ON ADAPTIVE ESTIMATION AND POLE ASSIGNMENT

OF OVERPARAMETRIZED SYSTEMS

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ABSTRACT

In this paper, a first step is taken to avoid ill−conditioning in adaptive estimation and pole assignment schemes for the case when there is a signal model overparametrization. Such a situation can occur in practice when an unknown model order is guessed too high so as to be on the "safe" side. The methods proposed in the paper are relatively simple compared to on−line order determination, being based on introducing suitable excitation in the "regression" vectors of the parameter estimation algorithms to ensure parameter convergence. For the case when the models are nonunique in that pole−zero cancellations can occur, the algorithms seek to estimate the unique model where the cancellations occur at the origin. Applying estimates of this (unique) model turns out to avoid ill−conditioning in central tendency adaptive pole assignment. For the case of one pole−zero cancellation the convergence theory of the algorithm is complete. For the more general case, algorithms are readily devised which appear to work well but for which a complete theory is not available.

1. INTRODUCTION

In the practice of adaptive estimation and control there is a tendency to overparametrize signal models (plants) to be on the "safe" side. However, for overparametrized models, there is a danger of ill−conditioning of both the adaptive estimation and the adaptive control algorithms applied to such plants. Of course, there is the twin danger of underparametrization, particularly in the absence of appropriate preprocessing of signals. The effects of underparametrization could be catastrophic and since this is widely known, overparametrization emerges as a common problem. This paper shows that (after appropriate preprocessing), certain ill−conditioning associated with overparametrization can be avoided, without the need to perform on−line order determination with its associated significant increase in computational complexity.

For overparametrized signal models, there can be a lack of excitation in regression vectors employed in parameter estimation, and consequent ill−conditioning in the algorithms. Also, insufficient excitation can lead to identification of nonuniquely parametrized models which include pole−zero cancellations in the complex z−plane. When recursive estimates of the parameters of such nonuniquely parametrized models are applied for adaptive control, then ill−conditioning leading to excessive controls can easily rise, particularly in adaptive pole assignment schemes.

Adaptive pole assignment schemes are perhaps the simplest schemes for adaptively stabilizing linear plants which are possibly nonminimum phase [1]. Also, they are the most natural form of adaptive scheme to use in some applications where it is required that the adaptive scheme behave as closely as possible to a nominal optimal design. However, a severe limitation for their application in practice has been their failure when the signal models are overparametrized. Adaptive pole assignment requires the solution of a linear algebraic equation which becomes ill−conditioned when estimates of the plant have near pole−zero cancellation. Such is inevitable when the signal model is overparametrized. Some authors have proposed methods to cope with this difficulty using on−line estimation of plant order in some sense [2]. Such an approach increases the complexity of the adaptive scheme considerably.

The first contribution of this paper, in Section 2, is to introduce excitation signals into the regression vectors for recursive (least squares based) parameter estimation in such a manner as to avoid ill−conditioning even when the model is overparametrized. For the special case when there is a potential nonuniqueness in the signal model due to a pole−zero cancellation on the real axis, it is shown how the excitation can be designed so that the parameter estimates converge to those of a unique signal model, if one exists, otherwise to a model with a pole−zero cancellation at the origin. The introduced excitation does not excite the plant as in the case of added persistence of excitation signals. The estimation result of Section 2 is useful when applied in conjunction with the second contribution of the paper, in Section 3, which shows that when the parameter estimates converge so that the identified plant has a pole−zero cancellation at the origin, the associated central tendency adaptive pole assignment controller converges without ill−conditioning.

Section 4 gives a novel property of Sylvester matrices required in the proof of the theory of Section 3, and Section 5 gives an illustrative simulation study. Conclusions are drawn in Section 6.

2. ALGORITHMS AND RESULTS − WHITE NOISE CASE

Signal Model.

Consider the following single−input, single−output (SISO), input−output (stochastic) signal model class (plant), in terms of the unit delay operator q^{-1}, input u_k, output y_k and white noise disturbances w_k.

$$A(q^{-1})y_k = B(q^{-1})u_k + w_k$$
$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n},$$
$$B(q^{-1}) = b_1 q^{-1} + \ldots + b_m q^{-m} \qquad (2.1)$$

This can be rewritten as

$$y_k = \theta^T \bar{x}_k + w_k , \qquad \theta^T = [a_1 \ a_2 \ldots a_n \ b_1 \ b_2 \ldots b_m]$$
$$\bar{x}^T = [-y_{k-1} \cdots -y_{k-n} \ u_{k-1} \cdots u_{k-m}] \qquad (2.1)'$$

The conditions on w_k are more precisely

The sequence $\{w_k\}$ is independent of u_k with $E\{w_k \mid F_{k-1}\} = 0$, $E\{w_k^2 \mid F_{k-1}\} < \sigma_w^2 < \infty$ where F_k denotes the σ−algebra generated by $w_1, w_2, \ldots w_k$. $\qquad (2.2)$

Recursive Least Squares (RLS) Estimation

Consider that θ is estimated recursively minimising a least squares criterion:

$$\bar{J}_k = \frac{1}{k}[\sum_{i=1}^{k} (y - \theta^T \bar{x}_i)^2 + (\theta - \bar{\theta}_0)^T \bar{B}_0 (\theta - \bar{\theta}_0)]$$

via an RLS scheme as

$$\bar{\theta}_k = \bar{P}_k[\bar{B}_{k-1}\bar{\theta}_{k-1} + \bar{x}_k y_k] , \quad \bar{B}_k = \bar{B}_{k-1} + \bar{x}_k \bar{x}_k^T$$
$$\bar{P}_k = \bar{P}_{k-1} - \bar{P}_{k-1}\bar{x}_k \bar{x}_k^T \bar{P}_{k-1}(1 + \bar{x}_k^T \bar{P}_{k-1}\bar{x}_k)^{-1} = \bar{B}_k^{-1} \qquad (2.3)$$

for some initial conditions $\bar{\theta}_0$, $\bar{B}_0 > 0$.

Convergence Properties Review

To achieve a simple analysis making connection with Kalman filter theory as in [3], let us assume

$\{w_k\}$ is normally distributed, and the a priori probability density associated with θ is $N[\bar{\theta}_0, P_0]$ for some $\bar{P}_0 = \bar{B}_0^{-1} > 0$. $\qquad (2.4)$

This assumption is not needed for a more general theory based on stochastic Lyapunov functions $\tilde{\theta}_k^T \bar{B}_k \tilde{\theta}_k$ in [4,5]. where $\tilde{\theta}_k = \theta - \bar{\theta}_k$, but then the results are not quite as tidy.

For the model (2.1) (2.2), and RLS scheme (2.3) under (2.4), Kalman filter theory tells us that with $\tilde{\theta}_k = \theta - \bar{\theta}_k$,

$$\sigma^2 \bar{P}_k = E[\tilde{\theta}_k \tilde{\theta}_k^T \mid F_{k-1}], \quad \bar{\theta}_k = E[\theta \mid F_{k-1}] \qquad (2.5)$$

Moreover, from [3] there is almost sure convergence as

$$\lim_{k \to \infty} \bar{P}_k = \bar{P}^{LS}, \quad \lim_{k \to \infty} \bar{\theta}_k = \bar{\theta}^{LS} \quad \text{a.s.} \qquad (2.6)$$

for random variables $\bar{P}^{LS}$, $\bar{\theta}^{LS}$. With $\bar{x}_k$ sufficiently exciting in that $\bar{P}^{LS} = 0$, then [3] tells us that $\bar{\theta}^{LS} = \theta$. Also if $\bar{\theta}^{LS} = \theta$, then $\bar{\theta}_k \bar{\theta}_k^T \to 0$ as $k \to \infty$ and consequently under (2.5), $\bar{P}_k \to 0$ as $k \to \infty$. Thus the following strong connection between sufficiency of excitation of $\bar{x}_k$ and parameter convergence.

Lemma 2.1
For the RLS scheme (2.3) applied to the signal model (2.1) (2.2) under (2.4), then

$$\lim_{k \to \infty} \bar{P}_k = 0 \iff \lim_{k \to \infty} \bar{\theta}_k = \theta \qquad (2.7)$$

Proof: As above based on results in [3].

On Sufficient Excitation   In this subsection three specific excitation scenarios are studied using known results from [4,5,6]. These relate excitation of signal model inputs to outputs or states for reachable open−loop time−invariant plants, with or without (possibly time−varying) feedback. The first two cases are a review of known results for the case when there is no overparametrization, while the third case deals with the case of overparametrized models.

Case (i):   The simplest case to study is the nonoverparametrized case when

$$q^n A(q^{-1}), \quad q^m B(q^{-1}) \text{ are coprime,} \qquad (2.8)$$

and u_k is suitably exciting in that

$$\lim_{k \to \infty} (\sum_1^k \bar{u}_k \bar{u}_k^T)^{-1} = 0, \quad \bar{u}_k^T = [u_{k-1} \ u_{k-2} \ \cdots \ u_{k-n-m}] \qquad (2.9)$$

This latter condition is achieved when u_k includes at least $n + m/2$ distinct sinusoids decaying no faster than $1/k$, as when u_k is white or filtered white noise with a variance decaying no faster that $1/k$.

Under (2.8), the model (2.1) is uniquely parametrized, $\bar{x}_k$ is reachable from u_k alone, and excitation of the inputs as in (2.9) implies excitation of the states $\bar{x}_k$ [4,5]. Thus (2.8),(2.9) ensure that $\bar{P}_k \to 0$ as $k \to \infty$, and in turn $\bar{\theta}_k \to \theta$ as $k \to \infty$. For this case then, w_k need not be sufficiently exciting in any sense.

Case (ii):   Another simple case to study is when (2.8) does not necessarily hold, but w_k as well as u_k are suitably exciting in that (2.9) is satisfied and

$$(\sum_1^\infty E[w_k^2 | F_{k-1}])^{-1} = 0 \qquad (2.10)$$

Under (2.9),(2.10), the model (2.1) is uniquely parametrized, $\bar{x}_k$ is reachable from u_k, w_k, and is sufficiently exciting to guarantee that $\bar{P}_k \to 0$ as $k \to \infty$ [4,5] and in turn that $\bar{\theta}_k \to \theta$ as $k \to \infty$. For this case then, the convergence as such is independent of whether or not the coprimeness condition (2.8) is satisfied.

Case (iii):   The possibly overparametrized signal model situation of particular interest in this paper, is when (2.8) possibly fails and there is no a priori guarantee of sufficient excitation of w_k as in (2.10). In this case the model (2.1) may not be uniquely parametrized, having one or more pole−zero cancellations in the z−plane. Also, $\bar{x}_k$ may not be sufficiently exciting to ensure that $\bar{P}_k \to 0$ as $k \to \infty$. Convergence can take place to a signal model with pole−zero cancellation anywhere in the complex z−plane. We seek to avoid such a situation and propose an RLS algorithm with additional excitation in the regression vector. It is derived using an alternative signal model formulation.

Alternative Signal Model Formulation
Consider (2.1) re−organized as

$$y_k = \theta^T x_k + (w_k - \theta^T v_k), \quad x_k = \bar{x}_k + v_k \qquad (2.11)$$

where v_k is an excitation term to ensure that x_k is suitably exciting. Notice that v_k has no influence on y_k, u_k. Its selection in the next subsection is in accordance with a parameter estimation error measure, so that when parameter estimates are converging to their true values, v_k converges to zero.

RLS Estimation with Regression Vector Excitation
Consider that θ is estimated recursively minimising a least squares criterion:

$$J_k = \frac{1}{k}[\sum_{1=1}^k (y_1 - \theta^T x_1)^2 + (\theta - \theta_0)^T B_0(\theta - \theta_0)] \qquad (2.12)$$

via an RLS scheme as

$$\hat{\theta}_k = P_k[B_{k-1}\hat{\theta}_{k-1} + x_k y_k], \quad B_k = B_{k-1} + x_k x_k^T$$

$$P_k = P_{k-1} - P_{k-1}x_k x_k^T P_{k-1}(1 + x_k^T P_{k-1} x_k)^{-1}$$
$$= B_k^{-1} \qquad (2.13)$$

for some initial conditions θ_0, $B_0 > 0$.

Regression Vector Excitation Selection
Consider the signal model (2.1) formulated as (2.11). Let us assume that either there is no overparametrization in that (2.8) holds, or that there is the possibility of overparametrization which includes one pole−zero cancellation, leading to a nonuniquely parametrized model. In the latter case, we consider a unique parametrization with the properties (meaningful only when $n > 1$, $m > 1$)

$$a_n = b_m = 0, \quad q^{n-1}A(q^{-1}), \quad q^{m-1}B(q^{-1}) \text{ are coprime} \qquad (2.14)$$

For such a situation, we propose a v_k selection as follows,

The sequence $\{v_k\}$ is selected as an independent (Gaussian) zero mean white noise excitation term such that its covariance $\Sigma_v^2 = D \ k^{-1} \mathrm{tr}(P_{k-1})$ for $D = \mathrm{diag}[0 \ 0 \ \ldots \ 0 \ d_n^2 \ 0 \ \ldots 0 \ d_{n+m}^2]$ where $d_n^2 > 0$, $d_{n+m}^2 > 0$. Denote the nonzero elements as $\sigma^2_{n,k}$, $\sigma^2_{n+m,k}$. $\qquad (2.15)$

The Gaussian assumption on v_k is to keep the analysis simple, and is not a necessary condition. In practice a more efficient excitation would be where elements had values in a bounded domain. Also, results are readily derived for the case when v_k is deterministic but containing a sufficient number of frequency components.

The following results are now a consequence of a straightforward application of results from [4,5].

Lemma 2.2
Consider the linear signal model (2.1) (2.2) formulated as (2.11) with the v_k selection of (2.15). (i) Then under (2.8), x_k is the output of a linear time invariant system reachable from u_k. Moreover, with u_k selected so that for some $\alpha > 0$

$$\lim_{k \to \infty} \ln k^{1+\alpha} (\sum_1^k \bar{u}_i \bar{u}_i^T)^{-1} = 0, \qquad (2.16)$$

then

$$\lim_{k \to \infty} \ln k^{1+\alpha} P_k = 0, \quad \lim_{k \to \infty} \ln k^{1+\alpha} \bar{P}_k = 0 \qquad \text{a.s} \qquad (2.17)$$

(ii) Also under (2.14), x_k is the output of a linear time invariant system reachable from u_k and the elements $v_{n,k}$, $v_{n+m,k}$ of v_k. Then under (2.14) the excitation conditions on u_k of (2.4) and v_k, namely,

$$\lim_{k \to \infty} (\sum_1^k \sigma^2_{n,k})^{-1} = \lim_{k \to \infty} (\sum_1^k \sigma^2_{n+m,k})^{-1} = 0 \qquad (2.18)$$

translate to excitation of x_k as

$$\lim_{k \to \infty} P_k = 0 \qquad \text{a.s.} \qquad (2.19)$$

Proof:
(i) Under the coprimeness condition (2.8), then from [5] it is immediate that x_k is the output of a linear time−invariant system driven by u_k, v_k, w_k and is reachable from u_k alone. Now should u_k be acting alone, then Lemma (3.2) of [4] applies to give that for all k, and some $K > 0$

$$\sum_1^{k+m+n} x_i x_i^T \ \rangle \ K \sum_1^k \bar{u}_i \bar{u}_i^T$$

from which (for some $\alpha > 0$)

$$\lim_{k \to \infty} (\sum_1^k \bar{u}_i \bar{u}_i^T)^{-1} = 0, \implies \lim_{k \to \infty} (\sum_1^k x_i x_i^T)^{-1} = 0$$

$$\lim_{k \to \infty} \ln k^{1+\alpha} (\sum_1^k \bar{u}_i \bar{u}_i^T)^{-1} = 0 \implies \lim_{k \to \infty} \ln k^{1+\alpha} P_k = 0$$

Applying the results of Lemma (3.3) and its Remark 1 of [4], now tells us that when bounded variance white noise inputs v_k, w_k (independent of u_k) are also applied, the same implications hold, so that (2.16) implies (2.17a) as claimed. The result (2.17b) holds likewise.
(ii) Under (2.14), the elements $y_{k-1} \cdots y_{k-n-1}$, $u_{k-1} \cdots u_{k-m-1}$ of x_k are reachable from u_k alone [5], and the remaining elements $(y_{k-n} + v_{n,k})$, $(u_{k-m} + v_{n+m,k})$ of x_k

are reachable from the nonzero elements of v_k, namely $v_{n,k}$ and $v_{n+m,k}$. Thus x_k is reachable from u_k, v_k under (2.14). Applying again the Lemma (3.3) and its Remark 1 of [4] gives directly that (2.4), (2.18) together imply (2.19).

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Main Results of Section
Theorem 2.1
Consider the signal model (2.1) (2.2) which is possibly overparametrized in that either (2.8) or (2.14) holds. Consider an RLS scheme (2.13) based on the alternative model formulation (2.11) with v_k selected as in (2.15) and (2.4) holding. Consider also that u_k is sufficiently exciting in that (2.16) holds. Then there is parameter convergence as

$$\lim_{k \to \infty} \hat{\theta}_k = \theta \quad \text{a.s.} \tag{2.20}$$

where θ is the unique parameter associated with (2.1) under (2.8) or (2.14). Moreover (2.19) also holds.

Proof:
Part (i)   In the case that (2.14) is satisfied, so that $a_n = b_m = 0$, then $\theta^T v_k = 0$. Now Lemma (2.1) applies with x_k replacing $\bar{x}_k$, so that (2.20) holds if and only if (2.19) holds. Assume that (2.19) does not hold, then since $P_k < P_{k-1}$ for all k, $\text{tr}(P_k)$ converges to a nonzero element and from (2.15) the variances of v_k decay at a rate k^{-1}. Thus (2.18) holds, and in turn (2.19) holds under Lemma 2.2. This contradicts the assumption, so that (2.19) and (2.20) hold.

Part (ii)   In the case that (2.8) is satisfied, Lemma 2.2 tells us that (2.16) implies (2.17). As a consequence from (2.15) then

$$\lim_{k \to \infty} \sum_1^k \sigma^2_{n,k} < \infty \; , \quad \lim_{k \to \infty} \sum_1^k \sigma^2_{n+m,k} < \infty$$

In turn we claim that

$$\lim_{k \to \infty} \sum_1^k (\theta^T v_k)^2 < \infty \quad \text{a.s.} \tag{2.21}$$

This follows, since as is readily established under (2.15)

$$\sum_1^k (v^2_{k,n} - \sigma^2_{n,k}) \; , \quad \sum_1^k (v^2_{n+m,k} - \sigma^2_{n+m,k})$$

are martingales bounded in L_2 and converge almost surely. Under (2.21) and (2.17b) we now claim

$$\lim_{k \to \infty} \bar{P}_k \sum_1^k w_i(v_i^T \theta) = 0 \quad \text{a.s.} \tag{2.22}$$

To see this observe that

$$E[M_k \mid F_{k-1}] = M_{k-1} \; , \quad M_k = \sum_1^k w_i(v_i^T \theta)$$

$$E[M_k^2] = E\{\sum_{i=1}^k \sum_{j=1}^k \theta^T v_i \; E\{w_i w_j \mid F_{\min i-1, j-1}\} \; v_j^T \theta\}$$

$$< E[\sum_1^k (\theta^T v_i)^2] \sigma^2_w$$

Thus M_k is a martingale on F_{k-1}, bounded in L_2 under (2.21) and so converges almost surely, so that (2.17b) implies (2.22).
Now under (2.17b) Lemma (2.1) can be applied to yield the following

$$\lim_{k \to \infty} \bar{P}_k \sum_1^k \bar{x}_i w_i = 0 \; , \quad \lim_{k \to \infty} \bar{P}_k \sum_1^k \bar{x}_k(v_i^T \theta) = 0$$

The former results follows from (2.7) and the relationships

$$\bar{\theta}_k = \bar{P}_k \sum_1^k \bar{x}_i y_i \; , \quad \tilde{\theta}_k = -\bar{P}_k(\sum_1^k \bar{x}_i w_i - B_0)$$

The second result follows from the first since $(v_i^T \theta)$ has the same essential properties as w_i in (2.2) (2.4).
The results (2.22),(2.23) lead in turn to the following convergence results

$$\bar{P}_k B_k - \bar{P}_k \sum_1^k (\bar{x}_i \bar{x}_i^T + \bar{x}_i v_i^T + v_i \bar{x}_i^T + v_i v_i^T) \to I \quad \text{a.s}$$

as $k \to \infty$

$$\hat{\theta}_k - P_k \sum_1^k x_i y_i$$

$$- (\bar{P}_k B_k)^{-1} \bar{P}_k \sum_1^k (\bar{x}_i \bar{x}_i^T \theta + \theta^T v_i \bar{x}_i^T \theta$$

$$+ \bar{x}_i w_i + \theta^T v_i w_i) \to \theta \quad \text{a.s. as } k \to \infty$$

so that (2.20) holds as claimed. $\triangle\triangle\triangle$

Remarks:
1.   The specific v_k selection of the theorem is for the case when there is one possible pole−zero cancellation in the model. This is clearly one of the most important cases since in selecting a model order there is a tendency when in doubt to merely increase a likely order by one for safety.
2.   If a bank of estimators is employed conditioned on different model orders, then the results above tell us that only odd (or even) orders need to be covered. Such a saving is a factor of two.
3.   Rather than work with banks of estimators as in Remark 2 above, an ad hoc approach is to relax (2.15) and have $v_{k,1}$, $v_{k,2}$,... $v_{k,n}$ each independent and suitably exciting with the variance of $v_{k,i}$ increasing with i. Such would force pole−zero cancellations to occur near the origin, but would lead to biased estimates. We do not study this technique.
4.   The algorithms and results of this section have been analysed for the simplest of stochastic signal models, namely, when $(y_k - \theta^T x_k)$ is white. For more general autoregressive moving average exogonous input (ARMAX) models, extended least squares (ELS) based algorithms can be employed. We claim that the technique of introducing v_k also can be made to extended the capability of ELS based schemes. In particular, for those which are globally convergent under the coprimeness condition (2.9), [7] with the modifications they are globally convergent also when there is a possible overparametrization as when (2.14) holds. Essentially the same theoretical approach applies but the technical details are more tedious, so are not explored here.

3. ADAPTIVE POLE ASSIGNMENT
Pole Assignment   Let us seek an adaptive pole assignment scheme associated with the signal model (2.1) so that there is asymptotic convergence to

$$H(q^{-1})y_k = KB(q^{-1})r_k \tag{3.1}$$

where $H(q^{-1}) = 1 + h_1 q^{-1} + ... + h_{n+m} q^{-n-m}$ is specified by the desired closed loop poles, K is a constant, and r_k is a reference input. This can be achieved by the following controller [2].

$$H(q^{-1})E(q^{-1})u_k = -H(q^{-1})F(q^{-1})y_k + Kr_k \tag{3.2}$$

where $E(q^{-1}) = 1 + e_1 q^{-1} + ... + e_m q^{-m}$, $F(q^{-1}) = f_1 q^{-1} + ... + f_n q^{-n}$ are given from the solution of the Bezout equation.

$$A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) = 1 \; , \quad \text{or } S(n,m)\varphi = \alpha \tag{3.3}$$

where

$$\varphi^T = [e_1 \; e_2 ... e_m \; f_1 \; f_2 ... f_n] \; ,$$
$$\alpha^T = [-a_1 \; -a_2 ... -a_n \; 0 ... 0]$$

$$S(n,m) = \begin{bmatrix} 1 & 0 & . & . & 0 & 0 & . & . & . & 0 \\ a_1 & 1 & & & 0 & b_1 & & & & \\ . & a_1 & . & & . & . & b_1 & & & \\ . & & . & & . & . & . & . & & \\ a_n & & & 1 & b_m & & & . & . & \\ 0 & . & & & 0 & . & & & b_1 & \\ . & . & & & . & . & & & & \\ . & & . & & & . & . & & . & . \\ 0 & . & . & 0 & a_n & 0 & . & . & 0 & b_m \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \\ n+m \\ \\ \\ \\ \downarrow \end{matrix} \tag{3.4}$$

$$\leftarrow \quad m \quad \to \leftarrow \quad n \quad \to$$

It is known that solutions of (3.3) exist if and only if $\text{rank}[S \; \alpha] = \text{rank}[S]$. Also from [8]

$S(n,m)$ is nonsingular $\longleftrightarrow q^n A(q^{-1})$, $q^m B(q^{-1})$ are coprime (3.5)

In certainty equivalence adaptive pole—assignment, the estimates θ_k are used in lieu of θ in (3.4) to compute on—line estimates $\hat{\varphi}_k^{CE}$ of the controller parameters, so that in obvious notation

$$\hat{S}_k^{CE}(n,m)\hat{\varphi}_k^{CE} = \hat{\alpha}_k^{CE} \qquad (3.6)$$

When $q^n A(q^{-1})$, $q^m A(q^{-1})$ are coprime then $\hat{S}_k(n,m)$ is nonsingular and the solution of (3.6) exists. Otherwise it may not. Ill—conditioning in $\hat{S}_k^{-1}(n,m)$ can cause excessive values for $\hat{\varphi}_k$. One modification to avoid large $\hat{\varphi}_k$ is to select $\hat{\varphi}_k = \hat{\varphi}_{k-1}$ during ill—conditioning, but for unknown plants, it is not a priori clear how to quantify ill—conditioning to achieve a useful adaptive controller.

Central Tendency Adaptive Control   In central tendency adaptive pole assignment [9], ill—conditioning in calculating the controller parameters is avoided without requiring prior information concerning the plant or controller. Suppose there is a Gaussian a posteriori probability density for the model parameters θ as $N[\theta_k,\hat{\sigma}_k{}^2 P_k]$ where $\hat{\sigma}_k{}^2$ is an estimate of σ^2, then there is an associated nongaussian probability density for the pole assignment controller parameters φ. A central tendency selection $\hat{\varphi}_k^{CT}$ is one which maximizes this density, or at least avoids the tails of this density. Practical implementations are given in [9]. Associated with $\hat{\varphi}_k^{CT}$ is some parameter estimate θ_k^{CT} which is not in general θ_k. Thus, in obvious notation

$$\hat{S}_k^{CT}(n,m)\hat{\varphi}_k^{CT} = \hat{\alpha}_k^{CT} \qquad (3.7)$$

The estimate θ_k^{CT} has the property [9] that it is "close" to θ_k but "far" from hypersurfaces for which $q^n A_k(q^{-1})$, $q^m B_k(q^{-1})$ are not coprime. As a consequence, the following property is claimed for central tendency adaptive pole assignment.

The selections $\theta_k^{CT} \to \theta$ are such that $[\hat{S}_k^{CT}(n,m)]^{-1}$ exists for all k, and if θ belongs to the hypersurface (in θ—space) defined by $q^n A(q^{-1})$, $q^m B(q^{-1})$ not coprime, then as $k \to \infty$, θ_k^{CT} is contained in a cone centred at θ which excludes the tangent hyperplane at θ.   (3.8)

Remarks:
1. To give a geometric interpretation of (3.8), consider Figures 3.1, 3.2. The heavy arc is the pole zero cancellation singular region for θ estimates in Θ—space. The light shaded area is a zone of ill—conditioning control surrounding the singular arc. The double shaded area cones are the conic regions of possible central tendency control estimates θ_k^{CT} of (3.8). Figure 3.1 depicts the situation when the solution (2.14) is at the conic intersection and the cones avoid the ill—conditioned regions. Figure 3.2 depicts a solution not satisfying (2.14) when the central tendency estimates θ_k^{CT} become ill—conditioned when converging.
2. The property (3.8) can be viewed as a corollary of results rigorously proved in [9], although specific reference has not been made to (3.8) in [9]. The property is readily believable, but since it is not rigorously proved as such in [9] and is beyond the scope of this paper, we here add the "qualification" to central tendency control that (3.8) be satisfied.
   Then two cases are studied now.

Case (i) Known Model Order   We now claim the following
   Theorem 3.1
   Consider the signal model (2.1) for the case when it is not overparametrized, so that $q^n A(q^{-1})$, $q^m B(q^{-1})$ are coprime. Consider also RLS parameter estimates θ_k from (2.13), and associated certainty equivalence [or central tendency] pole assignment with controller parameters $\hat{\varphi}_k^{CE}$ [or $\hat{\varphi}_k^{CT}$] and Sylvester matrices $\hat{S}_k^{CE}(n,m)$ [or $\hat{S}_k^{CT}(n,m)]$. Consider also that r_k is sufficiently exciting so that $\hat{\sigma}_k{}^2 P_k \to 0$ as $k \to \infty$ and there is parameter convergences with $\theta_k \to \theta$ as $k \to \infty$. Then

$$\lim_{k \to \infty} [\hat{S}_k^{CE}(n,m)]^{-1} = S^{-1}(n,m), \text{ or}$$

$$\lim_{k \to \infty} [\hat{S}_k^{CT}(n,m)]^{-1} = S^{-1}(n,m)$$

$$\lim_{k \to \infty} \hat{\varphi}_k^{CE} = \varphi, \text{ or } \lim_{k \to \infty} \hat{\varphi}_k^{CT} = \varphi \qquad (3.9)$$

Proof: This is immediate from the coprimeness assumption and the property (3.5). Notice that there is no need of assumption (3.8).

Case (ii) Overparametrization   We now claim the following
   Theorem 3.2
   Consider the signal model (2.1) with $n > 1$, $m > 1$, and (2.14) holding. Consider also RLS parameter estimates θ_k of (2.13) and associated central tendency adaptive pole assignment controller parameters $\hat{\varphi}_k^{CT}$ given from (3.7) with (3.8) satisfied. Then under RLS convergence of θ_k^{CT} to θ as $k \to \infty$,

$$\lim_{k \to \infty} \det \hat{S}_k^{CT}(n,m) = 0, \quad \lim_{k \to \infty} \hat{\varphi}_k^{CT} = \varphi^* \qquad (3.10)$$

where φ^* is a unique solution of $S(n,m)\varphi^* = \alpha$ with zero elements as

$$\varphi^* = [e^*_1 \ e^*_2 ... e^*_{m-1} \ 0 \ f^*_1 \ f^*_2 ... f^*_{n-1} \ 0]^T \qquad (3.11)$$

Proof: The proof is given in the Appendix, based on Sylvester matrix property studied in Section 4.

Remarks:
1. It might be thought that the results can be more directly proved from properties of Diophantine equations. Although certain progress can be made along these lines, and indeed the results can be stated in such terms, it does not appear straightforward to complete any proof without resort to Sylvester matrix properties as in Section 4.
2. This theorem result is dependent on the nonstandard nature of the RLS algorithm with its internal perturbations v_k. In the presence of overparametrization, standard RLS estimation (when $v_k = 0$) will almost surely not converge to the unique limits solution (2.14). This means that there is inevitable ill—conditioning. This situation applies even when a central tendency adaptive control law is implemented, as depicted in Figure 3.2.
3. The above theorem result is also facilitated by the central tendency property (3.8) of the adaptive controller. In the presence of overparametrization, even when the modified RLS algorithm of the paper is implemented without (3.8) satisfied, there is a nonzero probability of ill—conditioning as suggested from Figure 3.1, at least during transients.
4. The theorem is developed in conjunction with the RLS estimation (2.4) which copes with possible overparametrization, by the a_n, b_m. Should some RLS based scheme cope with higher orders of overparametrization, the results of this theorem would still hold. The details of a more general proof are straightforward and are omitted here.
5. The result of the theorem also apply to the situation when (3.3) is replaced by

$$A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) = H(q^{-1}) \qquad (3.12)$$

with $H(q^{-1})$ having degree no greater than $n+m-2$. This is the usual situation when $A(q^{-1})$ is of degree $n-1$, and $B(q^{-1})$ is of degree $m-1$. The proof details are a mild variation on that given here when (3.3) applies. The restriction on $H(q^{-1})$ implies that the associated α in the algebraic form of (3.12) corresponding to (3.3.b) has its last two entries zero.
6. The above result covers the cases when the plant has a delay of unity or greater. In these cases $B(q^{-1})$ specializes as having a factor q^{-N} where $N \geqslant 1$.

4. A PROPERTY OF SYLVESTER MATRICES
   First recall (3.4) which associates with $\theta = [a_1 ... a_n \ b_1 ... b_m]^T$ a Sylvester matrix $S(n,m)$. Let us denote the adjoint of $S(n,m)$ as $M(n,m)$ and determinant as $D(n,m)$. Now consider a (linear) trajectory in θ—space parametrized in terms of a scalar variable ξ, as

$$\theta(\xi) = [a_1 ... a_{n-1} \ (a\xi) \ b_1 ... b_{m-1} \ (b\xi)]^T \ , \ |a| + |b| = 1 \qquad (4.1)$$

Also denote the Sylvester matrix associated with $\theta(\xi)$ as $S_\xi(a_n,b_m)$, its adjoint matrix as $M_\xi(a_n,b_m)$ and determinant as $D_\xi(a_n,b_m)$. Then we claim the following:

Lemma 4.1   Consider
$S_\xi^{-1}(a_n,b_m) = D_\xi^{-1}(a_n,b_m)M_\xi(a_n,b_m)$ for the case that

$D(n-1,m-1) \neq 0, \quad ab_{m-1} - a_{n-1}b \neq 0 \qquad (4.2)$

Then the following limits exist as $\xi \to 0$ for all $j < n+m-1$

$$\lim_{\xi \to 0} D_\xi^{-1}(a_n,b_m)[M_\xi(a_n,b_m)]_{i,j}$$

$$- \begin{cases} [S^{-1}(n-1,m-1)]_{i,j} & \text{for } i < m \quad (4.3a) \\ [S^{-1}(n-1,m-1)]_{i-1,j} & m < i < m+n \quad (4.3b) \\ 0 & \text{for } i = m \text{ or } i = m+n \quad (4.3c) \end{cases}$$

Proof: We consider in turn expressions for $D_\xi(a_n,b_m)$ and $M_\xi(a_n,b_m)$ in terms of order ξ and higher order terms denoted $O(\xi^2)$. Simple manipulations give an expression for $D_\xi(a_n,b_m)$ in terms of the elements of the last row of $S_\xi(a_n,b_m)$ and their minors, with the minors likewise expanded, as

$$D_\xi(a_n,b_m) = (-1)^n \xi(ab_{m-1} - a_{n-1}b)D(n-1,m-1) + O(\xi^2) \quad (4.4)$$

Noting that $[M_\xi(a_n,b_m)]_{i,j}$ is a (signed) determinant of a submatrix of $S_\xi(a_n,b_m)$, then again simple manipulations give an expression in terms of the elements of the last row of the submatrix of $S_\xi(a_n,b_m)$ and minors of this submatrix as

$$[M_\xi(a_n,b_m)]_{i,j} = (-1)^n(a\xi)[M_\xi(a_n,m-1)]_{i,j}$$
$$+ (b\xi)[M_\xi(a_n,b_m)]_{i,j} + O(\xi^2),$$
$$\text{for } i < m, \ j < n+m-1 \quad (4.5)$$

Here $M_\xi(a_n,m-1)$ denotes the adjoint matrix of the lower dimensioned Sylvester matrix $S_\xi(a_n,m-1)$ which associates with the vector $[I_{n+m-1} \ 0]\theta(\xi) = [a_1...a_{n-1} \ (a\xi) \ b_1...b_{m-1}]^T$. Using derivations similar to those giving (4.5), then

$$[M_\xi(a_n,m-1)]_{i,j} = b_{m-1}[M(n-1,m-1)]_{i,j} + O(\xi)$$
$$= b_{m-1}D(n-1,m-1)[S^{-1}(n-1,m-1)]_{i,j} + O(\xi) \text{ for } i < m, \ j < n+m-1 \quad (4.6)$$

where $S^{-1}(n-1,m-1)$ exists under the assumption (4.2). The dual form of (4.6) is

$$[M_\xi(n-1,b_m)]_{i,j} = (-1)^{n-1}a_{n-1}D(n-1,m-1)[S^{-1}(n-1,m-1)]_{i,j} + O(\xi)$$
$$\text{for } i < m, \ j < n+m-1 \quad (4.7)$$

Substitution of (4.6) (4.7) into (4.5) yields

$$[M_\xi(a_n,b_m)]_{i,j} = (-1)^n(ab_{m-1} - a_{n-1}b)\xi D(n-1,$$
$$m-1)[S^{-1}(n-1,m-1)]_{i,j} + O(\xi^2)$$
$$\text{for } i < m, \ j < n+m-1 \quad (4.8)$$

Dividing (4.8) by $D_\xi(a_n,b_m)$ from (4.4) and taking limits as $\xi \to 0$ under the assumption (4.2) leads to the result (4.3a). The result (4.3b) can be established along similar lines.

Similar arguments to the derivation of (4.5) (4.6) leads to

$$[M_\xi(a_n,b_m)]_{m,j} = (b\xi)[M_\xi(n-1,b_m)]_{m,j} + O(\xi^2),$$
$$\text{for } j < n+m-1 \quad (4.9)$$

$$[M_\xi(n-1,b_m)]_{m,j} = (b\xi)O(\theta), \text{ for } j < n+m-1 \quad (4.10)$$

where $O(\theta)$ denotes a quantity that is bounded in term of θ. From (4.9) (4.10),

$$[M_\xi(a_n,b_m)]_{m,j} = O(\xi^2), \text{ for } j < n+m-1 \quad (4.11)$$

Similarly, we have

$$[M_\xi(a_n,b_m)]_{n+m,j} = O(\xi^2) \text{ for } j < n+m-1 \quad (4.12)$$

Then dividing (4.11) and (4.12) by $D_\xi(a_n,b_m)$ from (4.4) and taking limits as $\xi \to 0$ under the assumption (4.2), we have (4.3c) as claimed. $\triangle\triangle\triangle$

Remarks:
1. This result can be generalized to the case other a_i, b_i converge to zero using the same technique as above. Details are omitted.
2. The above result can be expressed in terms of an $(n+m)$ by $(n+m-2)$ matrix.

$$[S^+(\theta,\xi)]_{i,j} = D_\xi^{-1}(a_n,b_m)[M_\xi(a_n,b_m)]_{i,j} \quad (4.13)$$

Thus under (4.2)

$$S^+(\theta,0) \triangleq \lim_{\xi \to 0} S^+(\theta,\xi) - \begin{bmatrix} I_{m-1} & 0 \\ 0 & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} S^{-1}(n-1,m-1) \quad (4.14)$$

Moreover, simple manipulations yield

$$S(n,m)\begin{bmatrix} I_{m-1} & 0 \\ 0 & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S(n-1,m-1) \\ 0 \end{bmatrix} \begin{matrix} \uparrow \\ n+m \\ \downarrow \end{matrix}$$

so that

$$S(n,m)S^+(\theta,0) - \begin{bmatrix} I_{n+m-2} \\ 0 \end{bmatrix} \begin{matrix} \uparrow \\ n+m \\ \downarrow \end{matrix} \quad (4.15)$$

Corollary 4.2
Consider that $A(q^{-1},\xi)$, $B(q^{-1},\xi)$ are the polynomials associated with the $\theta(\xi)$ as in (4.1). Consider also that $E(q^{-1},\xi)$, $F(q^{-1},\xi)$ are the solution to
$$A(q^{-1},\xi)E(q^{-1},\xi) + B(q^{-1},\xi)F(q^{-1},\xi) = 1 \quad (4.16)$$
with the degree of $E(q^{-1},\xi)$ being m, and the degree of $E(q^{-1},\xi)$ being n. Then under the same conditions as in Lemma 4.1, then the last coefficients of $E(q^{-1},\xi)$, $F(q^{-1},\xi)$ converge as

$$\lim_{\xi \to 0} e_m(\xi) = 0, \quad \lim_{\xi \to 0} f_n(\xi) = 0 \quad (4.17)$$

Proof: It is straightforward from the result of Lemma 4.1, and in particular from (4.14). However, to prove it by just using the properties of polynomial equations appears too formidable without resort to the result on the Sylvester matrix described in Lemma 4.1.

Lemma 4.3
Consider the Bezout equation (3.3) under (2.14) with $m > 1$. Then a unique solution φ^* of (3.7) exists as

$$\varphi^* = S^+(\theta,0)[I_{n+m-2} \ 0]\alpha \quad (4.18)$$

with the property that φ^* has nth and $(n+m)$th elements which are zero.

Proof: Under (2.14), of course $S^{-1}(n,m)$ does not exist, but (4.2) holds for a suitable selection of a,b. Also, the last two elements of α are zero. Now Lemma 4.1 holds under (4.2) so that (4.14),(4.15) apply. Thus with φ^* uniquely defined from (4.18)

$$S(n,m)\varphi^* - \begin{bmatrix} I_{n+m-2} & 0 \\ 0 & 0 \end{bmatrix} \alpha = \alpha$$

and (3.3) is satisfied. Application of (4.14) under (4.16) gives guarantees that the nth and $(n+m)$th elements of φ^* are zero. $\triangle\triangle\triangle$

5. SIMULATIONS
Presented at Conference.

6. CONCLUSIONS
Nonstandard adaptive estimation and control techniques have been proposed to avoid ill-conditioning which can arise when standard techniques are applied to signal models with overparametrization. The techniques avoid ill-conditioning and yield asymptotic optimality in the case when there is possibly one pole/zero cancellation in the assumed signal model. It seems reasonable to apply such techniques in conjunction with on-line model order estimation techniques since from finite data these possibly lead to overestimation of the order. The techniques have been studied for the case when exact pole zero cancellation occurs, but is known from simulations to avoid ill-conditioning when there are stable near pole zero cancellations. The results of the paper are a starting point from which to cope with higher order pole-zero cancellations.

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APPENDIX

Consider first the following lemma.

Lemma A.1

Under (2.13), (2.14),(3.8) [the conditions of Theorem 3.2]

$$\lim_{k \to \infty} S^+(\hat{\theta}_k^{CT}, \xi_k) = S^+(\theta, 0) \tag{A.1}$$

Proof: With (3.8) satisfied and (2.13)

$$\lim_{k \to \infty} \hat{\theta}_k^{CT} = \theta$$

so that when (2.14) holds

$$\lim_{k \to \infty} \hat{a}_{n,k}^{CT} = \lim_{k \to \infty} a_k \xi_k = 0$$

$$\lim_{k \to \infty} \hat{b}_{m,k}^{CT} = \lim_{k \to \infty} b_k \xi_k = 0$$

Since $|a_k| + |b_k| = 1$, recalling (4.1), then

$$\lim_{k \to \infty} \xi_k = 0 \tag{A.2}$$

Now, under (3.8), (2.13), (2.14), we have

$$\lim_{\xi_k \to 0} S^+(\hat{\theta}_k^{CT}, \xi_k) - S^+(\hat{\theta}_k^{CT}, 0)$$

$$- \begin{bmatrix} I_{m-1} & & 0 \\ 0 & & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} [\hat{S}_k^{CT}(n-1, m-1)]^{-1}. \tag{A.3}$$

Also under (3.8), (2.13), (2.14)

$$\lim_{k \to \infty} S^+(\hat{\theta}_k^{CT}, 0) = S^+(\theta, 0) \tag{A.4}$$

From continuity and (A.2), (A.1) is established.

△△△

Remark   Without (3.8) satisfied, it can not be guaranteed that $S^+(\hat{\theta}_k^{CT}, \xi_k)$ exists for all k. Also, with $S^+(\hat{\theta}_k, \xi_k)$ large then φ_k is large, and ill—conditioning is said to occur.

Proof of Theorem 3.2

Consider the central tendency adaptive control with (3.8) satisfied and define for all k

$$[\hat{S}_k^{CT}(n,m)]^{-1} = [S^+(\hat{\theta}_k^{CT}, \xi) \, *] \tag{A.5}$$

where * denotes terms not of interest. Thus for the signal model with m > 1, or in other words the last two entries of $\hat{\alpha}_k^{CT}$ zero

$$\hat{\varphi}_k^{CT} = [\hat{S}_k^{CT}(n,m)]^{-1} \hat{\alpha}_k^{CT}$$

$$= [\hat{S}_k^{CT}(n,m)]^{-1}[I_{n+m-2} \ 0]^T[I_{n+m-2} \ 0]\hat{\alpha}_k^{CT}$$

$$= S^+(\hat{\theta}_k^{CT}, \xi)[I_{n+m-2} \ 0]\hat{\alpha}_k^{CT} \tag{A.6}$$

Now applying (A.1) when taking limits as $k \to \infty$

$$\lim_{k \to \infty} \hat{\varphi}_k^{CT} - \lim_{k \to \infty} S^+(\hat{\theta}_k^{CT}, \xi_k)[I_{n+m-2} \ 0]\hat{\alpha}_k^{CT}$$

$$- S^+(\theta, 0)[I_{n+m-2} \ 0]\alpha$$

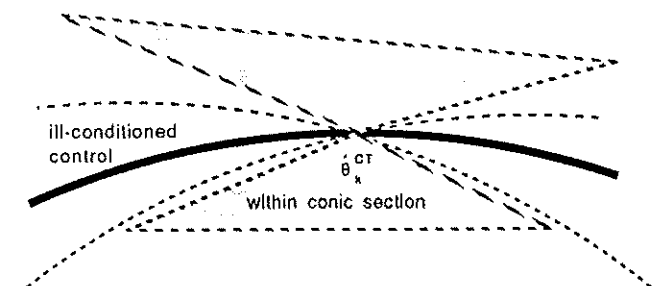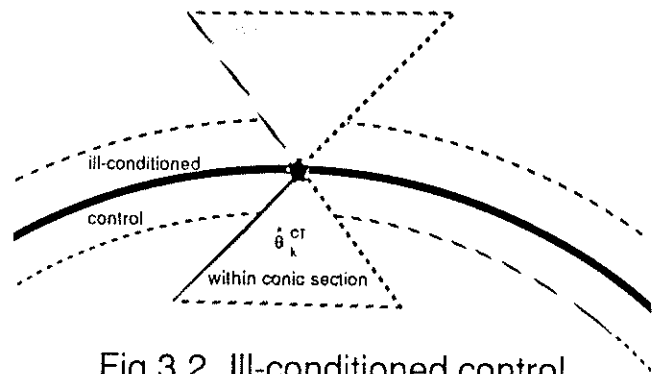The desired results (3.10) (3.11) follow from application of Lemma 4.3.

Fig. 3.1 Avoidance of ill-conditioning

Fig 3.2 Ill-conditioned control