\lambda}\, d\lambda - 2\epsilon\int_s^t \frac{\eta_\lambda^2}{r_\lambda}\, d\lambda + 2\int_s^t \tilde{\theta}_\lambda^T \frac{\psi_\lambda}{r_\lambda}\, d\omega_\lambda + \int_s^t \sigma^2 \frac{\|\psi_\lambda\|^2}{r_\lambda^2}\, d\lambda \qquad \text{(A.3)}$$

Consider now the integral $\int_0^t \frac{\|\psi_\lambda\|^2}{r_\lambda^2}\, d\lambda$. We have

$$\int_0^t \frac{\|\psi_\lambda\|^2}{r_\lambda^2}\, d\lambda \le \int_0^t \frac{dr_\lambda}{r_\lambda^2} = \frac{1}{r_0} - \frac{1}{r_\lambda} \le \frac{1}{r_0} \qquad \text{(A.4)}$$

(since $dr_\lambda = \|\psi_\lambda\|^2$ plus a possible positive contribution when $\|\psi_\lambda\|^2 > \sup_{s<\lambda} \|\psi_s\|^2$). We now define

$$X_t \triangleq \tilde{\theta}_t^T\tilde{\theta}_t + 2\int_0^t \frac{\xi_\lambda}{r_\lambda}\, d\lambda + 2\epsilon\int_0^t \frac{\eta_\lambda^2}{r_\lambda}\, d\lambda + \frac{\sigma^2}{r_0} - \int_0^t \sigma^2 \frac{\|\psi_\lambda\|^2}{r_\lambda^2}\, d\lambda \qquad \text{(A.5)}$$

By Lemma 3.1 and the definition of r_t, all terms in X_t have finite expectation. In addition, by (A.4) and Assumption 1 X_t is nonnegative.

Using (A.3) in (A.5) yields

$$X_t = X_s + \int_s^t \frac{\tilde{\theta}_\lambda^T \psi_\lambda}{r_\lambda}\, d\omega_\lambda \qquad \text{(A.6)}$$

Now denote by $\mathcal{F}_s$ the increasing σ–fields generated by ω_s.

Then for $t \ge s$:

$$E[X_t] = E[E[X_t \mid \mathcal{F}_s]] = E[X_s] = X_0 = \|\tilde{\theta}_0\|^2 \qquad \text{(A.7)}$$

since the second term in (A.6) is not $\mathcal{F}_s$– measurable. Therefore $\limsup_{t\to\infty} E[X_t] < \infty$ and in addition, for $t \ge s$:

$$E[X_t \mid \mathcal{F}_s] = X_s \qquad \text{(A.8)}$$

Therefore $(X_t, \mathcal{F}_t)$ is a Martingale and by the Martingale Convergence Theorem (Doob, 1953) it follows that

$$\lim_{t\to\infty} X_t = X < \infty \text{ a.s.} \qquad \text{(A.9)}$$

This proves (i) and (ii)

(b) To prove (iii), we must prove that for all $\epsilon > 0$ and $\Delta > 0$, there exists t_0 such that $\forall\, t \ge t_0$

$$\sup_{0\le T\le\Delta} \|\tilde{\theta}_{t+T} - \tilde{\theta}_t\| < \epsilon \qquad \text{(A.10)}$$

From (3.3) we can write

$$\|\tilde{\theta}_{t+T} - \tilde{\theta}_t\|^2 \le 2\left\{\left\|\int_t^{t+T} \frac{\psi_\tau \eta_\tau}{r_\tau}\, d\tau\right\|^2 + \left\|\int_t^{t+T} \frac{\psi_\tau}{r_\tau}\, d\omega_\tau\right\|^2\right\}$$

$$\le 2T\int_t^{t+T} \frac{\|\psi_\tau\|^2}{r_\tau^2}\, \eta_\tau^2\, d\tau + 2\left\|\int_t^{t+T} \frac{\psi_\tau}{r_\tau}\, d\omega_\tau\right\|^2 \qquad \text{(A.11)}$$

The last inequality is obtained by applying the Schwartz inequality. Consider now the first terms on the RHS of (A.11). From (ii) it follows that, given $\epsilon > 0$, $\Delta > 0$, there exists $t_0(\Delta,\epsilon)$ such that $\forall\, t \ge t_0$

$$\int_t^{t+T} \frac{\eta_\tau^2}{r_\tau}\, d\tau \le \frac{1}{4}\frac{\epsilon^2}{\Delta} \qquad \forall t\ge t_0 \text{ and } \forall T\ge 0$$

Since $\frac{\|\psi_\tau\|^2}{r_\tau} \le 1$ it follows that

$$2T\int_t^{t+T} \frac{\|\psi_\tau\|^2}{r_\tau^2}\, \eta_\tau^2\, d\tau \le \frac{1}{2}\epsilon^2 \qquad \forall\, t \ge t_0 \text{ and } \forall\, T \in [0,\Delta] \qquad \text{(A.12)}$$

Next denote $S_t \triangleq \int_0^t \frac{\psi_\tau}{r_\tau}\, d\omega_\tau$. It is readily shown (Gevers, 1987) that $\|S_t\|^2$ converges to a finite limit in view of (A.4).

Therefore, for given $\epsilon > 0$ and $\Delta > 0$, there exists $t_1(\epsilon,\Delta)$ such that $\forall\, t \ge t_1$,

$$\|S_{t+T} - S_t\|^2 \le \frac{1}{4}\epsilon^2, \quad \forall T\in[0,\Delta] \qquad \text{(A.13)}$$

Combining (A.11), (A.12) and (A.13) yields

$$\|\tilde{\theta}_{t+T} - \tilde{\theta}_t\|^2 \le \epsilon^2 \qquad \forall T\in[0,\Delta] \text{ and } \forall\, t \ge \max\{t_0, t_1\}.$$

This proves (iii).

Q.E.D.